Czechoslovak Mathematical Journal, 25 (100) 1975, Praha

A GENERALISATION OF ALMOST-COMPACTNESS, WITH AN ASSOCIATED GENERALISATION OF COMPLETENESS

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(Received June 19, 1973)

FLETCHER and NAIMPALLY [3] have recently shown that for quasi-uniform spaces one can define properties, called almost-completeness and almost-precompactness, which bear to almost-compactness a relationship similar to that of completeness and precompactness to compactness. Examination of Katětov's proof [4] of the existence of a canonical almost-compactification or H-closure of a given Hausdorff space, bearing in mind that a filter has an open base iff it is round in the associated Pervin P-proximity ([5], pp. 106–107, 19.7 and 19.14)), shows that it has many similarities to, but some differences from, the standard process of compactification of a (symmetrical) proximity space by maximal round filters. These considerations have led the present author to define, for P-proximity spaces and quasi-uniform spaces respectively, weak analogues of compactness and completeness, which we call P-compactness and P-completeness. These reduce on the one hand, for (symmetrical) proximities or uniformities, to ordinary compactness and completeness respectively, on the other, for the Pervin quasi-uniformity and its P-proximity, to almost-compactness and almost-completeness (though in general almost-completeness is not reducible to P-completeness). After discussing the elementary properties of these notions, we consider the problem of constructing Hausdorff P-compactifications and P-completions of a given Hausdorff P-proximity or quasi-uniform space. We show that these always exist, but are not unique, and are not necessarily in one-one relationship to each other for a given P-precompact space. Finally, we obtain some results concerning ordinary completions which arise naturally from our methods.

1. DEFINITIONS AND ELEMENTARY RESULTS.

Our notation is in general that of [5]. We point out two main differences: first, we always write relations on the right, so that, e.g., the set $\{x; \exists a \in A, (a, x) \in V\}$ is written $A \circ V$ or for brevity AV, not $V(A)$; secondly, we write P-proximities on a space X in terms of the relation $<$ defined in terms of ((5), 19.2) by (c.f. (5), 3.1, p. 15)

$$A < B \quad \text{iff} \quad A \text{ non } \beta(X - B).$$

We note that a P-proximity space is a topogeneous space as defined in (1). As in the case of an ordinary proximity, a filter $\mathscr{F}$ is said to be *round* iff $A \in \mathscr{F} \Rightarrow \exists A_1, A_1 \in \mathscr{F}, A_1 < A$. Just as for proximities, a round filter is maximal round iff $A < B$ implies that either $X \setminus A$ or B is in the filter ((5), 6.8, p. 35).

(1.1) **Definition.** If $\mathscr{F}$ is a filter, $\mathscr{F} \circ r$ will denote the filter $\{B; \exists A \in \mathscr{F}, A < B\}$.

(1.2) **Definition.** Let $(X, <)$ be a P-proximity space, with the associated topology. For $x \in X$, $A \subset X$, we say that *x is P-adherent to A* iff $A < B \Rightarrow x \in \bar{B}$, *$x$ is P-adherent to a filter $\mathscr{F}$* on X iff x is P-adherent to every set A of $\mathscr{F}$, and *$\mathscr{F}$ is P-convergent to x* iff $\{x\} < N$, $(X \setminus A) < (X \setminus N)$ imply $A \in \mathscr{F}$.

(1.3) **Proposition.** *The following are equivalent:*

(a) *x is P-adherent to $\mathscr{F}$;*

(b) *x is adherent to $\mathscr{F} \circ r$;*

(c) *$\exists \mathscr{F}_1 \supset \mathscr{F}, \mathscr{F}_1$ P-convergent to x.*

Proof. The equivalence of (a) and (b) is immediate. Now (a) can be written thus: if $A \in \mathscr{F}$ and $A < B$, then $\{x\} < N$ implies $B \not\subset (X \setminus N)$, as N is a neighbourhood of x. Equivalently, $A \in \mathscr{F} \Rightarrow A \nless (X \setminus N)$, or $(X \setminus C) < (X \setminus N) \Rightarrow C \cap A \neq \emptyset$, all $A \in \mathscr{F}$. The equivalence with (c) now follows in the standard way on varying A, N and C.

(1.4) **Proposition.** *The following are equivalent:*

(a) *Every filter has a P-adherent point;*

(b) *Every round filter has an adherent point;*

(c) *Every filter has a P-convergent refinement;*

(d) *Every maximal round filter has a limit.*

These follow immediately from (1.2) and (1.3) since $\mathscr{F}$ is round iff $\mathscr{F} = \mathscr{F} \circ r$, and $\mathscr{F} \to x \Leftrightarrow \mathscr{F} \circ r \to x$, while $x \in \operatorname{Ad} \mathscr{M}$, $\mathscr{M}$ maximal round, implies $\mathscr{M} \to x$.

(1.5) **Definition.** A space $(X, <)$ with the properties of (1.4) is called *P-compact*. We note that P-compactness is not a topological property since it depends on the proximity relation; however, it is clear that for a symmetrical proximity P-compactness reduces to ordinary compactness.

Now suppose that the P-proximity is derived from a quasiuniformity $\mathscr{V}$. We shall say that a set $A \subset X$ is *radially V-small* (r . $V -$ s) in X iff $\exists x \in X$, $xV \supset A$. As in (3), a *Cauchy filter* is one which contains a r . $V -$ s set for every $V \in \mathscr{V}$.

(1.6) **Definition.** $(X, \mathscr{V})$ is *P-complete* iff every Cauchy maximal round filter has a limit, and *P-precompact* iff every maximal round filter is Cauchy.

We have immediately, since $\mathscr{F}$ Cauchy $\Rightarrow \mathscr{F} \circ r$ Cauchy:

(1.7) **Proposition.** (i) *A quasi-uniform space is P-compact (in its associated P-proximity) iff it is both P-precompact and P-complete.*

(ii) *The following are equivalent*:

(a) *$(X, \mathscr{V})$ is P-complete;*

(b) *Every Cauchy round filter has an adherent point;*

(c) *Every Cauchy filter has a P-adherent point.*

As usual, we can express P-compactness and P-precompactness in terms of coverings, though this expression is not very elegant in general.

(1.8) **Proposition.** *$(X, <)$ is P-compact iff given any indexed family of pairs of sets $\{(G_i, H_i); i \in I\}$ such that the G_i form an open covering of X and $X \setminus H_i < < X \setminus G_i$ for each i, there exists a finite covering of X by $\{H_{i_r}; r = 1, 2, \ldots, n\}$ say.*

If there exists a filter $\mathscr{F}$ with no P-adherent point, then to each x we can associate an open neighbourhood G_x and a set A_x of $\mathscr{F}$ such that $A_x < X \setminus G_x$. Putting $H_x = X \setminus A_x$, no finite subfamily $\{H_{x_r}; r = 1, 2, \ldots, n\}$ can cover X. Conversely, if there exists a family $\{G_i, H_i); i \in I\}$ as in the statement above but with no finite cover of X by H-sets, the sets of the form $X \setminus (H_{i_1} \cup \ldots \cup H_{i_n})$, n finite but arbitrary, generate a filter with no P-adherent point, since the G_i cover X.

(1.9) **Proposition.** *The following are equivalent*:

(a) *The quasi-uniform space $(X, \mathscr{V})$ is P-precompact;*

(b) *For every $\mathscr{F}$ and every $V_0 \in \mathscr{V}$, $\exists x_0$ such that $A \in \mathscr{F}$, $V \in \mathscr{V} \Rightarrow AV \cap x_0V_0 \neq \emptyset$;*

(c) *If to each x is assigned a member V_x of $\mathscr{V}$ there is a finite covering of X by sets of the form $aV_0V_a^{-1}$.*

Proof. (a) $\Rightarrow$ (c).

Suppose (c) is false; then there exists a filter $\mathscr{F}$ containing all sets of the form $X \setminus aV_0V_a^{-1}$. Since $(X \setminus aV_0V_a^{-1})\, V_a \subset X \setminus aV_0$, the round filter $\mathscr{F} \circ r$ contains all sets of the form $X \setminus aV_0$. By Zorn's lemma $\mathscr{F} \circ r$ is contained in a maximal round filter $\mathscr{M}$, which cannot contain any r . V_0 − s. set.

(c) $\Rightarrow$ (b).

Suppose that for given $\mathscr{F}$, V_0 no x_0 exists as required. Then for each x, $\exists V_x \in \mathscr{V}$, $A_x \in \mathscr{F}$ with $xV_0 \cap A_xV_x = \emptyset$, so that $xV_0V_x^{-1} \cap A_x = \emptyset$ and $X \setminus xV_0V_x^{-1} \in \mathscr{F}$. This clearly means that no cover of the type stated in (c) can exist, for this choice of V_x.

(b) $\Rightarrow$ (a).

