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ON A HOLOMORPHIC SOLUTION OF A SINGULAR PARTIAL DIFFERENTIAL EQUATION WITH MANY SIMPLE POLES

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1. In the previous paper [3] and [4], the author tried to seek holomorphic solutions of some system of partial differential equations with one simple pole in several complex variables. On the other hand A. FURIOLI MARTINOLLI [2] considered a Darboux problem with singularities $xy = 0$, that is $x = 0$ and $y = 0$, in two real variables. Moreover, I. T. KIGURADZE [5] considered the existence and uniqueness of the solution of the problem

$$u^{(n)} = f(t, u, u', \ldots, u^{(n-1)}), \quad u^{(j-1)}(t_k) = 0$$
$$(j = 1, 2, \ldots, \nu_k;\ k = 1, 2, \ldots, m)$$

where the function $f(t, x_1, x_2, \ldots, x_n)$ has singularities for $t = t_k$ $(k = 1, 2, \ldots, m)$ in one real variable t.

The aim of this paper is to give a singular partial differential equation of order $m + n$ with $m + n$ simple poles which has a unique global holomorphic solution in a domain of the space $\boldsymbol{C}^2$ of two complex variables.

2. Singular ordinary differential equation. Let Ω be a bounded convex domain in the complex plane $\boldsymbol{C}$ of a complex variable x. Let L be a positive number larger than 1 such that any two points of Ω can be connected by a line segment in Ω with length smaller than L. Let m be a positive integer. Let $a_0, a_1, \ldots, a_{m-1}$ be m distinct points of Ω. We put

$$P(x) = (x - a_0)(x - a_1)\ldots(x - a_{m-1}). \tag{2.1}$$

Let $P_j(x)$ be the polynomial in x satisfying

$$P(x) = (x - a_j)\, P_j(x) \tag{2.2}$$

for $j = 0, 1, \ldots, m - 1$. Let $A_0(x), A_1(x), \ldots, A_{m-1}(x)$ and $D(x)$ be bounded holomorphic functions in Ω satisfying

(2.3) $$|A_j(x)| \leqq M \ (j = 0, 1, \ldots, m - 1), \quad |D(x)| \leqq N$$

in Ω for positive numbers M and N.

Lemma 1. *Assume that*

(2.4) $$mML^m < 1 .$$

Then the problem

(2.5) $$P(x) \frac{d^m u}{dx^m} = \sum_{j=0}^{m-1} P_j(x) A_j(x) \frac{d^j u}{dx^j} + P(x) D(x),$$

$$\frac{d^j u}{dx^j}(a_j) = 0 \quad (j = 0, 1, \ldots, m - 1)$$

has a unique holomorphic solution in Ω.

Proof. Let H_Ω be the set of all holomorphic functions $v(x)$ in Ω satisfying

(2.6) $$\|v\|_\Omega \sup_{x \in \Omega} \sum_{j=0}^{m-1} \left| \frac{\frac{d^j v}{dx^j}(x)}{x - a_j} \right| < +\infty .$$

Then we have

$$\left| \frac{d^j v}{dx^j}(x) \right| \leqq L \|x\|_\Omega \quad (j = 0, 1, \ldots, m - 1)$$

in Ω for $v \in H_\Omega$. For $v \in H_\Omega$, we define a holomorphic function Tv in Ω by putting

(2.7) $$Tv = \int_{a_0}^{x} ds_0 \int_{a_1}^{s_0} ds_1 \ldots \int_{a_{m-1}}^{s_{m-2}} \sum_{j=0}^{m-1} \left(\frac{P_j}{P} A_j \frac{d^j v}{dx^j} \right)(s_{m-1})\, ds_{m-1}$$

where the integral contours are line segments in Ω. Then we have $Tv \in H_\Omega$ and

(2.8) $$\|Tv\|_\Omega \leqq mML^m \|v\|_\Omega .$$

We define a holomorphic function v_0 in Ω by putting

(2.9) $$v_0(x) = \int_{a_0}^{x} ds_0 \int_{a_1}^{s_0} ds_1 \ldots \int_{a_{m-1}}^{s_{m-2}} D(s_{m-1})\, ds_{m-1} .$$

