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EPI-ARCHIMEDEAN GROUPS

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An *epi-archimedean* group is a lattice-ordered group for which each l-homomorphic image is archimedean. Such groups are abelian and have been called hyper-archimedean or para-archimedean. They are at the opposite end of the spectrum from the free abelian l-groups.

Each group in the class $\mathscr{E}$ of epi-archimedean groups can be represented as a subdirect sum of reals. Let $\mathscr{S}$ be the class of all l-groups which have a representation as a subdirect sum of reals in which each element has finite range. Then $\mathscr{S} \subseteq \mathscr{E}$ and, in fact, an l-group belongs to $\mathscr{S}$ if and only if it is an l-subgroup of a vector lattice in $\mathscr{E}$ with an order unit. $\mathscr{S}$ is closed with respect to cardinal sums, l-subgroups, l-homomorphic images and v-hulls. If $G \in \mathscr{S}$ then the v-hull G^v of G is an a-closure of G and G^v is the unique a-closure of G in $\mathscr{S}$ (Theorem 5.1). Also each $G \in \mathscr{S}$ has a unique essential closure in $\mathscr{S}$ (Theorem 2.3).

In Section 4 the subdirect sums of integers that have been studied by SPECKER, NOBELING and others are shown to be those l-subgroups of ΠZ_i that are generated by characteristic functions. These can also be characterized as rings of bounded integral functions. Such groups belong to $\mathscr{S}$ and the group generated by the set of all singular elements in an arbitrary l-group is such a group. We study these groups and also their v-hulls. Epi-archimedean f-rings are investigated in Sections 4 and 6.

DEFINITIONS AND NOTATION

Let G be an l-group. If $g \in G$ then $G(g)$ will denote the convex l-subgroup of G generated by g

$$G(g) = \{x \in G \mid |x| \leqq n|g| \text{ for some } n > 0\} .$$

If A is a subset of G then A' will denote the *polar* of A.

$$A' = \{x \in G \mid |x| \wedge |a| = 0 \text{ for all } a \in A\} .$$

The *cardinal sum* (*product*) of a set $\{G_i | i \in I\}$ of l-groups will be denoted by $\Sigma G_i(\Pi G_i)$ or if I is finite by $G_1 \boxplus \ldots \boxplus G_n$.

A *prime subgroup* M of G is a convex l-subgroup such that the convex l-subgroups that contain it form a chain. M is a minimal prime if and only if $0 < g \in M$ implies $g' \nsubseteq M$. A convex l-subgroup N of G that is maximal without some element g is prime and the intersection N^* of all convex l-subgroups of G that properly contain N covers N. (N^*, N) or just N is called a *value* of g. N is called *regular*.

An l-group G is *laterally complete* if each disjoint subset of G has a least upper bound. If G is a subdirect sum and a sublattice of a cardinal product of totally ordered groups then G is called *representable.*

Let H be an l-subgroup of G. Then G is an *a-extension* of H if for each $0 < g \in G$ there exists $0 < h \in H$ such that $nh > g$ and $ng > h$ for some $n > 0$. In this case g and h are said to be *a-equivalent.* G is an a-extension of H if and only if the map $L \to L \cap H$ is a one to one map of the set of all convex l-subgroups of G onto those of H. G is *a-closed* if it admits no proper a-extensions and an a-closed a-extension of G is called an *a-closure* of G.

H is a *large l-subgroup* of G or G is an *essential extension* of H if for each non-zero convex l-subgroup L of G, $L \cap H \neq 0$. Note that an a-extension is an essential extension.

Each archimedean l-group G has a unique essential closure G^e in the class $\mathscr{A}$ of archimedean l-groups [13]. (i.e., G^e is an essential extension of G that admits no proper essential extensions in $\mathscr{A}$). Also G is contained in a unique minimal vector lattice G^v in $\mathscr{A}$ called the *v-hull* of G. G is large in G^v (see [8] and [15]). We shall denote the Dedekind-MacNeille completion of G by $G^\wedge$, and the divisible closure or the injective hull of G by G^d.

Finally R will denote the additive group of real numbers with the natural order and $M \prec R$ denotes that M is a group that is o-isomorphic to a subgroup of R.

1. EPI-ARCHIMEDEAN l-GROUPS

The following theorem is basic for the theory developed in this paper.

Theorem 1.1. *For an l-group G the following are equivalent.*

1) *G is epi-archimedean.*

2) *Each proper prime subgroup of G is maximal and hence minimal.*

3) $G = G(g) \boxplus g'$ *for each* $g \in G$.

4) *If* $0 < f, g \in G$, *then* $[f - (mg \wedge f)] \wedge g = 0$ *for some* $m > 0$.

5) *G is l-isomorphic to an l-subgroup* G^* *of* ΠR_i *and for each* $0 < x, y \in G^*$ *there exists an* $n > 0$ *such that* $nx_i > y_i$ *for all* $x_i \neq 0$.

6) *If* $0 < f, g \in G$ *then* $f \wedge ng = f \wedge (n + 1) g$ *for some* $n > 0$.
Moreover each representation of an epi-archimedean l-group as a group of real valued functions must satisfy (5).

The history of this theorem is as follows: AMEMIYA [1] proved $1 \leftrightarrow 3$ for vector lattices; BAKER [2] proved $1 \leftrightarrow 2$ for vector lattices; PEDERSEN [19] proved $5 \leftrightarrow 3 \rightarrow 2$; LUXEMBURG and MOORE [17] proved $1 \rightarrow 3$ for vector lattices; ZANNEN [23] proved $3 \rightarrow 1$ for vector lattices; BIGARD [5] proved $2 \leftrightarrow 3$; BIGARD, CONRAD and WOLFENSTEIN [7] proved $1 \leftrightarrow 2 \leftrightarrow 3$.

Also conditions on pairs of elements from G^+ that are equivalent to (4) or (6) were derived by most of these authors.

Proof of Theorem. $(1 \rightarrow 2)$ G is archimedean and hence abelian. If M is a proper prime subgroup of G then G/M is an archimedean o-group and hence M is maximal.

$(2 \rightarrow 3)$ If $G \neq G(g) \boxplus g'$ then $G(g) \boxplus g' \subseteq M \subset G$ for some prime subgroup M which is necessarily minimal. But then $0 < g \in M$ and so $g' \nsubseteq M$, a contradiction.

$(3 \rightarrow 4)$ $f = f_1 + f_2 \in G(g) \boxplus g'$. Thus $mg \geqq f_1$ for some $m > 0$ and so $mg \wedge f = f_1$. Therefore

$$[f - (mg \wedge f)] \wedge g = f_2 \wedge g = 0\,.$$

$(4 \rightarrow 1)$ First G is archimedean. For suppose that $0 \leqq f, g \in G$ and $ng \leqq f$ for all n. If

$$0 = [f - (mg \wedge f)] \wedge g = (f - mg) \wedge g$$

then $mg = f \wedge (m + 1) g = (m + 1) g$ and so $g = 0$. Now if σ is an l-homomorphism of G then clearly $G\sigma$ also satisfies (4) and so $G\sigma$ is archimedean.

$(1, 2, 3 \rightarrow 5)$ G is abelian and each prime subgroup is maximal. Thus without loss of generality G is a subdirect sum of reals, $G \subseteq \Pi R_i$. Pick $0 < x, y \in G$. $y = a + b \in G(x) \boxplus x'$ and so $nx > a$ for some $n > 0$. Thus $nx_i > y_i$ provided $x_i \neq 0$. Note that we have shown that *every* representation of G as a subdirect sum of reals satisfies (5).

$(5 \rightarrow 6)$ There exists $n > 0$ such that $ng_i > f_i$ for all $g_i \neq 0$. Thus

$$(f \wedge ng)_i = \begin{Bmatrix} f_i & \text{if } g_i > 0 \\ 0 & \text{if } g_i = 0 \end{Bmatrix} = (f \wedge (n + 1) g)_i$$

and so $f \wedge ng = f \wedge (n + 1) g$.

$(6 \rightarrow 1)$ First G is archimedean. For if $mg \leqq f$ for all m then $ng = f \wedge ng = f \wedge (n + 1) g = (n + 1) g$ and so $g = 0$. Next each l-homomorphic image of G satisfies (6) and so is archimedean.

Let $\mathscr{E}$ be the class of all epi-archimedean l-groups. It follows at once from (2) that an l-group G belongs to $\mathscr{E}$ if and only if G is representable and each totally ordered l-homomorphic image is archimedean. One only needs the fact that the minimal prime subgroups of a representable group are normal.