Suppose (b) holds; let $\mathcal{M}$ be a maximal round filter and let $V_0 \in \mathscr{V}$ be given; $\exists V_1 \in \mathscr{V}$ with $V_1^2 \subset V_0$. Given any $A \in \mathcal{M}$, $\exists B \in \mathcal{M}$, $V \in \mathscr{V}$ with $BV \subset A$ since $\mathcal{M}$ is round. But by (b) $\exists x_1$ (independent of A, B) such that $BV \cap x_1 V_1 \neq \emptyset$, all $B \in \mathcal{M}$, $V \in \mathscr{V}$. Thus $A \in \mathcal{M} \Rightarrow A \cap x_1 V_1 \neq \emptyset$, so that $X \setminus x_1 V_1 \notin \mathcal{M}$. Since $\mathcal{M}$ is maximal it follows that $(x_1 V_1) V_1 \subset x_1 V_0$ is in $\mathcal{M}$, so that $\mathcal{M}$ is $\mathscr{V}$-Cauchy.

Local P-compactness and countable P-compactness. There is more than one plausible definition of local P-compactness; the following will be adopted for the purpose of this paper.

(1.10) **Definition.** $(X, <)$ is *locally P-compact* ($(X, \mathscr{V})$ *locally P-complete*) *at* x_0 iff there exists a neighbourhood N of x_0, in the induced topology, such that every filter (every Cauchy filter) with N as a member has a P-adherent point, not necessarily lying in N.

In general, if $\mathscr{F}$ is a countably- based filter, $\mathscr{F} \circ r$ is not countably based. For this reason, we define countable P-compactness only for spaces whose P-proximity is quasimetrisable.

(1.11) **Proposition.** *In a quasi-metric space, the following are equivalent*:

(a) *Every sequence has a P-adherent point*;

(b) *Every sequence has a P-convergent sub-sequence*;

(c) *Every countably-based filter has a P-adherent point*;

(d) *Every round countably-based filter has an adherent point*;

(e) *Every round countably-based filter has a round countably-based convergent refinement.*

By a P-adherent point or P-limit of a sequence we mean of course the corresponding idea in relation to the filter defined by the sequence. The proof of the proposition is straightforward and is omitted.

(1.12) **Definition.** A quasi-metrisable P-proximity space with the properties (1.11) will be called *countably P-compact*. Similarly, if for a quasi-uniform space with a countable uniformity base (i.e. quasi-metrisable) we insert 'Cauchy' in each of the statements of (1.11), we obtain a set of equivalent statements defining a *countably P-complete* (quasi-metrisable) space.

2. P-COMPACTIFICATION OF A GIVEN P-PROXIMITY SPACE.

By an *imbedding* i of a P-proximity space $(X, <)$ in another, $(X^+, <^+)$ we mean in this section a dense proximity imbedding; that is, a one-one map of X into X^+

such that Xi is dense in X^+ and $A < B$ iff $Ai <^+ Bi \cup (X^+ \setminus Xi)$. An imbedding will be called *relatively* T2 (relatively T1) iff any two distinct points x_1^+, x_2^+ of X^+, not both in Xi, have non-overlapping neighbourhoods (resp. each has a neighbourhood not containing the other) in the topology induced by $<^+$. (Cf. CSÁSZÁR, [2] where 'relatively separated' corresponds to 'relatively T0'.)

(2.1) **Definition.** An imbedding of $(X, <)$ in a P-compact space will be called a *P-compactification* of $(X, <)$.

It is trivial to construct a one-point P-compactification (indeed, compactification) of any P-proximity space if we neglect separation conditions, since all filters can be made to converge to one 'ideal' element. We seek relatively T2 compactifications, and show that we can always construct at least two such, in general topologically distinct; one of these, analogous to the Katětov H-closure, we call the *fine* P-compactification, and the other, which reduces for a symmetrical proximity to the classical compactification, the *natural* P-compactification. Some of the ideas used, more particularly in lemma (2.2) and Theorem (2.6), are outlined in (2), but are there applied only to a symmetrical structure derived from the original one.

If X, X^+, and $f : X \to X^+$ are any sets and any function, and $\mathscr{F}, \mathscr{F}^+$ filters on X, X^+ respectively, we write $\mathscr{F}f$ for $\{B^+ \subset X^+;\ \exists A,\ A \in \mathscr{F},\ Af \subset B^+\}$ and $\mathscr{F}^+f^{-1}$ for $\{A \subset X; \exists B^+ \in \mathscr{F}^+, B^+f^{-1} \subset A\}$. Then $\mathscr{F}^+f^{-1}$ is a filter iff $B^+ \in \mathscr{F}^+ \Rightarrow$ $\Rightarrow B^+f^{-1} \neq \emptyset$; this is easily seen to hold in particular if X^+ is a P-proximity space, $\mathscr{F}^+$ is round, and Xf dense in X^+, as then $\operatorname{Int} B^+ \neq \emptyset$. If f is one-one then $(\mathscr{F}f)f^{-1} = \mathscr{F}$, though in general f and f^{-1} are not inverse operators.

(2.2) **Lemma.** *If i is an imbedding of $(X, <)$ in $(X^+, <^+)$ then $(i \cdot r^+)|\mathscr{M}$ and $i^{-1}|\mathscr{M}^+$ are reciprocal one-one functions mapping the set $\mathscr{M}$ of all maximal round filters in X onto the set $\mathscr{M}^+$ of all such filters in X^+.*

(Here r^+ is the operator (1.1) for $(X^+, <^+)$.)

Corollary. *If $\mathscr{M}ir^+ \to x^+$ and $x^+ \in A^+ <^+ B^+$, then $B^+i^{-1} \in \mathscr{M}$.*

Proof. If $\mathscr{M}^+$ is maximal-round and $B_2^+ <^+ B_1^+$, both in $\mathscr{M}^+$, then $\emptyset \neq B_2^+i^{-1} <$ $< B_1^+i^{-1}$ so that $\mathscr{M}^+i^{-1}$ is a round filter. Given $A < B$ in X then either $X^+ \setminus Ai$ or $Bi \cup (X^+ \setminus Xi)$ is in $\mathscr{M}^+$, whence either $X \setminus A$ or B is in $\mathscr{M}^+i^{-1}$ which is therefore maximal round. Finally, $B_2^+i^{-1}i \subset B_2^+ <^+ B_1^+$ so that $(\mathscr{M}^+i^{-1})\,ir^+ \supset \mathscr{M}^+$, which is maximal.

Conversely, given $\mathscr{M}$ maximal round, $A_2 < A_1$, both in $\mathscr{M}$, then $A_2i <^+ A_1i \cup$ $\cup (X^+ \setminus Xi)$ so that $A_1i \cup (X^+ \setminus Xi) \in \mathscr{M}ir^+$, $A_1 \in (\mathscr{M}ir^+)\,i^{-1}$, and so $\mathscr{M} \subset$ $\subset (\mathscr{M}ir^+)\,i^{-1}$; but $\mathscr{M}ir^+ \subset \mathscr{M}i$ so, i being one-one, $(\mathscr{M}ir^+)\,i^{-1} \subset (\mathscr{M}i)\,i^{-1} = \mathscr{M}$. It remains to prove $\mathscr{M}ir^+$ maximal round; if not, let $\mathscr{M}^+ \supset \mathscr{M}ir^+$ be maximal round so that $\mathscr{M}^+i^{-1}$ is maximal round in X and includes $(\mathscr{M}ir^+)\,i^{-1} = \mathscr{M}$. Thus $\mathscr{M}^+i^{-1} = \mathscr{M}$ and we have shown that this implies $\mathscr{M}ir^+ = \mathscr{M}^+$.

The Corollary follows from the fact that B^+ is a neighbourhood of x^+ and so belongs to $\mathscr{M}ir^+$.

(2.3) **Notation.** $\mathscr{M}_0$ will denote the set of all those maximal round filters on $(X, <)$ which do not converge to any point of X. For given $A \subset X$, Aj denotes $\{\mathscr{M} \in \mathscr{M}_0; A \in \mathscr{M}\}$.

