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REMARKS ON INERTIA THEOREMS FOR MATRICES

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1. INTRODUCTION

In this note we give a unified treatment of two inertia results on the Ljapunov matrix equation

$$A^*H + HA = C\,, \quad C \geqq 0 \text{ (positive semidefinite)}\,, \quad H = H^*\,.$$

For a complex $n \times n$ matrix A the inertia, In A, of A is defined as the triple

$$\text{In } A = (\pi(A), \nu(A), \delta(A))$$

where $\pi(A)$, $\nu(A)$ and $\delta(A)$ are respectively the numbers of eigenvalues of A with positive, negative and vanishing real part. If $\{\lambda_j \mid j = 1, 2, \ldots, k\}$ is the set of distinct eigenvalues of A, then A can be written in the form (see e.g. [11])

$$A = \sum_{j=1}^{k} (\lambda_j P_j + N_j) \tag{1}$$

where $\{N_j\}$ is a set of nilpotent matrices and $\{P_j\}$ is a set of projection matrices such that

$$\sum_{j=1}^{k} P_j = I\,, \quad P_i P_j = P_j P_i = \delta_{ij} P_i\,, \quad P_i N_j = N_j P_i = \delta_{ij} N_i\,.$$

Equation (1) is easily derived from the Jordan form of A. We define

$$P_+ = \sum_{\text{Re}\lambda_j > 0} P_j \quad \text{and} \quad P_- = \sum_{\text{Re}\lambda_j < 0} P_j\,.$$

In the case $\delta(A) = 0$ we have $P_+ + P_- = I$. – H shall always denote a hermitian $n \times n$ matrix. $\delta(H) = 0$, then means H is nonsingular.

Our main tool will be the following theorem.

Theorem 1. *If A has no eigenvalues on the imaginary axis and*

$$A^*H + HA = C \tag{2}$$

holds, then

$$P_+^*H - HP_- = \frac{1}{2\pi}\int_{-\infty}^{\infty} [(A - iyI)^{-1}]^* C(A - iyI)^{-1} \, \mathrm{d}y . \tag{3}$$

Starting from (3) we will prove the following inertia theorems.

Theorem 2 [1, p. 432]. *Let A be a matrix with $\delta(A) = 0$. If*

$$A^*H + HA = C , \quad C \geqq 0 ,$$

then

$$\pi(H) \leqq \pi(A) \quad \text{and} \quad \nu(H) \leqq \nu(A) . \tag{4}$$

Theorem 3 [2], [9]. *If $A^*H + HA = C$, $C \geqq 0$ and*

$$\operatorname{rank}[C, A^*C, A^{*2}C, \ldots, A^{*n-1}C] = n , \tag{5}$$

then $\operatorname{In} A = \operatorname{In} H$ *and* $\delta(A) = \delta(H) = 0$.

There are applications of Theorem 3 to continued fractions [10] and to the linear vibration equation [9].

2. TWO LEMMAS

For the proof of Theorem 1 we need the following lemma.

Lemma 1. *Let P_1 and P_2 be two $n \times n$ matrices with*

$$\operatorname{rank} P_1 + \operatorname{rank} P_2 \geqq n . \tag{6}$$

If H satisfies

$$P_1^*HP_1 \geqq 0 \quad \text{and} \quad P_2^*HP_2 \leqq 0 , \tag{7}$$

then

$$\pi(H) \leqq \operatorname{rank} P_1 \quad \text{and} \quad \nu(H) \leqq \operatorname{rank} P_2 . \tag{8}$$

Proof. Let H have the spectral decomposition

$$H = \sum_{r=1}^{h} \mu_r Q_r$$

where μ_r are the eigenvalues of H and the Q_r's are hermitian projection matrices with $Q_r Q_s = \delta_{rs} Q_r$. We put

$$Q_+ = \sum_{\mu_r > 0} Q_r \quad \text{and} \quad Q_- = \sum_{\mu_r < 0} Q_r \, .$$