If $G \in \mathscr{E}$ has an order unit u, then $u' = 0$ and so by (3) $G = G(u)$. Thus u is a strong order unit. Also each l-ideal of $G \in \mathscr{E}$ that contains an order unit is a cardinal summand. It follows from (4) or (6) that $\mathscr{E}$ is closed with respect to l-subgroups, l-homomorphic images and cardinal sums. But $\prod_{i=1}^{\infty} R_i \notin \mathscr{E}$ and so $\mathscr{E}$ is not closed with respect to cardinal product. It follows from (2) that $\mathscr{E}$ is closed with respect to a-extensions, but Example 7.1 shows that $G \in \mathscr{E}$ need not have a unique a-closure.

If G is an epi-archimedean l-subgroup of an l-group H then it follows from (4) that there is a maximal l-subgroup of H that contains G and is epi-archimedean.

In Section 3 we show that each l-group H admits a largest epi-archimedean l-ideal. This is the epi-archimedean kernel of H introduced by Martinez [20].

Suppose that each proper l-homomorphic image of the l-group G is archimedean and let K be the intersection of all the non-zero l-ideals of G. Then there are three possibilities.

I. *If* $K = G$ *then* G *is l-simple.*

II. *If* $K = 0$ *then* G *is epi-archimedean.*

Proof. Let $\{L_\alpha \mid \alpha \in A\}$ be the set of non-zero l-ideals of G. Then there is a natural l-isomorphism of G into the abelian group $\Pi_A G/L_\alpha$ and so G is abelian. Thus if $P \neq 0$ is a prime subgroup of G then G/P is epi-archimedean and totally ordered and hence $G/P \prec R$. Thus each prime subgroup is a maximal l-ideal and so G is epi-archimedean.

III. *If* $0 \neq K \neq G$ *then* G *is an extension of* K *by an epi-archimedean l-group* $B \cong G/K$.

The following are examples of Case III.

a) G is a lexicographic extension of an l-simple l-group K by R. In particular, the lexicographic extension of R by R is such a group.

b) Let G be the wreath product of Z by Z. Then G is an extension of K by Z, where K is the direct product of a countable number of copies of Z. Let K have the cardinal order and let G be the lexicographic extension of K by Z.

c) Let G be the restricted wreath product of Z by Z. Then G is an extension of K by Z, where K is the direct sum, of a countable number of copies of Z, $K = \oplus Z_i$. Order K lexicographically and let G be the lexicographic extension of K by Z. Note that G is an o-group with one proper normal convex subgroup.

We have the following special cases.

Corollary. *If G is representable then G is epi-archimedean or G is an o-group with one or no proper normal convex subgroup.*

Proof. The intersection of all prime subgroups of G equals 0 and each minimal prime subgroup is normal. Thus if 0 is not prime it follows that $K = 0$.

Corollary. *If G is abelian then G is epi-archimedean or G is an o-group with rank* 2.

Lemma A. (a) *If G is an l-subgroup of $\Pi_I R_i$ and for each $0 < g \in G$ there exists $0 < r, s \in R$ such that $r < g_i < s$ for each $g_i \neq 0$ then $G \in \mathscr{E}$.*

(b) *If $G \in \mathscr{E}$ has an order unit u then there exists an l-isomorphism τ of G into $\Pi_I R_i$ with $u\tau = (1, 1, 1, \ldots)$. Moreover, each such representation $G\tau$ of G satisfies* (a).

(c) *If $G \in \mathscr{E}$ and G is an f-ring with no nilpotent elements then there exists a ring l-isomorphism τ of G into $\Pi_I R_i$. Moreover, each such representation $G\tau$ of G satisfies* (a).

Proof. (a) For $0 < x, y \in G$ there exist $0 < r, s \in R$ such that $r < x_i$ for all $x_i \neq 0$ and $y_i < s$ for all i. Pick an integer n such that $nr > s$. Then $nx_i > y_i$ for all $x_i \neq 0$ and so by (5) $G \in \mathscr{E}$.

(b) By Theorem 1.1 we may assume $G = G(u) \subseteq \Pi_I R_i$ and $u = (1, 1, 1, \ldots)$. If $0 < g \in G$ then by (5) there exist positive integers m, n such that $ng_i > u_i = 1$ for all $g_i \neq 0$ and $m = mu_i > g_i$ for all i. Thus $m > g_i > 1/n$ for all $g_i \neq 0$.

(c) Each prime l-ideal M of G is minimal and hence the join of principal polars. Thus M is a ring ideal and so G/M is o-isomorphic to a subring of R. Let $\{M_i \mid i \in I\}$ be the set of all prime ideals. Then there exists a ring l-isomorphism of G into $\Pi_I G/M_i$ and hence into $\Pi_I R_i$.

So (without loss of generality) we assume that G is an l-subring of $\Pi_I R_i$ and consider $0 < g \in G$. By (5) there exist positive integers m and n such that $mg > g^2$ and $ng^2 > g$. Thus $m > g_i > 1/n$ for all $g_i \neq 0$.

We have not been able to answer the following questions.

Does each $G \in \mathscr{E}$ have a representation that satisfies (a)?

Find an example of $G \in \mathscr{E}$ that is not contained in an epi-archimedean f-ring with zero radical.

Suppose $G \subseteq \Pi_I R_i$ satisfies (a). Does the l-subring of ΠR_i generated by G belong to $\mathscr{E}$?

Baker [2] defines an element g in ΠR_i to be a *step function* if g has finite range. Let $\mathscr{S}$ be the class of all l-groups G which have a representation as real valued step functions. Then by (5) $\mathscr{S} \subseteq \mathscr{E}$ and by Example 7.1 $\mathscr{S} \neq \mathscr{E}$.

Now clearly $\mathscr{S}$ is closed with respect to l-subgroups and cardinal sums. We shall show that $\mathscr{S}$ is closed with respect to l-homomorphic images and v-hulls and that the v-hull G^v of $G \in \mathscr{S}$ is the unique a-closure of G in $\mathscr{S}$.

Also both $\mathscr{S}$ and $\mathscr{E}$ are closed with respect to divisible hulls. For if $G \in \mathscr{S}$ then we may assume that G is an l-subgroup of step functions in ΠR_i and so $G \subseteq G^d \subseteq \Pi R_i$. If $x \in G^d$ then $nx \in G$ for some $n > 0$ and hence x must be a step function. If $G \in \mathscr{E}$

then we may assume that $G \subseteq G^d \subseteq \Pi R_i$. If $0 < x, y \in G^d$ then $nx, ny \in G$ for some $n > 0$ and hence by (5) there exists $m > 0$ such that $mnx_i > ny_i$ for all $x_i \neq 0$. Therefore $mx_i > y_i$ for all $x_i \neq 0$ and so by (5) again $G \in \mathscr{E}$.

Proposition 1.2. *If G is an epi-archimedean l-subgroup of $\Pi_I R_i$ that contains the long constants then G consists of step functions and so belongs to $\mathscr{S}$.*

Proof. If $0 < g \in G$ then by Lemma A there exists $0 < r, s \in R$ such that $r < g_i < < s$ for all $g_i \neq 0$. Suppose (by way of contradiction) that g has infinite range. Then $\{g_i | i \in I \text{ and } g_i \neq 0\}$ is an infinite subset of the compact set $[r, s]$ and so has a limit point a in $[r, s]$. Let $\bar{a}$ be the long constant $(a, a, a, \ldots)$. Each open interval of R that contains a must contain a component $g_i \neq a$ or 0 of g.

Case I. A sequence of the g_i converge to a from below. Then $(\bar{a} - g) \vee 0$ has a sequence of strictly positive components that converge to zero which contradicts Lemma A.

Case II. A sequence of the g_i converge to a from above. Then $(g - \bar{a}) \vee 0$ has a sequence of strictly positive components that converge to zero.

Corollary I. *Let G be an epi-archimedean vector lattice with an order unit u. Then there exists an l-isomorphism τ of G into $\Pi_I R_i$ with $u\tau = (1, 1, 1, \ldots)$ and $G\tau$ consists of step functions and so $G \in \mathscr{S}$.*

Corollary II. *An l-group G belongs to $\mathscr{S}$ if and only if G is an l-subgroup of an epi-archimedean vector lattice H with a unit.*

Proof. If $G \in \mathscr{S}$ then we may assume that $G \subseteq \Pi R_i$ and each g in G is a step function. Thus G is an l-subgroup of the group H of all step functions in ΠR_i. The converse follows from Corollary I.

An l-group G is *locally* $\mathscr{E}$ (*locally* $\mathscr{S}$) if each $G(g)$ belongs to $\mathscr{E}(\mathscr{S})$. Clearly locally $\mathscr{S}$ implies locally $\mathscr{E}$ and it follows from (4) that G is locally $\mathscr{E}$ if and only if $G \in \mathscr{E}$. By Corollary I each epi-archimedean vector lattice is locally $\mathscr{S}$.

Now Bernau [3] and Baker [2] both give an example of an epi-archimedean vector lattice that does not belong to $\mathscr{S}$. Therefore locally $\mathscr{S}$ does not imply $\mathscr{S}$. Thus any elementwise definition of $\mathscr{S}$ must involve an infinite number of elements; otherwise locally $\mathscr{S}$ implies $\mathscr{S}$.