(2.4) **Theorem.** *If* $X^* = X \cup \mathscr{M}_0$ *then writing* $A^* <^* B^*$ *iff* $\exists A, B, A < B(\subset X)$, $A^* \subset A \cup Aj$, $B \cup A^* \subset B^*$, *the identity map* i_0 *of* X *into* $(X^*, <^*)$ *is a relatively* T2 *P-compactification of* $(X, <)$ *which we call the fine P-compactification. If* $i : X \to (X^+, <^+)$ *is any other relatively* T2 *P-compactification, then the map* $g : X^* \to X^+$ *defined by* $g \mid X = i$, $\mathscr{M}g = \lim \mathscr{M}ir^+$ *for* $\mathscr{M} \in \mathscr{M}_0$, *is onto* X^+ *and proximity continuous.*

Proof. It is easily verified that $<^*$ is a P-proximity (e.g. if A^*, B^*, A, B are as above and $A < C < B$, then $A^* <^* C \cup A^* <^* B^*$ since $Aj \subset Cj$), that $A < B \Leftrightarrow \Leftrightarrow A <^* B \Leftrightarrow A <^* B \cup \mathscr{M}_0$, and that X is dense in X^* because $\{\mathscr{M}_0\} <^* B^*$ implies $\exists B \neq \emptyset$, $B \subset B^* \cap X$. We note that since $X < X$ we have $X <^* X$ so that X is (one-sidedly) remote from $X^* \setminus X = \mathscr{M}_0$. If $\mathscr{M}_1 \neq \mathscr{M}_2$, say $\mathscr{M}_1 \not\subset \mathscr{M}_2$ and $A_0 \in \in \mathscr{M}_1 \setminus \mathscr{M}_2$, then $\exists A_1 < A_0$, $A_1 \in \mathscr{M}_1$, while $X \setminus A_1 \in \mathscr{M}_2$ since $A_0 \notin \mathscr{M}_2$. Then as $\mathscr{M}_1$ and $\mathscr{M}_2$ are round we easily see that $\{\mathscr{M}_1\} <^* A_1 \cup \{\mathscr{M}_1\}$, $\{\mathscr{M}_2\} <^* <^* (X \setminus A_1) \cup \{\mathscr{M}_2\}$, so that $\mathscr{M}_1, \mathscr{M}_2$ have disjoint neighbourhoods in X^*, in fact, $X^* \setminus X$ is discrete. Similarly, given x_0 and $\mathscr{M}_0$, since $\mathscr{M}_0 \not\to x_0$ we can find A_0, A_1 with $A_0 \notin \mathscr{M}_0$, $\{x_0\} < A_1 < A_0$, and A_1, $(X \setminus A_1) \cup \{\mathscr{M}_0\}$ are disjoint neighbourhoods of x_0 and $\mathscr{M}_0$. Thus i_0 is a relatively T2 imbedding of $(X, <)$ in $(X^*, <^*)$.

We next show $(X^*, <^*)$ P-compact. If $\mathscr{M}^*$ is maximal round in this space, then by (2.2) $\mathscr{M}^* i_0^{-1}$ is maximal round in $(X, <)$. If $\mathscr{M}^* i_0^{-1} \to x_0$ in X then, as i_0 is an imbedding, $x_0 \in \operatorname{Ad} \mathscr{M}^*$ in X^* and in fact $\mathscr{M}^* \to x_0$ since $\mathscr{M}^*$ is maximal round. If $\mathscr{M}^* i_0^{-1}$ has no limit in X it is an element $\mathscr{M}_0$ of $X^* \setminus X$. Given any neighbourhood B^* of $\mathscr{M}_0$, $\exists B_1^*$, $\{\mathscr{M}_0\} <^* B_1^* <^* B^*$, and hence $\exists A_1, B_1, A_1 < B_1$, $\mathscr{M}_0 \in A_1 j \subset \subset B_1 j$, $B_1 \cup \{\mathscr{M}_0\} \subset B_1^*$. Thus B_1, as a subset of X^*, belongs to $\mathscr{M}_0 i_0$ and satisfies $B_1 <^* B^*$, whence $B^* \in \mathscr{M}_0 i_0 r^* = \mathscr{M}^*$. That is, $\mathscr{M}^* \to \mathscr{M}_0$, so in either case $\mathscr{M}^*$ converges in X^*.

Finally, any element x^+ of $X^+ \setminus Xi$ is the limit of at least one maximal round filter $\mathscr{M}^+$ (its neighbourhood filter if this is maximal round, or otherwise a refinement thereof). Let $\mathscr{M}_0 = \mathscr{M}^+ i^{-1}$, $\mathscr{M}^+ = \mathscr{M}_0 ir^+$. $\mathscr{M}_0$ cannot converge in X since $\mathscr{M}_0 \to x$ implies $\mathscr{M}_0 i$, and so $\mathscr{M}_0 ir^+ = \mathscr{M}^+$, convergent to $xi \neq x^+$, impossible as the imbedding i^+ is relatively T2. That is, $\mathscr{M}_0 \in \mathscr{M}_0$ and $\mathscr{M}_0 g = x^+$ (and is uniquely defined). The map g, which obviously covers Xi, is therefore onto X^+.

It remains to prove only that $A^+ <^+ B^+$ implies $A^+ g^{-1} <^* B^+ g^{-1}$. For this we take C^+, $A^+ <^+ C^+ <^+ B^+$, and write $B^+ i^{-1} = B$, $C^+ i^{-1} = C$. Then $A^+ g^{-1} = = A^+ i^{-1} i_0 \cup (A^+ \setminus Xi)\, g^{-1}$, since an argument like that above but starting with $\mathscr{M}_0$

rather than $\mathscr{M}^+$ shows that $\mathscr{M}_0 \in X^* \setminus Xi_0$ implies $\mathscr{M}_0 g \notin Xi$. By (2.2), Corollary, $(A^+ \setminus Xi)\, g^{-1} \subset Cj$, so that $A^+ g^{-1} \subset Ci_0 \cup Cj$ $(= C \cup Cj)$. Now $C < B$ as i is a proximity imbedding, and clearly $B \cup A^+ g^{-1} \subset B^+ g^{-1}$. Hence $A^+ g^{-1} <^* B^+ g^{-1}$ as required, and g is proximity continuous.

The following converse of the last part of (2.4) is almost immediate.

(2.5) **Proposition.** *If g is a proximity continuous map of $(X^*, <^*)$ onto a P-proximity space $(X^+, <^+)$, then $(X^+, <^+)$ is P-compact and $Xi_0 g$ is dense in X^+. Hence if $g \mid X$ is one-one and $A < B$ implies $Ag <^+ Bg \cup (X^+ \setminus Xg)$ then $i_0 g$, $(X^+, <^+)$ is a P-compactification of $(X, <)$.*

(We recall that i_0 is the identity imbedding of X in X^*, so that the symbol i_0 can be omitted when the context makes clear in which space we are working.) To prove (2.5) it is necessary only to remark that $\mathscr{F}^+$ round in $(X^+, <^+)$ implies $\mathscr{F}^+ g^{-1}$ round in $(X^*, <^*)$, so that $\mathrm{Ad}\,(\mathscr{F}^+ g^{-1}) \neq \emptyset$ and hence $\mathrm{Ad}\, \mathscr{F}^+ \neq \emptyset$ in X^+, g being proximity continuous. The rest follows at once from the data and definitions. In particular, we can construct another canonical P-compactification of $(X, <)$.

(2.6) **Theorem.** *If X, X^* are as in (2.4) then writing $A^* <_n B^*$ iff $\exists A, B$, $A < B (\subset X)$, $A^* \subset A \cup Aj$, $B \cup Bj \subset B^*$, the identity map i_0 of X into $(X^*, <_n)$ is a relatively* T2 *P-compactification, which we call the natural P-compactification of $(X, <)$, such that $x_0 \in (X^* \setminus X_0)^-$ iff X is not locally P-compact at x_0; hence $(X^*, <_n)$ is in general topologically distinct from $(X^*, <^*)$. If '$<$' is symmetrical then so is '$<_n$', which therefore yields the ordinary compactification if $(X, <)$ is also separated.*

We first remark that if $A_1^* <_n B^*$, A_1, B_1 being associated sets in X as in the definition, and also $A_2^* <_n B^*$ by using A_2, B_2, then we can take B_3 with $A_2 < B_3 < B_2$. Now if $\mathscr{M}$ is maximal round in $(X, <)$ and does not contain $B_2 > B_3$ then it must contain $X \setminus B_3$; so if such an $\mathscr{M}$ contains $(B_1 \cup B_3)$ it must contain $B_1 \supset (B_1 \cup B_3) \cap \cap (X \setminus B_3)$. That is, $(B_1 \cup B_3) j \subset B_1 j \cup B_2 j \subset B^*$. Since $A_1 \cup A_2 < B_1 \cup B_3$ and $A_1 j \cup A_2 j \subset (A_1 \cup A_2) j$ we see that $(A_1^* \cup A_2^*) <_n B^*$. The other axioms for a P-proximity are easily checked. It is clear that $<_n$ is coarser than $<^*$ so that the identity map of X^* on itself satisfies all the conditions of (2.5) and hence i_0, $(X^*, <_n)$ is a P-compactification of $(X, <)$. The relative T2 property is proved very much as in (2.4). It is clear that $x_0 \in X$ has a neighbourhood (in X^*) entirely included in X iff $\exists B$, $\{x_0\} < B \subset X$, such that $Bj = \emptyset$, i.e., every maximal round filter containing B is convergent in X, and X is locally P-compact at x_0. This is not in general true (e.g. for the space of rational numbers with the ordinary metric uniformity and proximity) so that $(X^*, <_n)$ is in general topologically distinct from $(X^*, <^*)$.

Now suppose that $(X, <)$ is a symmetric proximity space, and that $A^* \subset A \cup Aj$, $A < B$, $B \cup Bj \subset B^*$. Choose $C, A < C < B$, so $X \setminus B < X \setminus C < X \setminus A$. Now

$B \notin \mathscr{M}$ and $X \setminus B < X \setminus C$ imply $X \setminus C \in \mathscr{M}$, so that $X^* \setminus B^* \subset (X \setminus B) \cup \{\mathscr{M}; B \notin \notin \mathscr{M}\} \subset (X \setminus C) \cup (X \setminus C)j$. Since also $(X \setminus A) \cup (X \setminus A)j \subset X^* \setminus (A \cup Aj)$ we have $(X^* \setminus B^*) <_n (X^* \setminus A^*)$, so that the relation $<_n$ is symmetrical.