Then we have $v_0 \in H_\Omega$ and

(2.10) $$\|v_0\|_\Omega \leqq mL^m N$$

since the integral contours are line segments with length smaller than $L > 1$. We define a sequence $\{v_\nu;\ \nu = 1, 2, \ldots\}$ of holomorphic functions in Ω by putting

$$(2.11)\qquad (v_{\nu+1})(x) = (Tv_\nu)(x) \quad (\nu = 0, 1, 2, \ldots).$$

By (2.8) and (2.10) we have $v \in H_\Omega$ and

$$(2.12)\qquad \|v_\nu\|_\Omega \leqq (mML^m)^\nu\, mL^m N$$

for $\nu = 0, 1, 2, \ldots$ We define a sequence $\{u_\nu(x);\ \nu = 1, 2, \ldots\}$ of holomorphic functions in Ω by putting

$$(2.13)\qquad u_\nu(x) = v_0(x) + v_1(x) + \ldots + v_{\nu-1}(x) \quad (\nu = 1, 2, \ldots).$$

Then we have

$$(2.14)\qquad u_{\nu+1} = Tu_\nu + v_0 \quad (\nu = 1, 2, \ldots).$$

By (2.4), the sequence $\{u_\nu;\ \nu = 1, 2, \ldots\}$ converges uniformly to a holomorphic function $u(x)$ in Ω. By (2.14) u is a holomorphic solution in Ω of the integral equation

$$(2.15)\qquad u = Tu + v_0\,.$$

Hence u is a holomorphic solution in Ω of the problem (2.5).

Now let u and v be two holomorphic solutions of the problem (2.5). We put $w = u - v$ in Ω. Let Ω' be a relatively compact convex subdomain of Ω containing $a_0, a_1, \ldots, a_{m-1}$. Then $w \in H_{\Omega'}$ and we have

$$(2.16)\qquad w = Tw\,.$$

By (2.8) we have

$$(2.17)\qquad \|w\|_{\Omega'} \leqq (mML^m)^\nu\, \|w\|_{\Omega'}$$

for $\nu = 0, 1, 2, \ldots$. By (2.4) we have $\|w\|_{\Omega'} = 0$. By the theorem of identity w is identically zero in Ω.

Lemma 2. *Assume that there hold* (2.4) *and*

$$(2.18)\qquad A_j(a_j) \neq 0 \quad (j = 0, 1, \ldots, m-1).$$

Then the ordinary differential equation

$$(2.19)\qquad P(x)\frac{\mathrm{d}^m u}{\mathrm{d}x^m} = \sum_{j=0}^{m-1} P_j(x)\, A_j(x)\, \frac{\mathrm{d}^j u}{\mathrm{d}x^j} + P(x)\, D(x)$$

has a unique holomorphic solution in Ω.

Proof. Let u be a holomorphic solution in Ω of the equation (2.19). Substituting $x = a_j$ in (2.19), we have

$$\frac{d^j u}{dx^j}(a_j) = 0 \quad (j = 0, 1, \ldots, m - 1) \tag{2.20}$$

by (2.18). Hence u is the unique holomorphic solution in Ω of the problem (2.5).

3. Singular partial differential equation. Let Ω_1 and Ω_2 be bounded convex domains in the complex plane. Let L be a positive number larger than 1 such that any two points of Ω_1 and any two points of Ω_2 can, respectively, be connected by line segments in Ω_1 and Ω_2 with length smaller than L. Let m and n be positive integers. Let $a_0, a_1, a_2, \ldots, a_{m-1}$ be m distinct points of Ω_1. Let $b_0, b_1, b_2, \ldots, b_{n-1}$ be n distinct points of Ω_2. We put

$$I = \{0, 1, \ldots, m\} \times \{0, 1, 2, \ldots, n\} - \{(m, n)\}, \tag{3.1}$$

$$\begin{aligned} P(x, y) = {} & (x - a_0)(x - a_1) \ldots \\ & \ldots (x - a_{m-1})(y - b_0)(y - b_1) \ldots (y - b_{n-1}) . \end{aligned} \tag{3.2}$$

Let $P_{jk}(x, y)$ be a polynomial in x and y for $(j, k) \in I$ satisfying

$$\begin{gathered} P(x, y) = (x - a_j)(y - b_k) P_{jk}(x, y) \\ (j = 0, 1, \ldots, m - 1, \ k = 0, 1, \ldots, n - 1), \end{gathered} \tag{3.3}$$

$$\begin{gathered} P(x, y) = (x - a_j) P_{jn}(x, y), \quad P(x, y) = (y - b_k) P_{mk}(x, y) \\ (j = 0, 1, \ldots, m - 1, \ k = 0, 1, \ldots, n - 1) . \end{gathered} \tag{3.4}$$