Finally note that Example 7.1 shows that an epi-archimedean l-group with an order unit need *not* belong to $\mathscr{S}$. Let G be as in this example. Then since G has an order unit so does its v-hull G^v. Thus if $G^v \in \mathscr{E}$ then $G^v \in \mathscr{S}$ and hence so does G, a contradiction. Thus the v-hull of an epi-archimedean group need *not* be epi-archimedean.

Theorem 1.3. *$\mathscr{S}$ is closed with respect to cardinal sums, l-subgroup, l-homomorphic images and v-hulls.*

Proof. The first two are clear. Let K be an l-ideal of $G \in \mathscr{S}$. Then without loss of generality G is an l-subgroup of the group H of all step functions in ΠR_i. Let

$$K\mu = \bigcap\{L \mid K \subseteq L \text{ and } L \text{ is an } l\text{-ideal of } H\}.$$

This is the l-ideal of H that is generated by K and $K\mu \cap G = K$. Moreover,

$$\frac{G}{K} = \frac{G}{K\mu \cap G} \simeq \frac{K\mu + G}{K\mu} \subseteq \frac{H}{K\mu}$$

but $H/K\mu$ is an epi-archimedean vector lattice with an order unit and so belongs to $\mathscr{S}$.

It follows from a result of BLEIER [8] that the v-hull G^v of G is the intersection of all the l-subspaces of H that contain G.

Another proof. $G \subseteq G^d \subseteq (G^d)^\wedge \subseteq \Pi R_i$ and G^v is the l-subspace of $(G^d)^\wedge$ generated by G. Clearly $G \subseteq H \cap (G^d)^\wedge$ an l-subspace of $(G^d)^\wedge$. Thus $G^v \subseteq H$ and so $G^v \in \mathscr{S}$.

Corollary. *If $G \in \mathscr{E}$ has a unit then $G \in \mathscr{S}$ if and only if the v-hull of G belongs to $\mathscr{E}$.*

Proof. If u is a unit for G then it is also a unit for G^v and so by Corollary I of Proposition 1.2 $G^v \in \mathscr{E}$ implies $G^v \in \mathscr{S}$ and so $G \in \mathscr{S}$. Conversely if $G \in \mathscr{S}$ then so does G^v and hence $G^v \in \mathscr{E}$.

Lemma. *If $0 \neq A$ is a subgroup of an archimedean o-group B and α is an o-isomorphism of A into R then there exists a unique extension of α to an o-isomorphism of B into R.*

This follows from the fact that each o-isomorphism of a subgroup of R into a subgroup of R is a multiplication by a positive real number.

Lemma. *If G is an l-subgroup of an l-group H and M is a regular subgroup of G then $M = G \cap N$ for a regular subgroup N of H.*

Proof. Let Y be the convex l-subgroup of H generated by M. Then $Y \cap G = M$. Now M is maximal without some $g \in G$ and so $g \notin Y$. Thus $Y \subseteq N$ a value of g in H. $N \cap G \supseteq M$ and $g \notin N \cap G$ a convex l-subgroup of G. Therefore $N \cap G = M$.

Proposition 1.4. *If G is a large l-subgroup of $H \in \mathscr{E}$ and τ is an l-isomorphism of G into $\Pi_I R_i$ then there exists an extension of τ to an l-isomorphism of H into $\Pi_I R_i$.*

Proof. We may assume (without loss of generality) that $G_i = \{g \in G \mid (g\tau)_i = 0\} \neq 0$ for each $i \in I$. Thus each G_i is a regular subgroup of G. Pick H_i regular in H such that $H_i \cap G = G_i$. Then

$$\frac{G}{G_i} = \frac{G}{H_i \cap G} \simeq \frac{H_i + G}{H_i} \subseteq \frac{H}{H_i}$$

and since $H \in \mathscr{E}$, $H/H_i \prec R$. The map $g \to (g\tau)_i$ is an l-homomorphism of G into R_i with kernel G_i. Thus

$$H_i + g \to H_i \cap G + g = G_i + g \to (g\tau)_i$$

is an o-isomorphism of $(H_i + G)/H_i$ into R_i and so there exists a unique extension to an o-isomorphism α_i of H/H_i into R_i. Now for $h \in H$ and $g \in G$ we consider the maps

$$h \to (\ldots, H_i + h, \ldots) \in \Pi H/H_i \to (\ldots, (H_i + h)\,\alpha_i, \ldots) \in \Pi R_i$$
$$g \to (\ldots, H_i + g, \ldots) \to (\ldots, (g\tau)_i, \ldots) = g\tau\,.$$

Thus we have extended τ to an l-homomorphism of H into ΠR_i with kernel $\bigcap H_i$. Now $(\bigcap H_i) \cap G = \bigcap (H_i \cap G) = \bigcap G_i = 0$ and since G is large in H we have $\bigcap H_i = 0$. Thus this extended map is an l-isomorphism of H into ΠR_i.

Corollary. *Each $G \in \mathscr{E}$ admits an essential closure H in $\mathscr{E}$, and if $G \subseteq \Pi R_i$ then $G \subseteq H \subseteq \Pi R_i$.*

We shall show in section 7 that H need not be unique even if $G \in \mathscr{S}$, but each $G \in \mathscr{S}$ has a unique essential closure in $\mathscr{S}$.

Suppose that G is archimedean and has a strong unit u then there exists an l-isomorphism τ of G such that

$$G\tau \subseteq \Pi_I R_i \quad \text{and} \quad u\tau = (1, 1, 1, \ldots)$$

Proposition 1.5. *$G \in \mathscr{S}$ if and only if each $g\tau$ is a step function in this representation.*

Proof. $(\leftarrow)$ Trivial.

$(\rightarrow)$ The v-hull H of G is an essential extension of G and so by Proposition 1.4 τ can be extended to an l-isomorphism ϱ of H into ΠR_i. Now the long constant belongs to $H\varrho$ and so $H\varrho$ consists of step functions and hence so does $G\tau$ (see Proposition 1.2).

If B is an essential extension of an l-group A and u is an order unit in A then u is also an order unit in B. For suppose (by way of contradiction) that $0 < b \in B$ and $b \wedge u = 0$. Then $b'' \cap u'' = 0$, where the polar operations are in B, and since $b'' \cap \cap A \neq 0$ we have $a \wedge u = 0$ for some $0 < a \in A$, a contradiction.

Proposition 2.1. *If $G \in \mathcal{S}$ then there exists an essential extension of G in $\mathcal{S}$ that contains an order unit.*

Proof. We may assume that G is an l-subgroup of

$$H = \text{all step functions in } \Pi_I R_i .$$

Let W be the l-subspace of H generated by G and $u = (1, 1, \ldots)$ and let B be an l-ideal of W that is maximal with respect to $B \cap G = 0$. Then

$$G \simeq \frac{B \oplus G}{B} \subseteq \frac{W}{B} \in \mathcal{S}$$

and $B + u$ is a unit in W/B. Now if J/B is a non-zero l-ideal of W/B then $J \supset B$ and hence $J \cap G \neq 0$. Thus W/B is an essential extension of $(B \oplus G)/B$.

Corollary 1. *If $G \in \mathcal{S}$ and G is an l-subgroup of $\Pi_I R_i$ then there exists $w \in \Pi R_i$ such that each $w_i > 0$ and for which $Gw = \{gw \mid g \in G\}$ consists of step functions.*

Proof. Let K be an essential extension of G in $\mathcal{S}$ that contains an order unit u. By Proposition 1.4 we may assume that $G \subseteq K \subseteq \Pi R_i$ and we may also assume that $G_i = \{g \in G \mid g_i \neq 0\} \neq 0$ for each $i \in I$. Since u is a strong order unit for K we have $u_i > 0$ for each $i \in I$.

Let w be the multiplicative inverse of u in the ring ΠR_i. The map $x \to xw$ is an l-automorphism of the group ΠR_i and $Kw \in \mathcal{S}$ and contains $(1, 1, 1, \ldots)$. Thus by Proposition 1.5 Kw consists of step functions and hence so does Gw.

Corollary II. *If G is an l-subgroup of $\Pi_I R_i$ then so is $K = G + \Sigma R_i$. Moreover, if $G \in \mathcal{S}$ or $\mathcal{E}$ then so does K.*

Proof. Since ΣR_i is an l-ideal of $\prod R_i$ it follows that K is an l-subgroup of ΠR_i. If $0 < x, y \in K$ then they differ in only a finite number of places from elements in G and so it follows from (5) of Theorem 1.1 that if $G \in \mathcal{E}$ so does K.

If $G \in \mathcal{S}$ then by Corollary I there is a $w \in \Pi R_i$ such that each $w_i > 0$ and Gw consists of step functions. Clearly $(\Sigma R_i)\, w = \Sigma R_i$ and hence Kw also consists of step functions. Therefore $K \in \mathcal{S}$.