3. P-COMPLETION OF A GIVEN QUASI-UNIFORM SPACE.

In this section, an imbedding i of a quasi-uniform space $(X, \mathscr{V})$ in another, $(X^+, \mathscr{V}^+)$, means a dense (quasi-) uniform imbedding, that is, the sets $\{(x_1, x_2); (x_1 i, x_2 i) \in V^+\}$, for $V^+ \in \mathscr{V}^+$, form a base of $\mathscr{V}$. We remark that it may well happen that even a complete metric space can be imbedded in a larger P-complete Hausdorff space. For example, consider the space of positive integers with all mutual distances $d(m, n) = 1$ $(m \neq n)$. Introducing an element w with $d(w, n) = 1/n$, $d(n, w) = 1$ all n, we have a compact quasi-metric Hausdorff space, which is *a fortiori* P-complete. To avoid such "unnecessary" P-completions we make the following definition.

(3.1) **Definition.** An imbedding i of $(X, \mathscr{V})$ in a P-complete space $(X^+, \mathscr{V}^+)$ will be called a *P-completion* of $(X, \mathscr{V})$ iff $\mathscr{M}^+$ maximal round and Cauchy in $(X^+, \mathscr{V}^+)$ implies $\mathscr{M}^+ i^{-1}$ Cauchy in $(X, \mathscr{V})$.

We prove a theorem closely analogous to (2.4); there does not however appear to be any analogue (in general) of (2.6) for P-completions.

(3.2) **Notation.** We denote by $\mathscr{M}_1$ the set of all those Cauchy maximal round filters in $(X, \mathscr{V})$ which do not converge, and by X_1^* the set $X \cup \mathscr{M}_1$.

(3.3) **Lemma.** *Let $\mathscr{W}$ be a quasi-uniformity for X_1^* such that*

(i) *If $\mathscr{M}^*$ is any Cauchy maximal round filter on $(X_1^*, \mathscr{W})$ then $\{A \cap X; A \in \mathscr{M}^*\}$ is Cauchy in $(X, \mathscr{V})$;*

(ii) $\{W \cap (X \times X);\ W \in \mathscr{W}\} = \mathscr{V}$;

(iii) *Given any $\mathscr{M}_0 \in \mathscr{M}_1$ and any $W_0 \in \mathscr{W}$, $\exists A \in \mathscr{M}_0$ such that $\mathscr{M}_0 \circ W_0 \supset A$. Then the identity map i_0 of $(X, \mathscr{V})$ into $(X_1^*, \mathscr{W})$ is a P-completion of $(X, \mathscr{V})$.*

We see from (iii) that X is dense in $(X_1^*, \mathscr{W})$ so that i_0 is an imbedding in the sense of this section, hence also in that of section 2, with respect to the induced P-proximities. Thus if $\mathscr{M}^*$ is Cauchy maximal round in $(X_1^*, \mathscr{W})$, $\mathscr{M}^* i_0^{-1}$ is maximal round by (2.2) and Cauchy by datum (i). As before, if $\mathscr{M}^* i_0^{-1} \to x_0$ in $(X, \mathscr{V})$ then $\mathscr{M}^* \to x_0$ in $(X_1^*, \mathscr{W})$. Otherwise, $\mathscr{M}^* i_0^{-1}$ is an element $\mathscr{M}_0$ of $\mathscr{M}_1$. By datum (iii), $\mathscr{M}_0 i_0 \to \mathscr{M}_0$ in $(X_1^*, \mathscr{W})$, hence also $\mathscr{M}^* = \mathscr{M}_0 i_0 r^*$ converges to $\mathscr{M}_0$, where r^* is the rounding operation with respect to $\mathscr{W}$ so that $\mathscr{M}_0 i_0 r^* = \{B^* \subset X_1^*;\ \exists A \in \mathscr{M}_0,\ \exists W \in \mathscr{W},\ A i_0 W \subset B^*\}$. Thus $(X_1^*, \mathscr{W})$ is P-complete.

(3.4) **Theorem.** *Let* $f : \mathscr{M}_1 \to 2^X$, *be any function such that* $\mathscr{M} \circ f \in \mathscr{M}$ *for all* $\mathscr{M} \in \mathscr{M}_1$, *and for* $V \in \mathscr{V}$ *let* $W(f, V) \subset X_1^* \times X_1^*$ *be defined as follows, where* $x_1, x_2 \in X, \mathscr{M}_1, \mathscr{M}_2 \in \mathscr{M}_1$:

$(x_1, x_2) \in W(f, V)$ *iff* $(x_1, x_2) \in V$;
$(x_1, \mathscr{M}) \notin W(f, V)$ *for any* $\mathscr{M} \in \mathscr{M}_1$;
$(\mathscr{M}_1, \mathscr{M}_2) \in W(f, V)$ *iff* $\mathscr{M}_1 = \mathscr{M}_2$;
$(\mathscr{M}_1, x_1) \in W(f, V)$ *iff* $x_1 \in (\mathscr{M}_1 \circ f)\, V$.

Then the sets $W(f, V)$, *where* f, V *are chosen in all possible ways, form the base of a quasi-uniformity* $\mathscr{W}^*$ *for* X_1^* *such that the identity map* i_0 *of* $(X, \mathscr{V})$ *into* $(X_1^*, \mathscr{W}^*)$ *is a relatively* T2 *P-completion. If* $i, (X^+, \mathscr{W}^+)$ *is any other relatively* T2 *P-completion of* $(X, \mathscr{V})$, *then the map* $g : (X_1^*, \mathscr{W}^*) \to (X^+, \mathscr{W}^+)$ *defined by* $g \mid X = i$, $\mathscr{M}g = \lim \mathscr{M}ir^+$ *for* $\mathscr{M} \in \mathscr{M}_1$, *is uniquely defined, onto* X^+ *and (quasi)-uniformly continuous.*

Proof. It is easily verified that the the sets $W(f, V)$ do form the base of a uniformity: e.g. if $V_1 \overset{2}{\subset} V_0$ and f_0 is given, then $[W(f_0, V_1)]^2 \subset W(f_0, V_0)$ while if f_1, V_1 and f_2, V_2 are given and $V = V_1 \cap V_2$, and f is defined by $\mathscr{M}f = (\mathscr{M}f_1) \cap (\mathscr{M}f_2)$, we have $W(f, V) \subset W(f_1, V_1) \cap W(f_2, V_2)$. The uniformity $\mathscr{W}^*$ clearly satisfies conditions (ii) and (iii) of (3.3). It also satisfies (i), since for given $V_0 \in \mathscr{V}$ we can take $V_1 \in \mathscr{V}$, $V_1 \overset{2}{\subset} V_0$ and then f_1 such that $\mathscr{M} \circ f_1$ is always r . V_1 – s. Let $W_1 = W(f_1, V_1)$; then any round Cauchy filter $\mathscr{F}^*$ on $(X_1^*, \mathscr{W}^*)$ contains a set of one of the forms $x_1 W_1$, $\mathscr{M}_1 W_1$, and so $\mathscr{F}^* i_0^{-1}$ contains a set of one of the forms $(x_1 W_1) \cap \cap X = x_1 V_1, (\mathscr{M}_1 \circ f_1)\, V_1$. In either case we have a r . V_0 – s set in X so that $\mathscr{F}^* i_0^{-1}$ is Cauchy. By (3.3), $(X_1^*, \mathscr{W}^*)$ is a P-completion of $(X, \mathscr{V})$. The relative T2 property is proved very much as in (2.4).

If $(X^+, \mathscr{W}^+)$ is another relatively T2 P-completion, the proof that g is uniquely defined and onto X^+ is just as in (2.4), the filters concerned being necessarily Cauchy in their respective spaces. If $W_0^+ \in \mathscr{W}^+$ is given let $W_1^+ \in \mathscr{W}^+$ satisfy $W_1^+ \overset{2}{\subset} W_0^+$. Since $\mathscr{M}g = \lim \mathscr{M}ir^+$ we can choose f_1 such that, for every $\mathscr{M}$ of $\mathscr{M}_1$, $\mathscr{M}f_1$ is a member A of $\mathscr{M}$ with $Ai \subset \mathscr{M}gW_1^+$, and since $(X^+, \mathscr{W}^+)$ is a P-completion of $(X, \mathscr{V})$ we can find $V_1 \in \mathscr{V}$ with $(x_1, x_2) \in V_1 \Rightarrow (x_1 i, x_2 i) \in W_1^+$. Then $(\mathscr{M}, x) \in \in W(f_1, V_1) \Rightarrow x \in (\mathscr{M}f_1)\, V_1 \Rightarrow xi \in (\mathscr{M}f_1 i)\, W_1^+ \subset \mathscr{M}gW_1^+ \overset{2}{\subset} \mathscr{M}gW_0^+$. Other cases being trivial, we conclude that $(x_1^*, x_2^*) \in W(f_1, V_1) \Rightarrow (x_1^* g, x_2^* g) \in W_0^+$, so that g is quasi-uniformly continuous.