Let $A_{jk}(x, y)$ and $D(x, y)$ be bounded holomorphic functions in $\Omega = \Omega_1 \times \Omega_2$ of the space $\boldsymbol{C}^2$ of two complex variables x and y for $(j, k) \in I$ satisfying

$$|A_{jk}(x, y)| \leqq M , \tag{3.5}$$

$$|D(x, y)| \leqq N \tag{3.6}$$

in Ω for positive numbers M and N.

Proposition 2. *Assume that*

$$2mnML^{m+n} < 1 . \tag{3.7}$$

Then the problem

$$(3.8)\qquad P(x,y)\frac{\partial^{m+n}u}{\partial x^m\,\partial y^n}=\sum_{(j,k)\in I}P_{jk}(x,y)\,A_{jk}(x,y)\frac{\partial^{j+k}u}{\partial x^j\,\partial y^k}+P(x,y)\,D(x,y)\,,$$

$$\frac{\partial^j u}{\partial x^j}(a_j,y)=0\quad(j=0,1,\dots,m-1)\,,$$

$$\frac{\partial^k u}{\partial y^k}(x,b_k)=0\quad(k=0,1,\dots,n-1)\,.$$

has a unique holomorphic solution $u(x, y)$ in Ω. This solution u satisfies $u\in H_\Omega$ and

$$(3.9)\qquad \|u\|_\Omega \leqq 2mnL^{m+n}N\,.$$

Proof. Let u be a holomorphic solution in Ω of the problem (3.8). Since

$$\frac{\partial^j u}{\partial x^j}(a_j,y)=0\quad(j=0,1,\dots,m-1)$$

in Ω_2, we have

$$(3.10)\qquad u(x,y)=\int_{a_0}^{x}\mathrm{d}s_0\int_{a_1}^{s_0}\mathrm{d}s_1\dots\int_{a_{m-1}}^{s_{m-2}}\frac{\partial^m u}{\partial x^m}(s_{m-1},y)\,\mathrm{d}s_{m-1}$$

in Ω. Since $(\partial^k u/\partial y^k)(x,b_k)=0$ $(k=0,1,\dots,n-1)$, we have

$$(3.11)\qquad \frac{\partial^{j+k}u}{\partial x^j\,\partial y^k}(x,b_k)=0\quad(j=0,1,\dots,m,\ k=0,1,\dots,n-1)$$

in Ω_1. By (3.10) and (3.11), we have

$$(3.12)\qquad u(x,y)=\int_{a_0}^{x}\mathrm{d}s_0\int_{a_1}^{s_0}\mathrm{d}s_1\dots\int_{a_{m-1}}^{s_{m-2}}\mathrm{d}s_{m-1}\int_{b_0}^{y}\mathrm{d}t_0\int_{b_1}^{t_0}\mathrm{d}t_1\dots$$
$$\dots\int_{b_{n-1}}^{t_{n-2}}\left(\sum_{(j,k)\in I}\left(\frac{P_{jk}}{P}A_{jk}\frac{\partial^{j+k}u}{\partial x^j\,\partial y^k}\right)(s_{m-1},t_{n-1})+D(s_{m-1},t_{n-1})\right)\mathrm{d}t_{n-1}\,.$$

Let H_Ω be the set of all holomorphic functions $v(x, y)$ in Ω satisfying

$$(3.14)\qquad \|v\|_\Omega=\sup_{(x,y)\in\Omega}\sum_{(j,k)\in I}\left|\frac{P_{jk}(x,y)\dfrac{\partial^{j+k}u}{\partial x^j\,\partial y^k}(x,y)}{P(x,y)}\right|<+\infty\,.$$