If $G \in \mathscr{S}$ has a basis then ([11], p. 3.15) there is an l-isomorphism σ of G such that

$$\Sigma_I T_i \subseteq G\sigma \subseteq \Pi_I R_i\,, \quad \text{where} \quad 0 \neq T_i \subseteq R$$

and by Corollary I we may assume that $G\sigma$ consists of step functions. It follows that *the set of all step functions in* ΠR_i *is the essential closure of* $G\sigma$ *in* $\mathscr{S}$.

Let X be a *Stone space* (that is, a compact extremely disconnected Hausdorff topological space) and let $S(X)$ be the group of all step functions in the l-group $C(X)$ of all continuous real valued functions on X. Then $S(X)$ is the subspace of $C(X)$ that is generated by the characteristic functions on the clopen subsets of X and $C(X)$ is an essential extension of $S(X)$.

Proposition 2.2. *$S(X)$ is essentially closed in $\mathscr{E}$ and hence in $\mathscr{S}$.*

Proof. Suppose that $S(X) \subseteq K \in \mathscr{E}$, where K is an essential extension of $S(X)$. Now $S(X)$ and K have the same Boolean algebra of polars [13] and hence the same associated Stone space, namely X. Thus (see [4]) we can embed K into $C(X)$ so that $(1, 1, 1, \ldots)$ is mapped onto itself. This induces the identity map on $S(X)$ and so we may assume that

$$S(X) \subseteq K \subseteq C(X)\,.$$

Here we use the fact that $(1, 1, 1, \ldots)$ is also an order unit for K and hence a strong order unit. Now by Proposition 1.2 it follows that K consists of step functions and so $K = S(X)$.

Theorem 2.3. *Each $G \in \mathscr{S}$ has a unique essential closure in $\mathscr{S}$ namely the l-group $S(X)$ of all step functions in $C(X)$, where X is the Stone space associated with the Boolean algebra of polars of G. Moreover $S(X)$ is essentially closed in $\mathscr{E}$.*

Proof. Let H be an essential extension of G in $\mathscr{S}$. By Proposition 2.1 there is an essential extension K of H in $\mathscr{S}$ that has a unit u. Also the v-hull of K belongs to $\mathscr{S}$, and so we may assume that K is a vector lattice. Thus we can imbed K into $C(X)$ so that $u = (1, 1, 1, \ldots)$ and so that $C(X)$ is an essential extension of K.

$$G \subseteq H \subseteq K \subseteq C(X)\,.$$

Thus by Proposition 1.2 K consists of step functions and so

$$G \subseteq H \subseteq K \subseteq S(X)\,.$$

Thus $S(X)$ is an essential extension of G that is essentially closed in $\mathscr{S}$.

Now suppose that T is an essential closure of G in $\mathscr{S}$. Then clearly T is a vector lattice with an order unit u and so there is an l-isomorphism σ of T onto a large subgroup of $C(X)$ such that $u\sigma = (1, 1, 1, \ldots)$. Then as above $S(X)$ is an essential extension of $T\sigma$ and so $T\sigma = S(X)$.

Note that $S(X) \in \mathscr{S}$ but $S(X)^\wedge = C(X) \notin \mathscr{E}$ unless $S(X) = C(X)$. Thus $\mathscr{S}$ and $\mathscr{E}$ are not closed with respect to Dedekind-MacNeille completions. For $S(X)$ is dense in $C(X)$ and so $S(X)^\wedge$ is the l-ideal generated by $S(X)$ which is $C(X)$ (see [10]).

3. SOME PROPERTIES OF EPI-ARCHIMEDEAN l-GROUPS

Most of the theory in this section is not new, but the proofs given here are shorter than those in print.

Proposition 3.1. *If G is a laterally complete epi-archimedean l-group then $G \simeq T_1 \boxplus \ldots \boxplus T_s$ where each $T_i \subseteq R$.*

Proof. Suppose (by way of contradiction) that $a_1, a_2, \ldots$ is an infinite disjoint subset of G then so is $a_1, 2a_2, 3a_3, \ldots$ Let $x = \bigvee a_k$ and $y = \bigvee ka_k$. Then clearly x and y do not satisfy (5) of Theorem 1.1, a contradiction. Thus G has a finite basis and so $G \simeq T_1 \boxplus \ldots \boxplus T_s$.

Proposition 3.2. *If G is an epi-archimedean l-group, $2G = G$ and each countable bounded disjoint subset has a least upper bound, then $G \simeq \Sigma T_\lambda$, where each $T_\lambda \subseteq R$. Thus if G also has an order unit then $G \simeq T_1 \boxplus \ldots \boxplus T_n$.*

Proof. It suffices to show that each $G(g)$ has a finite basis; for then G has a basis and so we may assume

$$\Sigma T_\lambda \subseteq G \subseteq \Pi T_\lambda$$

and since each $G(g)$ has a finite basis it follows that $G = \Sigma T_\lambda$.

If $G(g)$ does not have a finite basis then there exists a countable disjoint subset $g_1, g_2, \ldots$ of $G(g)$. Since each g_k is divisible by 2 we may assume that $g_k \leqq g$ for all k and, hence, without loss of generality, $g = \bigvee g_k$. Now let $h = \bigvee(1/2^k)\, g_k$. Then h is a unit in $G(g)$ and hence a strong unit, but clearly $nh = \bigvee(n/2^k)\, g_k \ngeqq g$ for any n, which contradicts (5) of Theorem 1.1.

Remarks. We can replace the hypothesis epi-archimedean by archimedean and each order unit in each $G(g)$ is a strong order unit.

If G is the l-group of all bounded sequences of integers then $G \in \mathscr{S}$ and each bounded disjoint subset has a least upper bound. Thus the hypothesis $2G = G$ cannot be dispensed with. Note that $(1, 0, 0, \ldots), (0, 1/2, 0, 0, \ldots), (0, 0, 1/2^2, 0, 0), \ldots$ has no least upper bound in G^d; so we cannot use the divisible hull of G.

Corollary I. *If G is a laterally complete epi-archimedean vector lattice then* $G = R_1 \boxplus \dots \boxplus R_n$.

This is also a corollary of Proposition 3.1.

Corollary II. (Bigard, Bernau). *If G is a σ-complete epi-archimedean vector lattice then $G = \Sigma R_\lambda$. Thus if G has a unit then $G = R_1 \boxplus \dots \boxplus R_n$.*

Proposition 3.3. (Bernau). *If G is an epi-archimedean vector lattice with countable dimension as a real vector space then $G = \Sigma_I G(f_i)$ and so $G \in \mathscr{S}$.*

Proof. Let $g_1, g_2, \dots$ be a positive basis for the vector space G and let

$$f_1 = g_1$$

$$f_2 = b_2\,, \text{ where } g_2 = a_2 + b_2 \in G(f_1) \boxplus G(f_1)'$$

....

$$f_{n+1} = b_{n+1}\,, \text{ where } g_{n+1} = a_{n+1} + b_{n+1} \in G(f_1 + \dots + f_n) \boxplus G(f_1 + \dots + f_n)'$$

....

Then $f_1, f_2, \dots$ are disjoint and $g_1, \dots, g_n \in G(f_1 + \dots + f_n) = G(f_1) \boxplus \dots \boxplus G(f_n)$. Now $x \in G$ is a linear combination of a finite number of the g_i, say $g_1, \dots, g_n$, and so $x \in G(f_1) \boxplus \dots \boxplus G(f_n)$. Thus $G = \Sigma G(f_i)$. By Corollary I to Proposition 1.2 each $G(f_i) \in \mathscr{S}$ and hence $G \in \mathscr{S}$.

Baker [2] and Bernau [3] both show that an epi-archimedean vector lattice with uncountable dimension need not belong to $\mathscr{S}$.

Proposition 3.4. (Bigard) *An l-group G is epi-archimedean if and only if G is* (l-isomorphic to) *a group of real valued functions on a topological space X with pointwise addition and order and such that*

a) *G separates points, and*

b) *the support of each $g \in G$ is compact and open.*

Proof. ($\rightarrow$) Let E be the set of all maximal l-ideals of G and let τ be the natural l-isomorphism of G into $\Pi_{P \in E} G/P$

$$g\tau = (\dots, P + g, \dots)\,.$$

For each $g \in G$ let $\sigma(g) = \{P \in E \mid g \notin P\}$ the support of g. The $\sigma(g)$ form a basis of open sets for a topology on E. This is the hull kernel topology on E. If $P_1 \neq P_2$ then $(P_1 \setminus P_2) \cap G \neq \square$ and so G separates points.