We show that

$$Q_+ \mathbf{C}^n \cap P_2 \mathbf{C}^n = \{0\} \, . \tag{9}$$

Suppose that $Q_+ u = P_2 v$, then

$$(HQ_+ u, Q_+ u) = \sum_{\mu_r > 0} \mu_r (Q_r u, Q_r u) \geqq 0 \, . \tag{10}$$

On the other hand

$$(HQ_+ u, Q_+ u) = (HP_2 v, P_2 v) = (P_2^* H P_2 v, v) \leqq 0 \, .$$

Thus $(HQ_+ u, Q_+ u) = 0$ and by (10) $(Q_r u, Q_r u) = 0$ for each r with $\mu_r > 0$ and therefore $Q_+ u = 0$. (9) implies $\operatorname{rank} Q_+ + \operatorname{rank} P_2 \leqq n$. Similarly $\operatorname{rank} Q_- + \operatorname{rank} P_1 \leqq n$. The inequalities (8) are now immediate consequences of (6).

Lemma 2. *Let P_1 and P_2 be two $n \times n$ matrices with $P_1 + P_2 = I$ and $P_i P_j = \delta_{ij} P_i$, $i, j = 1, 2$. If H satisfies*

$$P_1^* H - H P_2 > 0 \text{ (positive definite)} \, , \tag{11}$$

then In H *is given by*

$$\delta(H) = 0 \, , \quad \pi(H) = \operatorname{rank} P_1 \, , \quad \nu(H) = \operatorname{rank} P_2 \, .$$

Proof. Suppose $Hv = 0$, then $(v(P_1^* H - HP_2), v) = 0$ and because of (11) $v = 0$. This means $\delta(H) = 0$ and $\pi(H) + \nu(H) = n$, so that in (8) the equality signs hold.

3. PROOFS

Proof of Theorem 1. Let Γ be a positively-orientated simple closed curve that consists of a segment of the imaginary axis and of a left semi-circle of radius R around the origin. If R is greater than the spectral radius of A, then

$$\frac{1}{2\pi i} \int_\Gamma (zI - A)^{-1} \, dz = P_- \quad \text{and} \quad \frac{1}{2\pi i} \int_\Gamma (zI + A)^{-1} \, dz = P_+ \, . \tag{12}$$

The integrals in (12) exist and the formulas follow easily from (see e.g. [11])

$$(zI - A)^{-1} = \sum_{j=1}^{k} [(z - \lambda_j)^{-1} P_j + M_j] .$$

For $z \in \Gamma$ we write (2) as

$$(zI + A^*)^{-1} H + H(A - zI)^{-1} = (A^* + zI)^{-1} C(A - zI)^{-1} ,$$

divide both sides of this equation by $2\pi i$ and integrate around Γ. The integrals on the left-hand side are evaluated with (12) and since the right-hand side is $O(z^{-2})$ at infinity, we obtain (3). – Let us remark that (3) is a generalisation of a result of SMITH [5, p. 425] which was stated for the case of a stable matrix A, i.e. $P_+ = 0$, $P_- = I$.

Proof of Theorem 2. For $C \geqq 0$ the matrix

$$M = \int_{-\infty}^{\infty} [(A - iyI)^{-1}]^* C(A - iyI)^{-1} \, \mathrm{d}y \tag{13}$$

is also positive semidefinite, so $P_+^* H - HP_- \geqq 0$. If we put $P_1 = P_+$ and $P_2 = P_-$ and observe that rank $P_+ = \pi(A)$ and rank $P_- = \nu(A)$, then the inequalities (4) follow from Lemma 1.