Then we have

$$(3.15)\qquad \sum_{(j,k)\in I}\left|\frac{\partial^{j+k}v}{\partial x^j\,\partial y^k}(x,y)\right|\leqq L^2\|v\|_\Omega$$

for $v \in H_\Omega$. For $v \in H_\Omega$, we define a holomorphic function Tv in Ω

$$(3.16)\qquad (Tv)(x, y) = \int_{a_0}^{x} ds_0 \int_{a_1}^{s_0} ds_1 \ldots \int_{a_{m-1}}^{s_{m-2}} ds_{m-1} \int_{b_0}^{y} dt_0 \int_{b_1}^{t_0} dt_1 \ldots$$

$$\ldots \int_{b_{n-1}}^{t_{n-2}} \left(\sum_{(j,k)\in I} \left(\frac{P_{jk}}{P} A_{jk} \frac{\partial^{j+k} u}{\partial x^j \partial y^k} \right) (s_{m-1}, t_{n-1}) \right) dt_{n-1} .$$

Then we have $Tv \in H_\Omega$ and

$$(3.17)\qquad \|Tv\|_\Omega \leqq mnML^{m+n}\|v\|_\Omega .$$

We define a holomorphic function v_0 in Ω by putting

$$(3.18)\qquad v_0(x, y) = \int_{a_0}^{x} ds_0 \int_{a_1}^{s_0} ds_1 \ldots \int_{a_{m-1}}^{s_{m-2}} ds_{m-1} \int_{b_0}^{y} dt_0 \int_{b_1}^{t_0} dt_1 \ldots$$

$$\ldots \int_{b_{n-1}}^{t_{n-2}} D(s_{m-1}, t_{n-1})\, dt_{n-1} .$$

Then we have $v_0 \in H_\Omega$ and

$$(3.19)\qquad \|v_0\|_\Omega \leqq mnL^{m+n}N .$$

We define a sequence of holomorphic functions $\{v_\nu;\ \nu = 1, 2, \ldots\}$ and $\{u_\nu;\ \nu = 1, 2, \ldots\}$ by putting (2.11) and (2.13) for this T. By (3.17) and (3.19), we have

$$(3.20)\qquad \|v_\nu\|_\Omega \leqq (mnML^{m+n})^\nu\, mnL^{m+n}N$$

for $\nu = 0, 1, \ldots$. By (3.7) and (3.20), the sequence $\{u_\nu;\ \nu = 1, 2, \ldots\}$ converges uniformly to a holomorphic function u in Ω. u is a unique holomorphic solution of the integral equation $u = Tu + v_0$, that is, the problem (3.8). By (3.19) and (3.20), we have (3.9).

Theorem 1. *Assume that there hold* (3.7),

$$(3.21)\qquad A_{jn}(a_j, y) \neq 0 \quad (j = 0, 1, \ldots, m-1)$$

in Ω_2,

$$(3.22)\qquad \sup_{y\in\Omega_2} \left| \frac{A_{jk}(a_j, y)}{A_{jn}(a_j, y)} \right| < \frac{1}{nL^n} \quad (j = 0, 1, \ldots, m-1,\ k = 0, 1, \ldots, n-1),$$

$$(3.23)\qquad A_{mk}(x, b_k) \neq 0 \quad (k = 0, 1, \ldots, n-1)$$

in Ω_1,

$$(3.24)\qquad \sup_{x\in\Omega_1} \left| \frac{A_{jk}(x, b_k)}{A_{mk}(x, b_k)} \right| < \frac{1}{mL^m} \quad (j = 0, 1, \ldots, m-1,\ k = 0, 1, \ldots, n-1)$$

and

$$A_{jk}(a_j, b_k) \neq 0 \tag{3.25}$$

for $(j, k) \in I$. Then the singular partial differential equation

$$P(x, y) \frac{\partial^{m+n} u}{\partial x^m \partial y^n} = \sum_{(j,k)\in I} P_{jk}(x, y) A_{jk}(x, y) \frac{\partial^{j+k} u}{\partial x^j \partial y^k} + P(x, y) D(x, y) \tag{3.26}$$

has a unique holomorphic solution $u(x, y)$ in Ω.