We have the following analogue of (2.5).

(3.5) **Proposition.** *If* g *is a (quasi)-uniformly continuous map of* $(X_1^*, \mathscr{W}^*)$ *onto a quasi-uniform space* $(X^+, \mathscr{W}^+)$ *such that*

(i) $g \mid X$ *is one-one and a uniform equivalence of* $(X, \mathscr{W}^* \mid X)$ $(=(X, \mathscr{V}))$ *with* $(Xg, \mathscr{W}^+ \mid (Xg))$;

(ii) $\mathscr{M}^+$ *maximal round and Cauchy in* $(X^+, \mathscr{W}^+)$ *implies* $\mathscr{M}^+(i_0 g)^{-1}$ *Cauchy in* $(X, \mathscr{V})$; *then* $i_0 g, (X^+, \mathscr{W}^+)$ *is a P-completion of* $(X, \mathscr{V})$.

We observe that X dense in $(X_1^*, \mathscr{W}^*)$ implies Xg dense in $(X^+, \mathscr{W}^+)$. Thus (by two applications of 2.2) $\mathscr{M}^+$ Cauchy and maximal round in $(X^+, \mathscr{W}^+)$ implies that $\mathscr{M}^+(i_0 g)^{-1}\, i_0 r^*$ is Cauchy and maximal round, hence convergent, in $(X_1^*, \mathscr{W}^*)$, say to x_0^*, so that $\mathscr{M}^+\,(i_0 g)^{-1}\, i_0$ and $\mathscr{M}^+ = \mathscr{M}^+\,(i_0 g)^{-1}\, i_0 g r$ converge to x_0^* and $x_0^* g$ respectively. Hence $(X^+, \mathscr{W}^+)$ is P-complete.

4. THE RELATION BETWEEN P-COMPACTIFICATION AND P-COMPLETION.

The following easy proposition justifies the name "P-precompact".

(4.1) *If* $i, (X^+, \mathscr{W}^+)$ *is a P-completion of* $(X, \mathscr{V})$, *then it is a P-compactification of* $(X, \mathscr{V})$ *as a P-proximity space iff* $(X, \mathscr{V})$ *is precompact.*

Suppose $(X, \mathscr{V})$ precompact. If $\mathscr{M}_0^+$ is maximal round in X^+ then by (2.2), as a uniform imbedding is certainly a proximity imbedding, $\mathscr{M}_0 = \mathscr{M}_0^+ i^{-1}$ is maximal round in $(X, \mathscr{V})$, and so Cauchy. Hence, $\mathscr{M}_0 i$ (or strictly, its trace on Xi) is Cauchy in the equivalent space Xi, as a subspace of $(X^+, \mathscr{W}^+)$, and so certainly $\mathscr{M}_0 i$ is Cauchy in $(X^+, \mathscr{W}^+)$. This implies $\mathscr{M}_0^+ = \mathscr{M}_0 i r^+$ also Cauchy, so that $(X^+, \mathscr{W}^+)$ is P-precompact, hence P-compact being P-complete.

Conversely, if $(X^+, \mathscr{W}^+)$ is P-compact, and $\mathscr{M}_0$ maximal round in $(X, \mathscr{V})$, $\mathscr{M}^+ = \mathscr{M}_0 i r^+$ is maximal round in $(X^+, \mathscr{W}^+)$ by (2.2), hence convergent and so Cauchy, so that $\mathscr{M}_0 = \mathscr{M}^+ i^{-1}$ is Cauchy by the definition (3.1).

We have therefore a map of the set of all P-completions of $(X, \mathscr{V})$ into the set of its P-compactifications, and we naturally ask whether this is a one-one onto map. There is one case when we can assert that every P-compactification gives a P-completion, namely when $(X, \mathscr{V})$ is totally bounded; that is, for every $V \in \mathscr{V}$, X can be covered by a finite system of sets X_r with $X_r \times X_r \subset V$ (hence also $\subset V \cap V^{-1}$) for each r. For (just as for uniformities) every P-compactification of $(X, \mathscr{V})$ has a unique compatible totally-bounded quasi-uniformity, $\mathscr{W}_0^+$ say, and the quasi-uniformity for X with base given by sets $\{(x_1, x_2);\ (x_1 i, x_2 i) \in W\}$ for $W \in \mathscr{W}_0^+$ is clearly the unique totally bounded uniformity compatible with the given P-proximity of X, so must coincide with $\mathscr{V}$. We show however that this result cannot be generalised even to pre-compact quasi-uniform spaces (i.e. spaces always covered by a finite number of r . $V-$s sets for any $V \in \mathscr{V}$).

(4.2) **Example.** *A precompact space* $(X, \mathscr{V})$ *whose natural P-compactification cannot be induced by any P-completion of* $(X, \mathscr{V})$.

The elements of X consist of two sequences (p_n), (q_n), $n = 1, 2, \ldots$, of distinct points.

The quasi-uniformity $\mathscr{V}$ has an enumerable base $\{V_t;\ t = 1, 2, \ldots\}$, where V_t contains the diagonal pairs (x, x) and those pairs (p_m, p_n) and (p_m, q_n) with $n \geqq m \geqq t$ (and no others). It is clear that $\overset{2}{V}_n = V_n$ and that $V_{n+1} \subset V_n$ so that we have a quasi-uniformity, which makes X pre-compact as it is covered by $p_n V_n$ and $2(n - 1)$ single point sets.

Let $Q_n = \{q_s;\ s \geqq n\}$; then as $Q_n < Q_n$ any maximal round filter $\mathscr{M}$ on $(X, \mathscr{V})$ contains either Q_n or $X \setminus Q_n$. Now $A < X \setminus Q_n$ implies that A consists only of a finite number of points. It follows that either $\mathscr{M}$ is a trivial filter defined by, and converging to, a single point of X, or it is a refinement of the filter with base $\{Q_n;\ n = 1, 2, \ldots\}$: in fact, the ultrafilters refining this filter are easily seen to be round and to constitute the set $\mathscr{M}_0 = \mathscr{M}_1$ of (Cauchy) non-convergent maximal round filters.

Suppose that there exists a quasi-uniformity $\mathscr{W}^+$ on $X^+ = X^* = X \cup \mathscr{M}_0$ which defines the natural P-compactification of $(X, \mathscr{V})$ and makes the identity imbedding i_0 a uniform imbedding. Then given V_{t_0}, $\exists W_0^+$, $W_0^+ \cap (X \times X) \subset V_{t_0}$. Take $W_1^+ \in \mathscr{W}^+$ with $\overset{3}{W}{}_1^+ \subset W_0^+$, and find t_1, $V_{t_1} \subset W_1^+ \cap (X \times X)$. By supposition, for any m, $p_m V_{t_1} <_n p_m V_{t_1} W_1^+$, so $\exists C_m, D_m$ in X with $C_m < D_m$, $p_m V_{t_1} \subset C_m \cup C_m j$, $D_m \cup D_m j \subset p_m V_{t_1} W_1^+$. Then $p_m V_{t_1} \subset C_m$ (being a subset of X) so that $(p_m V_{t_1}) j \subset C_m j$. If now $m \geqq t_1$ then all $M_0 \in \mathscr{M}_0$ contain Q_m so that $M_0 \subset Q_m j \subset (p_m V_{t_1}) j \subset C_m j \subset D_m j \subset p_m \overset{2}{W}{}_1^+$. We know that $\mathscr{M}_0 i_0 r^*$, hence certainly $\mathscr{M}_0 i_0$, converges to $\mathscr{M}_0$ in $(X^*, <^*)$, so that also $\mathscr{M}_0 i_0 \to \mathscr{M}_0$ in the coarser P-proximity $<_n$, supposed induced by $\mathscr{W}^+$.

That is, given $M_0 \in \mathscr{M}_0$, $\exists A_0 \in \mathscr{M}_0$, $A_0 \subset M_0 W_1^+ \subset p_m \overset{3}{W}{}_1^+ \subset p_m W_0^+$ (all $m \geqq t_1$), so that as $A_0 \subset X$, $A_0 \subset p_m V_{t_0}$, all $m \geqq t_1$. This is clearly a contradiction.

We next show that there always exists on the set $X^* = X_1^*$ a P-completion of a P-precompact space $(X, \mathscr{V})$, in general different from (hence necessarily coarser than) that of (3.4), which yields the fine P-compactification $(<^*)$ of (2.4); thus the relation of P-completions to P-compactifications is not in general one-one, for by (2.4) the P-proximity given by the P-completion $\mathscr{W}^*$ of (3.4) cannot be strictly finer than $<^*$.