Suppose that $\sigma(g) = \bigcup \sigma(g_\lambda)$ for $g, g_\lambda \in G$. If $g \notin \bigvee G(g_\lambda)$ then $g \notin P \supseteq \bigvee G(g_\lambda)$ for some value P of g. Thus $P \in \sigma(g) = \bigcup \sigma(g_\lambda)$ and so $g_\lambda \notin P$ for some λ, a contradiction.

Thus $g \in \bigvee G(g_\lambda)$ and so $g \in G(g_{\lambda_1}) + \ldots + G(g_{\lambda_n})$. But then $\sigma(g) \subseteq \sigma(g_{\lambda_1}) \cup \ldots \ldots \cup \sigma(g_{\lambda_n})$ and so $\sigma(g)$ is compact.

($\leftarrow$) G is an l-group of functions on X with compact open support. For $0 < f, g \in G$ and $n > 0$ let

$$V_n = \{x \in \sigma(g) \mid ng(x) > f(x)\} = \sigma(g) \cap \sigma((ng - f)^+)$$

which is open. Now $\sigma(g) = \bigcup V_n$ and since $\sigma(g)$ is compact

$$\sigma(g) = V_{n_1} \cup V_{n_2} \cup \ldots \cup V_{n_k}.$$

Let m = maximum of $n_1, n_2, \ldots, n_k$. Then $mg(x) > f(x)$ for all $x \in \sigma(g)$ and so by Theorem 1.1 G is epi-archimedean.

Remarks. The topology on E is Hausdorff. The set of *all* functions on X with compact open support need not be an l-group.

We next discuss the epi-archimedean kernel of an l-group C. This concept and theory are due to JORGE MARTINEZ [20]. We have removed his hypothesis that G be representable. Recall that a *value* of $g \in G$ is a regular subgroup G_γ of G such $g \in G^\gamma \setminus G_\gamma$. Let

$$E = \{g \in G \mid \text{each value of } g \text{ is a minimal prime}\} \text{ and}$$

$$\mathcal{N} = \text{set of all prime subgroup of } G \text{ that are not minimal.}$$

Theorem 3.5. (Martinez) *$E = \bigcap \mathcal{N}$ and so E is a convex l-subgroup of G that is invariant under all l-automorphisms of G. Moreover, E is epi-archimedean and contains each convex l-subgroup of G that is epi-archimedean; E is the epi-archimedean kernel of G.*

Proof. If $g \in E$ and $N \in \mathcal{N}$ then $g \in N$; otherwise g has a value that contains N. Thus $E \subseteq \bigcap \mathcal{N}$. Conversely if $g \in \bigcap \mathcal{N}$ and $g \in G^\gamma \setminus G_\gamma$ then clearly G_γ is minimal and so $\bigcap \mathcal{N} \subseteq E$.

Therefore $E = \bigcap \mathcal{N}$ a convex l-subgroup of G and if τ is an l-automorphism of G then τ induces a permutation on the set $\mathcal{N}$ and hence $E\tau = E$. If P is a prime subgroup of G that does not contain E then clearly P is minimal and so $P \cap E$ is a minimal prime in E. Since each prime in E is of this form (see [11] Theor. 1.14) it follows by Theorem 1.1 that E is epi-archimedean.

Finally consider $0 < g \in K$ an epi-archimedean convex l-subgroup of G and suppose (by way of contradiction) that $g \in G^\gamma \setminus G_\gamma$ where γ is not minimal. Then $G_\gamma \supset G_\delta$. Pick $0 < x \in G^\delta \setminus G_\delta$. Then by replacing x by $g \wedge x$ we may assume that $g \geqq x$ and so $x \in K$. But then $K \supset G_\gamma \cap K \supset G_\delta \cap K$ and so K is not epi-archimedean, a contradiction. Thus each value of g is a minimal prime and so $K \subseteq E$.

Let B be the group of all bounded functions in $\Pi_I Z_i$. If $g \in \Pi Z_i$ then $S(g)$ will denote the support of g

$$S(g) = \{i \in I|\ g_i \neq 0\}$$

and if $X \subseteq I$ then χ_X will denote the characteristic function on X.

$$(\chi_X)_i = 1 \text{ if } i \in X \text{ and } 0 \text{ otherwise}.$$

Each $0 \neq g \in B$ has a unique representation

$$g = n_1 \chi_{X_1} + \dots + n_k \chi_{X_k}$$

where the n_i are distinct non-zero integers and the X_i are disjoint subsets of I.

The next propostion is more or less implicit in [21] but this formulation and proof is due to LASZLO FUCHS.

4.1. *For a subgroup G of B the following are equivalent.*

a) $g = n_1\chi_{X_1} + \dots + n_k\chi_{X_k} \in G$ *implies* $\chi_{X_i} \in G$ *for* $i = 1, \dots, k$, *where of course this is the unique representation of* g.
b) $g \in G$ *implies* $\chi_{S(g)} \in G$.
c) G *is pure in* B *and a subring of* B.

A subgroup G of B that satisfies a), b), and c) is called *a Specker group*.

Proof. (a → b) Clear, since $\chi_{S(g)} = \chi_{X_1} + \dots + \chi_{X_k}$.

(b → a) We use induction on k.

$$(n_1 - n_k)\chi_{X_1} + \dots + (n_{k-1} - n_k)\chi_{X_{k-1}} = g - n_k\chi_{S(g)} \in G.$$

Thus by induction $\chi_{X_1}, \dots, \chi_{X_{k-1}} \in G$ and so since

$$n_k\chi_{X_k} = g - n_1\chi_{X_1} - \dots - n_{k-1}\chi_{X_{k-1}} \in G$$

we have χ_{X_k} also belongs to G.

(a → c) We first show that if $g = n_1\chi_{X_1} + \dots + n_k\chi_{X_k} \in B$ and $mg \in G$ for some $m \neq 0$ then $g \in G$ and so G is pure.

$$mg = mn_1\chi_{X_1} + \dots + mn_k\chi_{X_k}.$$

Thus by a) the χ_{X_i} belong to G and so $g \in G$.

Now G is generated by characteristic functions. Thus it suffices to show that if $\chi_X, \chi_Y \in G$ then $\chi_X\chi_Y \in G$. For then it follows that G is closed with respect to multiplication. Note that $\chi_X\chi_Y = \chi_{X\cap Y}$, and

$$\chi_X + \chi_Y = \chi_{(X\cup Y)\setminus(X\cap Y)} + 2\chi_{X\cap Y} .$$

Thus by a) $\chi_{X\cap Y} \in G$.

(c → a) If $g = n\chi_X \in G$ then since G is pure $\chi_X \in G$. Now consider $g = n_1\chi_{X_1} + \dots$ $\dots + n_k\chi_{X_k}$ and use induction on k.

$$g^2 - n_k g = (n_1^2 - n_k n_1)\,\chi_{X_i} + \dots + (n_{k-1}^2 - n_k n_{k-1})\,\chi_{X_{k-1}}$$

and $g^2 - n_k g \in G$ since G is a ring. Thus by induction $\chi_{X_1}, \dots, \chi_{X_{k-1}} \in G$. But

$$n_k\chi_{X_k} = g - n_1\chi_{X_1} - \dots - n_{k-1}\chi_{X_{k-1}}$$

and so by purity again it follows that $\chi_{X_k} \in G$.

Note that the group of all bounded continuous functions from a topological space X into Z is Specker. Also the intersection of Specker groups is Specker and the join of a chain of Specker groups is Specker.

Clearly a Specker group is generated by characteristic functions. In the next proposition we make use of the cardinal order of ΠZ_i and the fact that B is an l-ideal of ΠZ_i.

4.2. *For a subgroup G of $\Pi_I Z_i$ that is generated by its set S of characteristic functions the following are equivalent.*

a) *G is Specker.*

b) *G is an l-subgroup of B.*

c) *S is closed with respect to multiplication.*

d) *S is closed with respect to $\wedge$.*

Proof. If $x, y \in S$ then $xy = x \wedge y$ and hence (c) and (d) are equivalent and clearly (b) implies (d).

(a → b) If $g = n_1\chi_{X_1} + \dots + n_k\chi_{X_k} \in G$ then the $\chi_{X_i} \in G$ and so it follows that $g \wedge 0 \in G$.

(d → a) If $0 \neq g \in G$ then $g = m_1\chi_{Y_1} + \dots + m_t\chi_{Y_t}$ where the m_i are integers and the $\chi_{Y_i} \in S$. Here we do not assume that the Y_i are disjoint subsets of I.

$$\chi_{Y_1}\chi_{Y_2} = \chi_{Y_1\cap Y_2} = \chi_{Y_1} \wedge \chi_{Y_2} \in S .$$

Thus $\chi_{Y_1} - \chi_{Y_1\cap Y_2} = \chi_{Y_1\setminus Y_2} \in G$ and so we have

$$m_1\chi_{Y_1} + m_2\chi_{Y_2} = m_1\chi_{Y_1\setminus Y_2} + (m_1 + m_2)\,\chi_{Y_1\cap Y_2} + m_2\chi_{Y_2\setminus Y_1} .$$

It follows that g has a representation

$$g = n_1\chi_{X_1} + \dots + n_k\chi_{X_k}$$

where the n_i are distinct non-zero integers, the X_i are disjoint subsets of I *and* each $\chi_{X_i} \in S$.