Proof. We put

$$\varphi_j(y) = \frac{\partial^j u}{\partial x^j}(a_j, y) \quad (j = 0, 1, \ldots, m-1) \tag{3.27}$$

in Ω_2. Substituting $x = a_j$ in (3.23), we have

$$\sum_{k=0}^{n-1} P_{jk}(a_j, y) A_{jk}(a_j, y) \frac{\mathrm{d}^k}{\mathrm{d}y^k} \varphi_j(y) = 0 \tag{3.28}$$

in Ω_2. By (3.22) and Lemma 2, we have

$$\frac{\partial^j u}{\partial x^j}(a_j, y) = 0 \quad (j = 0, 1, \ldots, m-1) \tag{3.29}$$

in Ω_2. Similarly, we have

$$\frac{\partial^k u}{\partial y^k}(x, b_k) = 0 \quad (k = 0, 1, \ldots, n-1) \tag{3.30}$$

in Ω_1. By (3.29) and (3.30), u is the unique holomorphic solution of the problem (3.8).

4. Non-linear equation. Let $f(x, y, \ldots, u_{jk}, \ldots)$ be a holomorphic function in

$$F = \{(x, y, \ldots, u_{jk}, \ldots) \in \Omega \times \boldsymbol{C}^{mn-1};\ |u_{jk}| < R,\ (j, k) \in I\} \tag{4.1}$$

such that

$$|f(x, y, \ldots, u_{jk}, \ldots)| \leqq N\,, \tag{4.2}$$
$$|f(x, y, \ldots, u_{jk}, \ldots) - f(x, y, \ldots, v_{jk}, \ldots)| \leqq N \sum_{(j,k)\in I} |u_{jk} - v_{jk}|\,.$$

Let ε_0 be a positive number with $\varepsilon_0 \leqq 1$ satisfying

$$\varepsilon_0 < \frac{\min(1, R)}{4mnL^{m+n+2}N}\,. \tag{4.3}$$

For any positive number ε with $\varepsilon \leqq \varepsilon_0$, consider the non-linear problem

$$(4.4) \qquad P(x\ y,)\frac{\partial^{m+n}u}{\partial x^m\,\partial y^n} = \sum_{(j,k)\in I} P_{jk}(x, y)\, A_{jk}(x, y)\frac{\partial^{j+k}u}{\partial x^j\,\partial y^k} + \varepsilon\, P(x, y) f\left(x, y, \ldots, \frac{\partial^{j+k}u}{\partial x^j\,\partial y^k}, \ldots\right),$$

$$\frac{\partial^j u}{\partial x^j}(a_j, y) = 0 \quad (j = 0, 1, \ldots, m-1),$$

$$\frac{\partial^k u}{\partial y^k}(x, b_k) = 0 \quad (k = 0, 1, \ldots, n-1)$$

in Ω. Assume (3.7). Let v_0 be the holomorphic solution in Ω of the problem

$$(4.5) \qquad P(x, y)\frac{\partial^{m+n}v_0}{\partial x^m\,\partial y^n} = \sum_{(j,k)\in I} P_{jk}(x, y)\, A_{jk}(x, y)\frac{\partial^{j+k}v_0}{\partial x^j\,\partial y^k} + \varepsilon\, P(x, y) f(x, y, \ldots, 0, \ldots),$$

$$\frac{\partial^j v_0}{\partial x^j}(a_j, y) = 0 \quad (j = 0, 1, \ldots, m-1),$$

$$\frac{\partial^k v_0}{\partial y^k}(x, b_k) = 0 \quad (k = 0, 1, \ldots, n-1).$$

By Proposition 2, we have

$$(4.6) \qquad \|v_0\|_\Omega \leqq 2\varepsilon mnL^{m+n}N.$$

We want to construct sequences $\{v_\nu;\ \nu = 0, 1, 2, \ldots\}$ and $\{u_\nu;\ \nu = 0, 1, 2, \ldots\}$ of holomorphic functions in Ω satisfying

$$(4.7) \qquad P(x, y)\frac{\partial^{m+n}v_\nu}{\partial x^m\,\partial y^n} = \sum_{(j,k)\in I} P_{jk}(x, y)\, A_{jk}(x, y)\frac{\partial^{j+k}v_\nu}{\partial x^j\,\partial y^k} + \varepsilon P(x, y)\,. \cdot\left\{f\left(x, y, \ldots, \frac{\partial^{j+k}u_\nu}{\partial x^j\,\partial y^k}, \ldots\right) - f\left(x, y, \ldots, \frac{\partial^{j+k}u_{\nu-1}}{\partial x^j\,\partial y^k}, \ldots\right)\right\},$$

$$\frac{\partial^j v_\nu}{\partial x^j}(a_j, y) = 0 \quad (j = 0, 1, \ldots, m-1),$$