(4.3) **Theorem.** *If $(X, \mathscr{V})$ is P-precompact and $X^* = X \cup \mathscr{M}_0 = X \cup \mathscr{M}_1$ as before, then defining, for any $V \in \mathscr{V}$ and any $A \subset X$, $W_1(A, \mathscr{V}) = W(f_A, \mathscr{V})$ where $\mathscr{M} f_A = A$ if $A \in \mathscr{M}$, X otherwise, the sets $W_1(A, \mathscr{V})$ form the sub-base of a quasi-uniformity $\mathscr{W}_1^*$ for X^*, in general distinct from $\mathscr{W}^*$, which induces on X^* the P-proximity $<^*$ derived from that induced on X by $\mathscr{V}$ as in* (2.4), *and which makes i_0, $(X^*, \mathscr{W}_1^*)$ a P-completion of $(X, \mathscr{V})$.*

To show that we have a sub-base of a uniformity with $\mathscr{W}_1^* \subset \mathscr{W}^*$ we need check only that given A_0, V_0 we can find a set W_1 of $\mathscr{W}_1^*$ with $\overset{2}{W}_1 \subset W_1(A_0, V_0)$. As before, this is satisfied, if $\overset{2}{V}_1 \subset V_0$, by $W_1 = W_1(A_0, V_1)$. Since $\mathscr{W}_1^* \subset \mathscr{W}^*$, i_0 is clearly a dense uniform imbedding of $(X, \mathscr{V})$ in $(X^*, \mathscr{W}_1^*)$ so that conditions (i) and (ii) of (3.3) are satisfied by (2.2), since $(X, \mathscr{V})$ is P-precompact. Since (iii) has been proved to hold for $\mathscr{W}^*$ it must hold also for the coarser quasi-uniformity $\mathscr{W}_1^*$. Thus i_0, $(X^*, \mathscr{W}_1^*)$ is a P-completion of $(X, \mathscr{V})$. (Alternatively, we could use (3.5) with g as the identity mapping.)

Now suppose $A^* <^* B^*$ and that A, B are as in (2.4), so that $\exists V \in \mathscr{V}$, $AV \subset B$. Consider $A^*W_1(A, V)$; we have at once $A^*W_1(A, V) \cap \mathscr{M}_0 = A^* \cap \mathscr{M}_0$. If now $x_1 \in A^* \cap X \subset A$ and $(x_1, x_2) \in W_1(A, V)$ then $x_2 \in AV \subset B$, while if $\mathscr{M}_0 \in A^* \cap \mathscr{M}_0$ and $(\mathscr{M}_0, x) \in W_1(A, V)$ then $\mathscr{M}_0 \in Aj$ so that $A \in \mathscr{M}_0$, $\mathscr{M}_0 f_A = A$, $x \in AV \subset B$. Thus $A^*W_1(A, V) \subset A^* \cup B \subset B^*$, so that the P-proximity defined by $\mathscr{W}_1^*$ is at least as fine as that given by $<^*$. It therefore certainly gives a relatively T2 imbedding by i_0 and so by (2.4) must coincide with $<^*$.

We note that in defining the sets $W_1(A, V)$ it is sufficient to select V from a base of $\mathscr{V}$. To show that $\mathscr{W}^*, \mathscr{W}_1^*$ are in general distinct, we consider the following development of the example of (4.2). The elements of X consist of a sequence (p_n), $n = 1, 2, \ldots$, and a set Q of all points q_r, where r is any rational satisfying $n \leqq r \leqq$ $\leqq n + \frac{1}{2}$ for some positive integer n. $\mathscr{V}$ has as base the sets V_t, $t = 1, 2, \ldots$, where V_t consists of all pairs (x, x) $(x \in X)$, all pairs p_m, p_n with $n \geqq m \geqq t$, and all pairs (p_m, q_r) (r rational), with $r \geqq m \geqq t$, together with all pairs (q_r, q_s) with $|r - s| < 2^{-t}$. Much as before, this makes X a precompact quasiuniform space, and if $Q_n = \{q_r, r \geqq n\}$, n integral, we still have $Q_n < Q_n$, the subspace Q having in fact the ordinary metric uniformity. The maximal round filters now fall into two sets; those which refine the filter with base $\{Q_n;\ n = 1, 2, \ldots\}$ and those which contain $X \setminus Q_n$ for some n. The latter may or may not be convergent; those which are not convergent may be labelled $\mathscr{M}_\alpha$, where α is any irrational satisfying $n < \alpha < n + \frac{1}{2}$ for some positive integer n, and $\{q_r;\ \alpha - \delta < r < \alpha + \delta\} \in \mathscr{M}_\alpha$ for all $\delta > 0$. Clearly $\mathscr{M}_\alpha$ is not P-convergent in $(X, \mathscr{V})$.

Define f_1 by $\mathscr{M}_\alpha f_1 = \{q_r;\ n \leqq r \leqq n + \frac{1}{2}\}$ if $n < \alpha < n + \frac{1}{2}$; for those non-convergent $\mathscr{M}$ of the other type $\mathscr{M}f_1$ may be an arbitrarily chosen set Q_n. Now consider any basic set $W = \bigcap_{s=1}^{k} W_1(A_s, V_{t(s)})$ of $\mathscr{W}_1^*$. Let γ be any subset (possibly empty) of the set $1, 2, \ldots, k$ and let $B_\gamma = \{\alpha;\ A_s \in M_\alpha \text{ iff } s \in \gamma\}$. For some γ_0, B_{γ_0} is not bounded above, and hence $\{r;\ q_r \in \bigcap_{s \in \gamma_0} A_s\}$ is not bounded above. Hence, taking any fixed $\alpha_0 \in B_{\gamma_0}$ we can find r so that $q_r \notin \mathscr{M}_{\alpha_0} f_1 V_1$ while $q_r \in \bigcap_{s \in \gamma_0} A_s$ so that $q_r \in \mathscr{M}_{\alpha_0} W$; it follows that $W(f_1, V_1)$ includes no set of $\mathscr{W}_1^*$ and that $\mathscr{W}_1^* \neq \mathscr{W}^*$.

Finally, we show that (in spite of (4.2)) there does in general exist a relatively T2 P-completion of a P-precompact space which is topologically coarser than those given by (3.4) and (4.3).

(4.4) **Theorem.** *With* $(X, \mathscr{V})$ *and* X^* *as in* (4.3), *defining* (*for any* $V \in \mathscr{V}$, $A \subset X$) f_A *as in* (4.3) *and* $W^+(A, V)$ *as follows:*

$(x_1, x_2) \in W^+(A, V)$ *iff* $(x_1, x_2) \in V$;

$(x, \mathcal{M}) \notin W^+(A, V)$;

$(\mathcal{M}, x) \in W^+(A, V)$ *iff* $x \in (\mathcal{M}f_A)\, V$;

$(\mathcal{M}_1, \mathcal{M}_2) \in W^+(A, V)$ *iff* $(\mathcal{M}_1 f_A)\, V \in \mathcal{M}_2$;

and writing $\mathcal{W}^+$ *for the quasi-uniformity with these sets as a sub-base, then* i_0, $(X^*, \mathcal{W}^+)$ *is a relatively* T2 *P-completion of* $(X, \mathcal{V})$, *in general topologically distinct from* $(X^*, \mathcal{W}^*)$.

To check that $\mathcal{W}^+$ is a quasi-uniformity we remark that if $\overset{2}{V_1} \subset V_0$ and $W_1^+ = W^+(A, V_1) \cap W^+(AV_1, V_1)$ then $\overset{2}{W_1^+} \subset W^+(A, V_0)$. Since $\mathcal{W}^+ \subset \mathcal{W}_1^*$ but differs from it only as regards pairs $(\mathcal{M}_1, \mathcal{M}_2)$ the proof of (4.3) shows that it gives a P-completion by i_0 (we recall that $(X, \mathcal{V})$ is given P-precompact). Easy examples (e.g., when $(X, \mathcal{V})$ is the space of rationals in [0, 1] with the ordinary metric uniformity) show that the subspace $\mathcal{M}_0 = \mathcal{M}_1$ is not in general discrete in $\mathcal{T}(\mathcal{W}^+)$ as it is in $\mathcal{T}(\mathcal{W}^*) = \mathcal{T}(\mathcal{W}_1^*)$, but the relative T2 property is easily established as before.

We remark that, except when the original space is totally bounded, we have not been able to construct any P-completion without imposing the apparently unnatural condition that Xi is remote from (though dense in) its complement; neither has the author been able to find any analogue for non P-precompact spaces of the constructions of (4.3) and (4.4), so that in general our methods have yielded only one P-completion, that of (3.4).

5. THE QUASI-METRIC CASE.

Except in specially simple cases, the constructions of (3.4), (4.3) and (4.4), applied to a quasi-metric space, yield quasi-uniformities which have no enumerable base and are therefore not quasi-metrisable. The following result may therefore be of some interest, though it yields only a relatively T1 imbedding.