Note that each Specker group belongs to $\mathscr{S}$. Also if L is an l-ideal of a Specker group G then clearly L satisfies (b) of 4.1 and so L is also Specker.

4.3. *Each l-ideal L of a Specker group G is a ring ideal.*

Proof. Since $G \in \mathscr{S}$, $G = G(g) \boxplus g'$ for each $g \in G$. So each $G(g)$ is a ring ideal, but L is the join of a directed (by inclusion) set of such $G(g)$ and so L is a ring ideal.

Theorem. (Nobeling [21]) *If $G \subset H$ are Specker groups then $H = G \oplus F$ where F is a free abelian group with characteristic basis.*

Laszlo Fuchs (unpublished) and Paul Hill [16] have derived simpler proofs of this remarkable result.

Actually, as we now show, Specker groups occur quite natually, in the theory of l-groups. Recall that an element s in an l-group H is *singular* if $s > 0$ and

$$0 \leqq g < s \quad \text{implies} \quad g \wedge (s - g) = 0 \quad \text{for each} \quad g \in H\,,$$

and let S be the set of all singular elements in H. Then in [10] it is shown that:

4.4. *The subgroup $[S]$ of H generated by S is an abelian l-ideal of H.*

4.5. *There exists an l-isomorphism τ of $[S]$ onto a subdirect sum of $\Pi_I Z_i$ and for each such mapping τ, $[S]\,\tau$ is Specker and hence a subring of $\Pi_I Z_i$.*

Proof. In [10] it is shown that τ exists and for each $s \in S$, $s\tau$ is characteristic. Thus $[S]\,\tau$ is Specker by (b) of 4.2.

It follows that a group G is (isomorphic to) a Specker group if and only if there exists a set S of generators of G and a lattice order for G in which each $s \in S$ is singular. An l-group G is l-isomorphic to a Specker group if and only if G is generated as a group by singular elements.

4.6. *If τ is an l-homomorphism of $[S]$ then $[S]\,\tau$ is also an l-group that is generated by singular elements as a group.*

Proof. It is shown in [10] that if $s \in S$ then $s\tau = 0$ or $s\tau$ is singular.

Definition. An *S-group* is an l-group G that is generated (as a group) by singular elements. Such a group G is free abelian, belongs to $\mathscr{S}$ and each l-homomorphic image of G is also an S-group. A subgroup H of G that is generated by its set T of singular elements is an l-subgroup and hence an S-group if and only if T is closed with respect to $\wedge$.

4.7. *Let $G = [S]$ be an S-group. Then there exists a unique multiplication on G so that it is a ring for which $st = s \wedge t$ for all $s, t \in S$. Moreover G is an f-ring with zero radical, each l-ideal of G is a ring ideal and each l-homomorphism of the group G is a ring homomorphism.*

Proof. We may assume that G is an l-subgroup of $\Pi_I Z_i$ and each $s \in S$ is characteristic. Since G is Specker it is a subring of ΠZ_i and so $st = s \wedge t$ for $s, t \in S$.

Now suppose that . and $*$ are multiplications for G so that it is a ring for both and

$$s \,.\, t = s \wedge t = s * t \quad \text{for all} \quad s, t \in S\,.$$

Then

$$(ms)\,.\,(nt) = mn(s\,.\,t) = mn(s * t) = (ms) * (nt)$$

for all $m, n \in Z$ and it follows that $g \,.\, h = g * h$ for all $g, h \in G$.

It follows from [14] that this multiplication on G has a unique extension to the v-hull G^v of G.

Note that if u is an order unit in an S-group $G = [S]$ then $\chi_{S(u)}$ is an order unit and a singular element.

4.8. *If $G = [S]$ is an S-group and $s \in S$ is an order unit for G then the multiplication in* 4.7 *is the unique multiplication so that G is an f-ring with identity s.*

Proof. Clearly $s = \bigvee S$. Thus if $a \in S$ then $sa = s \wedge a = a$ and so s is the identity in the above multiplication. In [12] it is shown that there is at most one such multiplication.

Corollary. *If G is an l-group with a singular element u as a strong order unit then G is an S-group and there is a unique multiplication on G so that it is an f-ring with identity u.*

Proof. Let S be the set of all singular elements of G. Then $[S]$ is an l-ideal of G that contains a strong order unit of G and so $G = [S]$.

Suppose that $G = [S]$ is an S-group with no order unit. Then without loss of generality G is a subdirect sum and a subring of $\Pi_I Z_i$. Let

$$H = G \oplus Z(1, 1, 1, \ldots)\,.$$

Then H is an l-subgroup of ΠZ_i. In fact H is an S-group with G as an l-ideal and with $(1, 1, 1, \ldots)$ as a unit. Or one can define H by

$$H = Z \oplus G$$

and let H^+ be the subsemigroup of H generated by all the elements of the form

$$(n, 0), (n, -s), (0, s), (0, 0) \quad \text{where} \quad 0 < n \in Z \quad \text{and} \quad s \in S .$$

Here $(1,0)$ is an order unit; in fact $(1, 0)$ is the join of all the singular elements in G. Then by 4.8 H is an f-ring with identity $(1, 0)$ and, of course, this is just the standard way of adjoining an identity to the ring G.

Let F be the group of all functions in $\Pi_I R_i$ with finite range. Each $0 \neq g \in F$ has a unique representation

$$g = a_1\chi_{X_1} + \ldots + a_k\chi_{X_k}$$

where the a_i are distinct non-zero reals and the X_i are disjoint subsets. The proofs of the next two propositions are almost identical with the proofs of 4.1, 4.2 and 4.3 and we shall omit them.

4.9. *For a subspace G of F the following are equivalent.*

a) *$g = a_1\chi_{X_1} + \ldots + a_k\chi_{X_k} \in G$ implies each $\chi_{X_i} \in G$.*

b) *$g \in G$ implies $\chi_{S(g)} \in G$.*

c) *G is a subring of F.*

d) *G is generated as a subspace of F by a set of characteristic functions and G is an l-subgroup of F.*

e) *G is generated as a subpsace of F by a set S of characteristic functions and S is closed with respect to $\wedge$.*

A subspace G of F that satisfies a)–e) will be called a *Specker space.*

4.10. *Each l-ideal of a Specker space is a ring ideal and, of course, a Specker space.*

4.11. *If H is an l-subgroup of $\Pi_I R_i$ consisting of step functions and $u = (1, 1, 1, \ldots) \in H$ then H satisfies condition* a) *of 4.9. Thus if G is an epi-archimedean vector lattice with order unit u then G is (l-isomorphic to) a Specker space and there exists a unique multiplication so that G is an f-ring with identity u.*

Proof. Each $0 < h \in H$ has a unique representation

$$h = a_1\chi_{X_1} + \ldots + a_k\chi_{X_k}$$

where $0 < a_1 < a_2 < \ldots < a_k$ are real numbers and the X_i are disjoint subsets of I.

Pick positive integers m and n so that $na_{k-1} < m < na_k$. Then

$$(nh - mu) \vee 0 = t\chi_{X_k} \quad \text{where} \quad 0 < t = na_k - m .$$

Now pick a positive integer $q > 1/t$. Then

$$\chi_{X_k} = qt\chi_{X_k} \wedge u \in G .$$

Thus each of the χ_{X_i} belongs to G.

Let τ be an l-isomorphism of G into $\Pi_I R_i$ such that $u\tau = (1, 1, 1, \ldots)$. By Proposition 1.2 $G\tau$ consists of step functions and so by the above is a Specker space. Thus $G\tau$ is a subring of $\Pi_I R_i$ with identity $u\tau$. Finally it is shown in [12] that there exists at most one multiplication so that G is an f-ring with identity u.

Note that if G is an epi-archimedean vector lattice and $0 < g \in G$, then $G(g)$ satisfies 4.11. Thus "locally" G is an f-ring with no nilpotent elements.

4.12. (Bleier). *If G is an epi-archimedean vector lattice that is finitely generated as a vector lattice then* $G \simeq \sum_{i=1}^{n} R_i$.

Remark. Note that Example 7.1 shows that an epi-archimedean l-group generated by two elements need not belong to $\mathscr{S}$.

Proof. If $g_1, \ldots, g_n$ generate G then clearly $G = G(u)$ where $u = |g_1| + \ldots + |g_n|$. So by 4.11 we may assume that G is a Specker subspace of $\Pi_I R_i$. Then G is generated by a finite number of characteristic functions and in fact by a finite number of disjoint characteristic functions.