Proof of Theorem 3. We first show that $\delta(A) = 0$. Assume the contrary, then there is a u, $u \neq 0$, and a real α such that $Au = i\alpha u$. Let r be a nonnegative integer, then $A^*(A^{*r}HA^r) + (A^{*r}HA^r) A = A^{*r}CA^r$. Hence

$$(A^{*r}CA^r u, u) = (-i\alpha + i\alpha)(A^{*r}HA^r u, u) = 0 .$$

$C \geqq 0$ implies $u^*A^*C = 0$ for $r = 0, 1, \ldots, n - 1$. Thus rank $(C, A^*C, \ldots$ $\ldots, A^{*n-1}C) < n$, which contradicts to (5). Now that we know that $\delta(A) = 0$, we can write equation (3). We next show that $M > 0$ where M is given by (13). Suppose u is a vector such that $(Mu, u) = 0$. Then $((A^* + iyI)^{-1} C(A - iyI)^{-1} u, u) =$ $= 0$ or $C(A - iyI)^{-1} u = 0$ for all real y. Therefore

$$C(zI - A)^{-1} u = 0 \tag{14}$$

holds for all complex z which are not eigenvalues of A. Multiplying (14) by z^r and integrating around a curve which surrounds the eigenvalues of A we find that $CA^r u = 0$, $r = 0, 1, \ldots, n - 1$. (5) implies $u = 0$ which means $M > 0$. Theorem 3 now follows directly from Lemma 2.

The important special case of Theorem 3 where C is a positive definite matrix is due to TAUSSKY [7] and OSTROWSKI and SCHNEIDER [4].

4. STEIN'S EQUATION

Theorems corresponding to those on Ljapunov's equation (2) can be derived for Stein's equation

$$A^*HA - H = C\,. \tag{15}$$

If A is given in the form (1), we define

$$P_c = \sum_{|\lambda_j|<1} P_j \quad \text{and} \quad P_x = \sum_{|\lambda_j|>1} P_j\,.$$

Let Δ be the positively orientated unit circle. Suppose A has no eigenvalue of modulus 1. Then

$$P_c = \frac{1}{2\pi i}\int_\Delta (zI - A)^{-1}\,dz \quad \text{and} \quad P_x = \frac{1}{2\pi i}\int_\Delta A(zA - I)^{-1}\,dz\,. \tag{16}$$

For $z \in \Delta$ write (15) as

$$HA(zA - I)^{-1} + (A^* - zI)^{-1} H = (A^* - zI)^{-1} C(zA - I)^{-1}\,.$$

Using (16) we obtain

$$HP_x - P_c^*H = \frac{1}{2\pi i}\int_\Delta (A^* - zI)^{-1} C(zA - I)^{-1}\,dz = \tag{17}$$

$$= \frac{1}{2\pi}\int_\Delta [(A - e^{-i\theta}I)^{-1}]^* C(A - e^{-i\theta}I)^{-1}\,d\theta\,.$$

Equation (17) is a generalisation of another result of Smith [6, p. 214]. There it was assumed that $P_c = I$ and $P_x = 0$. – By the same method we used for Theorem 3 we can refine a theorem which is mentioned in [8].

Theorem 4. *If* $A^*HA - H = C$, $C \geqq 0$ *and* $\operatorname{rank}(C, A^*C, \ldots, A^{*n-1}C) = n$, *then A has no eigenvalues of modulus* 1. *The number of eigenvalues of A with modulus less [greater] than* 1 *is equal to the number of negative [positive] eigenvalues of H.*

The results derived in this note for the equations (2) and (15) can not be extended to the more general matrix equation

$$\sum_{\varrho,\sigma=0}^{m} c_{\varrho\sigma} A^{*\varrho} H A^\sigma = C, \quad c_{\varrho\sigma} = \overline{c_{\sigma\varrho}}\,,$$

as the following example shows. Take

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$A^T H_1 A + H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} > 0, \quad A^T H_2 A + H_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} > 0,$$

but $\operatorname{In} H_1 \neq \operatorname{In} H_2$. — Generalisations of inertia theorems of a different type are contained in [3].

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