(5.1) **Theorem.** *Let* (X, ϱ) *be a quasi-metric space; write* $V_n = \{(x_1, x_2); \varrho(x_1, x_2) < 2^{-n}\}$. *Let* $\mathcal{C}$ *be the set of all the Cauchy round countably-based filters on* (X, ϱ) *with empty adherence. For each such filter* $\mathcal{F}$ *select a base of* $\mathcal{F}$, $\{B_n; n = 1, 2, \ldots\}$ *in such a way that* B_n *is* $r\,.\,V_n$*-small and* $B_1 > B_2 > \ldots$ *and write* $\mathcal{F}f_n$ *for the set* B_n *selected from* $\mathcal{F}$. *Let* $X_0 = X \cup \mathcal{C}$ *and define* $W_n \subset X_0 \times X_0$ *by*

$(x_1, x_2) \in W_n$ *iff* $(x_1, x_2) \in V_n$;

$(x_1, \mathcal{F}) \notin W_n$ *for every* $\mathcal{F} \in \mathcal{C}$;

$(\mathcal{F}, x) \in W_n$ *iff* $x \in (\mathcal{F}f_n)\, V_n$;

$(\mathcal{F}_1, \mathcal{F}_2) \in W_n$ *iff* $\mathcal{F}_1 = \mathcal{F}_2$.

Then $\{W_n : n = 1, 2, \ldots\}$ *is the base of a countably P-complete quasi-uniformity* $\mathscr{W}_0$ *on* X_0 *such that the identity imbedding* i_0 *is a dense uniform relatively* T1 *imbedding of* (X, ϱ) *in* $(X_0, \mathscr{W}_0)$. *If* $\mathscr{F}_0$ *is any round Cauchy countably-based filter on* $(X_0, \mathscr{W}_0)$, *then* $\mathscr{F}_0 i_0^{-1}$ *is a round Cauchy countably-based filter on* (X, ϱ).

Proof. It is easy to see that $W_{n+1}^2 \subset W_n$ so that $\mathscr{W}_0$ is indeed a quasi-uniformity, and that i_0 is a dense uniform imbedding; hence $\mathscr{F}_0$ round in $(X_0, \mathscr{W}_0)$ implies $\mathscr{F}_0 i_0^{-1}$ a round filter in (X, ϱ), countably-based if $\mathscr{F}_0$ is. By construction $(x_1 W_n)\, i_0^{-1}$ is r. V_n-small and $(\mathscr{F} W_n)\, i_0^{-1}$ always r. V_n^2-small, so that $\mathscr{F}_0$ round and Cauchy implies $\mathscr{F}_0 i_0^{-1}$ Cauchy.

Given $x \in X$, $\mathscr{F} \in \mathscr{C}$, then as $\mathrm{Ad}\mathscr{F} = \emptyset$ in X, $\exists m$, $X \setminus xV_m \in \mathscr{F}$. As $\mathscr{F}$ is round $\exists A \in \mathscr{F}$ and some n such that $AV_n \subset X \setminus xV_m$, and then some p with $\mathscr{F} f_p \subset A$. Thus for $q \geqq \max(m, n, p)$ we have $xW_q \cap FW_q = \emptyset$ so that the imbedding is relatively T2 as concerns such pairs of points; hence it is clearly relatively T1 in general as $\mathscr{C}$ is a discrete subspace of X_0. (However, if $\mathscr{F}_2$ is a round refinement of the round Cauchy filter $\mathscr{F}_1 \in \mathscr{C}$, $\mathscr{F}_1 W_m$ and $\mathscr{F}_2 W_n$ always meet as $\mathscr{F}_1 f_m$ and $\mathscr{F}_2 f_n$ are both members of $\mathscr{F}_1$).

It remains to prove that $(X_0, \mathscr{W}_0)$ is countably P-complete. Let $\mathscr{F}_0$ be round Cauchy and with countable base $\{A_n;\ n = 1, 2, \ldots\}$ in $(X_0, \mathscr{W}_0)$, where we may suppose $A_1 \supset A_2 \supset \ldots$. Then $\mathscr{F}_0 i_0^{-1}$ has similar properties in (X, ϱ) with basic sets $A_n i_0^{-1} = A_n \cap X$. As before, if $x_0 \in \mathrm{Ad}(\mathscr{F}_0 i_0^{-1})$ in (X, ϱ) then $x_0 i_0 = x_0 \in \mathrm{Ad}\mathscr{F}_0$ in $(X_0, \mathscr{W}_0)$. If no such x_0 exists then $\mathscr{F}_0 i_0^{-1} \in \mathscr{C}$ and for any p, $\exists n(p)$ such that $\emptyset \neq A_n \cap X \subset (\mathscr{F}_0 i_0^{-1}) f_p$ for all $n \geqq n(p)$. That is, A_n meets $(\mathscr{F}_0 i_0^{-1})\, W_p$ (in some point of X) for all $n \geqq n(p)$ and since the sets A_n form a nested base of $\mathscr{F}_0$ it follows that $\mathscr{F}_0 i_0^{-1} \in \mathrm{Ad}\mathscr{F}_0$ in $(X_0, \mathscr{W}_0)$. This completes the proof.

6. EXISTENCE OF T1 AND T2 COMPLETIONS.

The methods used above also yield considerable information about the existence of (ordinary) T1 or T2 completions of a given quasi-uniform space. For T1 we obtain a neat necessary and sufficient condition; for T2, however, the condition obtained seems unlikely to be of any real use as it would be extremely difficult to test in any particular case. Our arguments, being closely similar to those used in the constructions for P-completions, will be somewhat abbreviated.

(6.1) **Theorem.** *Let* $(X, \mathscr{V})$ *be a quasi-uniform space, such that* $\mathscr{T}(\mathscr{V})$ *is* T0 *on* X. *Then a necessary and sufficient condition that there exists a dense uniform imbedding* i *of* $(X, \mathscr{V})$ *in a complete space* $(X^+, \mathscr{W}^+)$, *with* $\mathscr{T}(\mathscr{W}^+)$ T1 *on* X^+, *is: for every* $\mathscr{V}$*-Cauchy filter* $\mathscr{F}$, $x_0 \in \mathrm{Ad}\mathscr{F}$ *in* $\mathscr{T}(\mathscr{V}^{-1})$ *implies* $x_0 \in \mathrm{Ad}\mathscr{F}$ *in* $\mathscr{T}(\mathscr{V})$. *If this is satisfied it is possible to choose* $(X^+, \mathscr{W}^+)$ *in such a way that* $\mathscr{F} i$ *Cauchy* *on $(X^+, \mathscr{W}^+)$ implies that $\mathscr{F}$ is a Cauchy filter on $(X, \mathscr{V})$. Moreover, if a Cauchy filter on $(X, \mathscr{V})$ has a limit in $\mathscr{T}(\mathscr{V}^{-1})$ then it has a unique limit in both $\mathscr{T}(\mathscr{V})$ and $\mathscr{T}(\mathscr{V}^{-1})$.*

Proof. We show first that the condition is necessary. If the imbedding i, $(X^+, \mathscr{W}^+)$ exists and $\mathscr{F}$ is Cauchy in $(X, \mathscr{V})$, $x_0 \in \mathrm{Ad}\mathscr{F}$ in $\mathscr{T}(\mathscr{V}^{-1})$ then $\exists \mathscr{F}_1 \supset \mathscr{F}$, $\mathscr{F}_1 \to x_0$ in $\mathscr{T}(\mathscr{V}^{-1})$. Then $\mathscr{F}_1^+ = \mathscr{F}_1 i$ is Cauchy in $(X^+, \mathscr{W}^+)$ and converges to $x_0 i$ in $\mathscr{T}(\mathscr{W}^{+-1})$ since it has a base lying entirely in Xi, on which $\mathscr{W}^{+-1}$ is equivalent to $\mathscr{V}^{-1}$. By completeness, a refinement $\mathscr{F}_2^+$ of $\mathscr{F}_1^+$ converges in $\mathscr{T}(\mathscr{W}^+)$ to some point x_1^+ say. We show that x_1^+ must be $x_0 i$. For, given $W_0^+ \in \mathscr{W}^+$, let $V_0 \in \mathscr{V}$ be such that $(x_1, x_2) \in V_0 \Rightarrow (x_1 i, x_2 i) \in W_0^+$; $\exists A_1 \in \mathscr{F}_1$, $A_2^+ \in \mathscr{F}_2^+$ such that $A_1 \subset \subset x_0 V_0^{-1}$, $A_2^+ \subset x_1^+ W_0^+$. Then $A_1 i \in \mathscr{F}_1^+$ and so meets A_2^+ in a point xi say, where $(x, x_0) \in V_0$ so that $(xi, x_0 i) \in W_0^+$. Hence $(x_1^+, x_0 i) \in (W_0^+)^2$. Since W_0^+ is arbitrary and X^+ is T1 in $\mathscr{T}(\mathscr{W}^+)$, we have $x_1^+ = x_0 i$ as stated. Moreover, $\mathscr{F}_2^+$ must have a base in Xi (as it refines $\mathscr{F}_1^+$) so that $\mathscr{F}_2^+ i^{-1}$ is a filter on X, refining $\mathscr{F}_1$ and $\mathscr{F}$, which converges to x_0 (so that $x_0 \in \mathrm{Ad}\mathscr{F}$) in $\mathscr{T}(\mathscr{V})$.