Corollary. *A finitely generated Specker space is l-isomorphic to* $\sum_{i=1}^{n} R_i$.

Corollary. *Each finitely generated epi-archimedean vector lattice is generated by two elements.*

Proof. $\sum_{i=1}^{n} R_i$ is generated by $(1, 1, 1, \ldots)$ and $(1, 2, 3, \ldots, n)$.

Corollary. *$R \boxplus R$ is the free epi-archimedean vector lattice on the one generator* $(1, -1)$. *There is no free epi-archimedean vector lattice on more than one generator.*

Proof. Suppose that F is a free epi-archimedean lattice on two generators, then we may assume that $F = \sum_{i=1}^{n} R_i$ for some $n > 0$. Then there must be a linear l-homomorphism of F onto $\sum_{i=1}^{n+1} R_i$ since $\sum_{i=1}^{n+1} R_i$ is generated by two elements, but this is impossible.

Let $G \subseteq \Pi_I R_i$ be a Specker space generated by a set S of characteristic functions. Then $[S]$ is a Specker group and so $[S]$ is an l-subgroup of $\Pi_I Z_i$ and G is the v-hull of $[S]$. Conversely let $[S] \subseteq \Pi_I Z_i$ be a Specker group and let G be the subspace of $\Pi_I R_i$ generated by $[S]$. Then G consists of all real linear combinations of the elements of S. Since S is closed with respect to multiplication it follows that G is a Specker space and the v-hull of $[S]$.

4.13. *For a vector lattice G the following are equivalent.*

a) *$G \in \mathscr{S}$ and G is an f-ring with no nilpotent elements.*

b) *G is the v-hull of an S-group.*

c) *G is l-isomorphic to a Specker space.*

Proof. We have shown that b) and c) are equivalent and clearly c) implies a). The fact that a) implies c) follows from the next proposition.

4.14. *If $G \in \mathscr{S}$ is an f-ring with no nilpotent elements then G can be embedded as a ring into a cardinal product of reals and each such representation consists of step functions. Thus if G is an f-algebra then it is l-isomorphic to a Specker space.*

Proof. By Lemma A we can embed G as an f-ring into $\Pi_I R_i$. Suppose (by way of contradiction) that $0 < g = (\ldots, g_i, \ldots) \in G$ has infinite range. By Corollary I of Proposition 2.1 there exists $w \in \Pi R_i$ such that $w_i > 0$ for all i and Gw consists of step functions. Now there is an infinite subset J of I for which the g_j are all distinct and each $g_j w_j = k$, a constant. Thus $g_j^2 w_j = k g_j$ and so $g^2 w$ is not a step function, a contradiction.

5. THIS SECTION CONSISTS OF THE FOLLOWING THEOREM

Theorem 5.1. *If $G \in \mathscr{S}$ then G^v is an a-closure of G. Moreover, G^v is the unique a-closure of G in $\mathscr{S}$. In particular $G \in \mathscr{S}$ is a closed if and only if G is a vector lattice.*

Proof. Case I. G has an order unit u. By Proposition 1.5 we may assume that G is an l-subgroup of $\Pi_I R_i$ consisting of step functions and containing $(1, 1, 1, \ldots)$. By 4.11 G satisfies condition a) of 4.9. Thus G^v is the Specker space generated by the set S of characteristic functions in G. If $0 < h \in G^v$ then $h = h_1 \chi_{X_1} + \ldots + h_k \chi_{X_k}$ where the h_i are non-zero reals and the X_i are disjoint subsets of I. Thus $g = \chi_{X_1} + \ldots \ldots + \chi_{X_k} \in G$ and clearly $G^v(h) = G^v(g)$ and so G^v is an a-extension of G.

If H is an a-extension of G^v then $H \in \mathscr{E}$ and so by Proposition 1. 4 we may assume that $G \subseteq G^v \subseteq H \subseteq \Pi_I R_i$, and by Proposition 1.2 H consists of step functions. Consider

$$0 < h = h_1 \chi_{X_1} + \ldots + h_k \chi_{X_k} \in H$$

where the X_i are disjoint subsets of I and $0 < h_1 < \ldots < h_k$. By 4.11 the $\chi_{X_i} \in H$. Now there exists $0 < g \in G^v$ such that $H(g) = H(\chi_{X_k})$. In particular, $\chi_{X_k} = \chi_{S(g)} \in G^v$ and since G^v is a vector space, $h_k \chi_{X_k} \in G^v$. Thus $h \in G^v$ and so G^v is a-closed.

Case II. G does not contain an order unit. $G^v \subseteq (G^d)^\wedge$ and so if $0 < h \in G^v$ then $h < g$ for some $g \in G$.

$$G = G(g) \boxplus g' \quad \text{and} \quad G^v = G^v(g) \boxplus g^* .$$

Now $h \in G^v(g) = G(g)^v$ and g is a unit for $G(g)$. Thus by Case I $G^v(g)$ is an a-extension of $G(g)$ and so h is a-equivalent to an element in G. Therefore G^v is an a-extension of G.

Now suppose that H is an a-extension of G^v and consider $0 < h \in H$. Then $H(h) = H(g)$ for some $0 < g \in G^v$ and $H \in \mathscr{E}$. Thus,

$$H = H(g) \boxplus g^\# \quad \text{and} \quad G^v = G^v(g) \boxplus g^*$$

where $\#, *$ are the polar operations in H and G^v respectively. Now $H(g)$ is an a-extension of $G^v(g)$ and $G^v(g)$ is a vector lattice in $\mathscr{S}$ with an order unit g. Thus by Case I $G^v(g) = H(g)$ and so $h \in G^v(g) \subseteq G^v$. Therefore G^v is a-closed.

Thus we have shown that if $G \in \mathscr{S}$ then G^v is an a-closed a-extension of G. Now let K be an a-closure of G in $\mathscr{S}$. Then K is a vector lattice and without loss of generality

$$G \subseteq G^v \subseteq \Pi_I R_i \quad \text{and} \quad G \subseteq K \subseteq \Pi_I R_i .$$

Thus $G^v \cap K$ is a vector lattice that contains G and so $G^v \cap K = G^v$. Therefore $G \subseteq G^v \subseteq K$ and so $G^v = K$.

6. EPI-ARCHIMEDEAN f-RINGS

Suppose that $G \in \mathscr{S}$ is an f-ring with no nilpotent elements and let X be the Stone space associated with the Boolean algebra of polars of G. By the embedding theorem of Bernau [4] G is (l-isomorphic to) a large subring of the ring $D(X)$ of continuous extended real valued functions on X. Here each $f \in D(X)$ is real on a dense open subset of X.

Lemma 6.1. $G \subseteq S(X)$.

Proof. $0 < g \in G$ is real on a dense open subset Y of X. Then $G(g)$ is a subring of $C(Y)$ and so by 4.14 $G(g) \subseteq S(Y)$. Therefore g is a real valued step function in $D(X)$ and hence $G \subseteq S(X)$.

Theorem 6.2. *If $G \in \mathscr{S}$ is an f-ring with no nilpotent element, then $S(X)$ is the essential ring closure of G in $\mathscr{S}$.*

Proof. Let H be an essential f-ring extension of G in $\mathscr{S}$. Then clearly H has no nilpotent elements, and the Stone space associated with H is X (see [13]). Thus by the Lemma we can embed H into $S(X)$ as an f-ring.

In order to get rid of the hypotheses that G contains no nilpotent elements we need the following concept. An l-group G is *strongly projectable* ("SP-group") if

$$G = C \boxplus C' \text{ for each polar } C \text{ of } G .$$

In [14] it is shown that each archimedean l-group G admits a unique SP-hull G^{SP}. Thus G^{SP} is the minimal essential extension of G that is an SP-group.

Proposition 6.3. a) *If* $G \in \mathscr{E}$ *then* $G^{SP} \in \mathscr{E}$.

b) *If* $G \in \mathscr{S}$ *then* $G^{SP} \in \mathscr{S}$.

Proof. a) For each polar C of G, $G/C' \in \mathscr{E}$ and so each G_C used in the construction of G^{SP} belongs to $\mathscr{E}$ (see [14] Theorem A). Now G^{SP} is a direct limit of these G_C and since we have a two element characterization of the groups in $\mathscr{E}$ (Theorem 1.1 (4) or (6)) the direct limit G^{SP} must also be epi-archimedean.

b) We show that the essential closure $S(X)$ of G in $\mathscr{S}$ is an SP-group. Then G^{SP} is the intersection of all l-subgroups of $S(X)$ that contain G and are SP-group and so $G^{SP} \in \mathscr{S}$.