$$\frac{\partial^k v_\nu}{\partial y^k}(x, b_k) = 0 \quad (k = 0, 1, \ldots, n-1),$$

$$(4.8) \qquad u_0 = 0\,, \quad u_\nu = v_0 + v_1 + \ldots + v_{\nu-1}$$

for $\nu = 1, 2, \ldots$ in Ω. Assume that $v_0, v_1, \ldots, v_{\nu-1}$ and $u_0, u_1, \ldots, u_\nu$ are well-defined so as to belong to H_Ω and satisfy

$$\|v_{\nu-1}\|_\Omega \leqq (2\varepsilon mnL^{m+n+2}N)^{\nu-1}\, 2\varepsilon mnL^{m+n}N \tag{4.9}$$

and

$$\|u_\nu\|_\Omega \leqq 4\varepsilon mnL^{m+n}N\,. \tag{4.10}$$

By (3.15) and (4.2), we have

$$\left|f\left(x, y, \ldots, \frac{\partial^{j+k}u_\nu}{\partial x^j\,\partial y^k}, \ldots\right) - f\left(x, y, \ldots, \frac{\partial^{j+k}u_{\nu-1}}{\partial x^j\,\partial y^k}, \ldots\right)\right| \leqq \tag{4.11}$$

$$\leqq N \sum_{(j,k)\in I}\left|\frac{\partial^{j+k}u_\nu}{\partial x^j\,\partial y^k} - \frac{\partial^{j+k}u_{\nu-1}}{\partial x^j\,\partial y^k}\right| \leqq L^2N\|v_{\nu-1}\|_\Omega\,.$$

By (4.11), Proposition 2 and (3.9), the problem (4.7) has the holomorphic solution v_ν in Ω satisfying

$$\|v_\nu\|_\Omega \leqq 2mnL^{m+n}\varepsilon L^2N\|v_{\nu-1}\|_\Omega\,. \tag{4.12}$$

By (4.3), (4.9), (4.10) and (4.12), we have

$$\|v_\nu\|_\Omega \leqq (2\varepsilon mnL^{m+n+2}N)^\nu\, 2\varepsilon mnL^{m+n}N \leqq (\tfrac{1}{2})^\nu\, 2\varepsilon mnL^{m+n}N \tag{4.13}$$

and

$$\|u_{\nu+1}\|_\Omega \leqq 4\varepsilon mnL^{m+n}N\,. \tag{4.14}$$

Hence we have

$$\left|\frac{\partial^{j+k}u_{\nu+1}}{\partial x^j\,\partial y^k}(x, y)\right| \leqq 4\varepsilon mnL^{m+n+2}N < R \tag{4.15}$$

in Ω. Thus we have proved that the sequences $\{v_\nu;\ \nu = 0, 1, 2, \ldots\}$ and $\{u_\nu;\ \nu = 0, 1, 2, \ldots\}$ are well-defined. By (4.13) the sequence $\{u_\nu;\ \nu = 0, 1, 2, \ldots\}$ converges uniformly to a holomorphic function $u(x, y)$ in Ω. $u(x, y)$ is a unique holomorphic solution of the problem (4.4). Summarizing the above result, we have the following Proposition and Theorem.

Proposition 3 *Assume* (3.7). *Let* ε_0 *be a positive number satisfying* (4.3). *Then for any positive number* ε *with* $\varepsilon \leqq \varepsilon_0$, *the problem* (4.4) *has a unique holomorphic solution in* Ω.

Theorem 2. *Assume* (3.7), (3.21), (3.22), (3.24) *and* (3.25). *Let* ε_0 *be a positive number satisfying* (4.3). *Then for any positive number* ε *with* $\varepsilon \leqq \varepsilon_0$, *the singular* *partial differential equation*

$$P(x, y)\frac{\partial^{m+n}u}{\partial x^m\,\partial y^n} = \sum_{(j,k)\in I} P_{jk}(x, y)\, A_{jk}(x, y)\frac{\partial^{j+k}u}{\partial x^j\,\partial y^k} + \varepsilon P(x, y) f\left(x, y, \ldots, \frac{\partial^{j+k}u}{\partial x^j\,\partial y^k}, \ldots\right) \tag{4.16}$$

has a unique holomorphic solution in Ω.

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