The same argument shows that if the given $\mathscr{F}$ converges to x_0 in $\mathscr{T}(\mathscr{V}^{-1})$, and is Cauchy in $\mathscr{V}$, it converges to x_0, and to no other point, in $\mathscr{T}(\mathscr{V})$, since otherwise $\mathscr{F}$ has a refinement with a base lying entirely outside some neighbourhood of x_0, so that we could find a convergent refinement $\mathscr{F}_2^+$ whose limit could not be $x_0 i$. Thus x_0 must be unique.

We now suppose the condition satisfied and construct the required completion. We let $\mathscr{U}^+$ be the set of all Cauchy ultrafilters on $(X, \mathscr{V})$ with no limit in $(X, \mathscr{T}(\mathscr{V}))$, and write $X_0^+ = X \cup \mathscr{U}^+$. We define $\mathscr{W}_0^+$ by means of basic sets $W_0(f, V)$, where f assigns to each $\mathscr{U} \in \mathscr{U}^+$ a member $\mathscr{U}f$ of $\mathscr{U}$ and, very much as in (3.4);

$(x_1, x_2) \in W_0(f, V)$ iff $(x_1, x_2) \in V$;
$(x_1, \mathscr{U}) \notin W_0(f, V)$;
$(\mathscr{U}_1, \mathscr{U}_2) \in W_0(f, V)$ iff $\mathscr{U}_1 = \mathscr{U}_2$;
$(\mathscr{U}, x) \in W_0(f, V)$ iff $x \in (\mathscr{U}f)\, V$.

This is a quasi-uniformity making the identity i_0 a dense uniform imbedding. To show $\mathscr{T}(\mathscr{W}_0^+)$ T1 we need prove only $x_1 \notin \{x_2\}^-$ if $x_1 \neq x_2$ and $\mathscr{U} \notin \{x\}^-$, other cases being obvious. If $x_1 \in \{x_2\}^-$ then $x_1 \in x_2 V^{-1}$ for all $V \in \mathscr{V}$, so by the data $x_2 \in \mathrm{Ad}\mathscr{F}$, where $\mathscr{F}$ is the filter with base $\{x_1\}$, in $\mathscr{T}(\mathscr{V})$; that is, $x_2 \in \{x_1\}^-$; as X is given T0 this gives $x_1 = x_2$. Again, given $\mathscr{U}, x$ we know $\mathscr{U} \nrightarrow x$ in $\mathscr{T}(\mathscr{V})$ so by the data $\mathscr{U} \nrightarrow x$ in $\mathscr{T}(\mathscr{V}^{-1})$. (As $\mathscr{U}$ is an ultrafilter, adherence implies convergence.) We can therefore find f_0, V_0 with $\mathscr{U}f_0 = X \setminus x V_0^{-1}$ so that $x \notin (\mathscr{U}f_0)\, V_0 = \mathscr{U} W_0(f_0\, V_0)$.

Now let $\mathscr{U}_0^+$ be an ultrafilter on X_0^+, Cauchy in $\mathscr{W}^+$. We have to prove that every set of $\mathscr{U}_0^+$ meets X so that $\mathscr{U}_0^+ i_0^{-1}$ is a filter on X, except in the trivial case when $\mathscr{U}_0^+$ has as base a single point of $\mathscr{U}_0^+$. We know that $\mathscr{U}_0^+$ has a member of one of the forms $x_0 W_0(f, V)$, $\mathscr{U}_0 W_0(f, V)$. The first lies entirely in X, and the second meets

$X_0^+ \setminus X$ only in $\mathscr{U}_0$ itself, which proves our statement. Just as in (3.4), every Cauchy filter $\mathscr{F}^+$ in $(X_0^+, \mathscr{W}_0^+)$ for which $\emptyset \notin \mathscr{F}^+ i_0^{-1}$ makes $\mathscr{F}^+ i^{-1}$ Cauchy in $(X, \mathscr{V})$. For $\mathscr{U}_0^+$, $\mathscr{U}_0^+ i_0^{-1}$ is clearly an ultrafilter on X. An argument like that of (3.3), but without any rounding operation, then shows that $\mathscr{U}_0^+ i_0^{-1} i_0 = \mathscr{U}_0^+$ converges in $(X_0^+, \mathscr{W}_0^+)$, either to $(\lim \mathscr{U}_0^+ i_0^{-1})\, i_0$, if this exists, or to $\mathscr{U}_0^+ i_0^{-1}$ itself, as an element of X_0^+, if this filter has no limit in $(X, \mathscr{T}(\mathscr{V}))$. Thus $(X_0^+, \mathscr{W}_0^+)$ is complete.

(6.2) **Theorem.** *A necessary and sufficient condition that there exists a dense uniform imbedding i of a quasi-uniform space $(X, \mathscr{V})$ in a complete space $(X^+, \mathscr{W}^+)$, with $\mathscr{T}(\mathscr{W}^+)$ T2 on X^+ and $\mathscr{F} i$ Cauchy in $(X^+, \mathscr{W}^+)$ iff $\mathscr{F}$ is Cauchy in $(X, \mathscr{V})$, is the following:*

There exists a family $\mathscr{A}$ of round Cauchy filters on $(X, \mathscr{V})$ such that

(i) *every neighbourhood filter on $(X, \mathscr{T}(\mathscr{V}))$ is in $\mathscr{A}$;*

(ii) *every Cauchy ultrafilter on $(X, \mathscr{V})$ is a refinement of just one member of $\mathscr{A}$. If this condition holds, and if $\mathscr{F}$ is a Cauchy filter on $(X, \mathscr{V})$ which P-converges to x_0, then $\mathscr{F}$ converges to x_0 and so x_0 is unique.*

Proof. To prove necessity we have only to take for $\mathscr{A}$ the family of all filters $\mathscr{N}^+ i^{-1}$, where $\mathscr{N}^+$ is the filter of neighbourhoods in $\mathscr{T}(\mathscr{W}^+)$ of an arbitrary point of X^+, for if $\mathscr{U}$ is a Cauchy ultrafilter on $(X, \mathscr{V})$ then $\mathscr{U} i$ is a Cauchy ultrafilter, hence convergent, on $(X^+, \mathscr{W}^+)$, with unique limit as $(X^+, \mathscr{T}(\mathscr{W}^+))$ is T2. Moreover, if a Cauchy filter $\mathscr{F}$ P-converges to x_0 and $\mathscr{U} \supset \mathscr{F}$, $\mathscr{U} i$ converges to x^+ say in $(X^+, \mathscr{W}^+)$. For all $V \in \mathscr{V}$, $x_0 V V^{-1} \in \mathscr{F}$ and hence for all $W^+ \in \mathscr{W}^+$, $(x_0 i W^+ \cap \cap X i)\, W^{+-1} \cap X i$, *a fortiori* $x_0 i W^+ W^{+-1} \cap X i$, is in $\mathscr{F} i$. Again, $x^+ W^+$ is in $\mathscr{U} i$, so these two sets meet, whence $(x_0 i)\, W^+$ meets $x^+ (W^+)^2$ for all W^+; by the assumed T2 property this gives $x^+ = x_0 i$.

For the sufficiency we proceed just as in (6.1) adjoining to X the set of all those members of $\mathscr{A}$ which are not neighbourhood filters in $(X, \mathscr{T}(\mathscr{V}))$, with the quasi-uniformity defined as in (6.1) with $\mathscr{F}(\in \mathscr{A})$ substituted for $\mathscr{U}$. (Alternatively, we could take the set of all members of $\mathscr{A}$, imbedding X by making $xi =$ neighbourhood filter $\mathscr{N}_x$; this is a closer parallel to the classical case but not so similar to our previous constructions.) The T2 condition is satisfied because if $\mathscr{F}_1, \mathscr{F}_2$ are distinct members of $\mathscr{A}$ there must exist $A_1 \in \mathscr{F}_1$, $A_2 \in \mathscr{F}_2$ with $A_1 \cap A_2 = \emptyset$, otherwise they would have a common ultra-refinement. As $\mathscr{F}_1, \mathscr{F}_2$ are round we can find $B_1 \in \mathscr{F}_1$, $B_2 \in \mathscr{F}_2$ and $V \in \mathscr{V}$ with $B_1 V \cap B_2 V = \emptyset$ and then take (if neither $\mathscr{F}_1$ nor $\mathscr{F}_2$ is a neighbourhood filter) f such that $\mathscr{F}_1 f = B_1$, $\mathscr{F}_2 f = B_2$. Similarly, taking $\mathscr{F}_1$ as $\mathscr{N}_x$ we obtain the T2 condition for $x, \mathscr{F}_2$. The proof of completeness is just as before, except that if $\mathscr{U}_0^+ i_0^{-1}$ fails to converge in $(X, \mathscr{V})$ then $\mathscr{U}_0^+$ converges in $(X^+, \mathscr{W}^+)$ to the unique member $\mathscr{F}$ of $\mathscr{A}$ such that $\mathscr{U}_0^+ i_0^{-1} \supset \mathscr{F}$. (Here $\mathscr{U}_0^+$ is of course supposed Cauchy.)

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