Let T be a polar in $S(X)$ and let

$$X_T = \{x \in X \mid t(x) \neq 0 \text{ for some } t \in T\} .$$

Then the closure Y of X_T is clopen and

$$T = \{s \in S(X) \mid \text{the support of } s \text{ is contained in } Y\} .$$

Therefore

$$S(X) = S(Y) \boxplus S(X \setminus Y) = T \boxplus S(X \setminus Y) .$$

Corollary I. *If* $G \in \mathscr{E}$ *is an f-ring then* G^{SP} *is also an epi-archimedean f-ring. Thus the radical of* G^{SP} *is a cardinal summand*

$$G^{SP} = \operatorname{rad} G^{SP} \boxplus H$$

where H *is an f-ring with no nilpotent element and* rad G^{SP} *has the zero multiplication.*

Corollary II. *If* $G \in \mathscr{S}$ *is an f-ring then the f-ring essential closure of* G *in* $\mathscr{S}$ *is of the form*

$$S(Y) \boxplus S(W)$$

where Y and W are Stone spaces. $S(W)$ has the natural multiplication and $S(Y)$ has the zero multiplication.

The proofs of these corollaries follow from the Proposition and from the theory in Section 6 of [14].

7. EXAMPLES AND OPEN QUESTIONS

Example 7.1. A divisible epi-archimedean l-group G with an order unit such that $G \notin \mathcal{S}$. Let $H = \prod_{i=1}^{\infty} R_i$ and let

$$G = \{h \in H \mid \text{ there exist rationals } r, s \text{ such that } h_i = r(\pi + 1/i) + s \text{ for almost all } i\} .$$

Clearly G is a divisible subgroup of H and $G \supseteq \sum_{i=1}^{\infty} R_i$.

a) G is an l-subgroup of H. For consider $g \in G$ where $g_i = r(\pi + 1/i) + s$ for almost all i. It suffices to show that almost all the g_i are positive or almost all of them are negative. For then $g \vee 0 \in G$ or $g \vee 0 \in \Sigma R_i \subseteq G$ and so G is an l-subgroup of H. If $r = 0$ then $g_i = s$ for almost all i. If $r > 0$ then $r(\pi + 1/i) + s < r(\pi + 1/j) + s$ for all $i > j$. Thus if $r(\pi + 1/j) + s < 0$ for some j then almost all the g_i are negative and otherwise almost all g_i are positive. If $r < 0$ then $r(\pi + 1/i) + s < r(\pi + 1/j) + s$ for all $j > i$. Thus if $r(\pi + 1/i) + s > 0$ for some i then almost all g_i are positive and otherwise almost all g_i are negative.

b) $G \in \mathcal{E}$. If $0 < g \in G$ then clearly the g_i are bounded from above and since $\lim g_i = r\pi + s \geqq 0$ it follows that the $g_i \neq 0$ are bounded away from zero. Thus by Lemma A $G \in \mathcal{E}$.

c) It follows from Proposition 1.5 that the l-subgroup of G generated by $(1, 1, 1, \ldots)$ and $(\pi + 1, \pi + 1/2, \pi + 1/3, \ldots)$ does not belong to $\mathcal{S}$ and hence $G \notin \mathcal{S}$.

This is a limiting example in many ways.

1) *$G^v \notin \mathcal{E}$ and so G^v is not an a-extension of G.* For if $G^v \in \mathcal{E}$ then by Proposition 1.2 $G^v \in \mathcal{S}$ and hence $G \in \mathcal{S}$.

2) *$G_i = \{g \in G \mid g_i = 0\}$ for $i = 1, 2, \ldots$ and ΣR_i are the prime l-ideals of G.*

Proof. Clearly the G_i are maximal l-ideals and since they are polars they are also minimal primes. Next the map

$$\Sigma R_i + g \to r\pi + s$$

where $g_i = r(\pi + 1/i) + s$ for almost all i, is an o-isomorphism of $G/\Sigma R_i$ onto $Q\pi + Q$ and so ΣR_i is also a maximal l-ideal. Now if $M \neq G$ is a prime l-ideal of G

and if for each i there is a $0 < g \in G$ such that $g_i > 0$ then $M = \Sigma R_i$. Otherwise $M \subseteq G_i$ for some i and hence $M = G_i$.

Next let E be the l-group of all eventually constant sequences of rational numbers. Then $E \in \mathscr{S}$ and it follows from (2) that G is an a-extension of E. Therefore

3) *There is an a-closure (essential closure) of E that belongs to $\mathscr{E}$ but not $\mathscr{S}$ and also an a-closure (essential closure) of E in $\mathscr{S}$. E^v is the a-closure of E in $\mathscr{S}$.*

4) *Let K be an a-closure of E that is not in $\mathscr{S}$. Then $K \in \mathscr{E}$ and K is not a vector lattice. Also K^v is not an a-extension of K.*

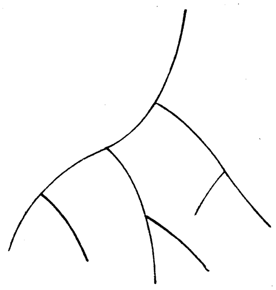
Proof. If K is a vector lattice then by Proposition 1.2 $K \in \mathscr{S}$.

Example 7.2. An f-ring G that belongs to $\mathscr{E}$ but not $\mathscr{S}$.

Let $l_1, l_2, \ldots$ be a sequence of positive rationals that converge to π and let G be the l-subring of $\prod_{i=1}^{\infty} R_i$ that is generated by $l = (l_1, l_2, \ldots)$, $(1, 1, 1, \ldots)$ and $\sum_{i=1}^{\infty} R_i$. Then G consists of ΣR_i + polynomial in l with integral coefficients. For if $0 \neq f(x) \in Z[x]$ then $f(\pi) \neq 0$ and so $f(\pi)$ and $f(l_i)$ agree in sign for almost all i. It follows from this that G is an l-subring of ΠR_i and that it satisfies Lemma A and hence belongs to $\mathscr{E}$.

If $G \in \mathscr{S}$ then so does G^v but then it follows from Proposition 1.5 that *this* representation of G consists of functions with finite range.

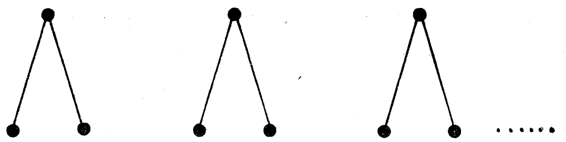
Theorem 1.1 assents that an l-group is epi-archimedean if and only if the set P of proper prime subgroups is trivially ordered with respect to inclusion. In general, P is a root system (that is, a po set such that the elements above any fixed element form a chain).



A maximal chain in P will be called a *root*.

If G is an l-group and E is the epi-archimedean kernel and if G/E is also epi-archimedean then [20] each root in P has length at most 2. The next example shows that the converse is false.

Example 7.3. For the free vector lattice G on two generators P looks like



one for each point on the unit circle (see [2]). But Bleier (Tulane Dissertation 1971) showed that G contains no l-ideals that are invariant under all l-automorphisms and hence $E = 0$.

Example 7.4 (Chambless) The group $C(X, Z)$ of all continuous integral valued functions on a compact Hausdorff space belongs to $\mathscr{S}$. This is because the range of such a function is a compact subset of Z and hence finite.

Example 7.5. The group G of eventually constant sequences of reals belongs to $\mathscr{S}$ but is not an SP-group. Note that G is a Specker space.

Example 7.6. $G = \Pi Z_i$ for all $i \in [0, 1]$ has the property that for each maximal l-ideal C, G/C is cyclic (see [10]).

The following conjecture is due to Jorge Martinez. If G is an epi-archimedean l-group and a subdirect sum of integers then is G/C cyclic for each prime subgroup C of G? We show that the answer is no.

1) *If the conjecture holds for a particular l-group G then it holds for each l-subgroup H of G.*

Proof. Let P be a prime subgroup of H. Then there exists a prime C in G such that $C \cap H = P$. Thus

$$H/P = H/(C \cap H) \simeq (C + H)/C \subseteq G/C \quad \text{cyclic}.$$

2) *If $0 < s$ is a singular element in the l-group H and (H^α, H_α) is a value of s, then $H_\alpha \lhd H^\alpha$ and H^α/H_α is cyclic.*

Proof. $H(s)$ is abelian and so $H_\alpha \cap H(s) \lhd H(s)$. Thus $H_\alpha \lhd H^\alpha$ (see [11]). Now $H_\alpha + s$ is singular in the archimedean o-group H^α/H_α and so H^α/H_α is cyclic.

3) *If C is a prime subgroup in the epi-archimedean l-group G, s is singular and $s \notin C$, then G/C is cyclic.*

Proof. (G, C) is a value of s.

4) *The conjecture is true for the group G of all bounded functions in $\Pi_I Z_i$.*

Proof. If C is a proper prime subgroup of G then $(1, 1, 1, \ldots) \in G \setminus C$.

Example 7.7. Let H be the l-group of all step functions in $\prod_{i=1}^{\infty} R_i$ and let α be the l-automorphism of ΠR_i obtained by multiplication by the element $(1, 2, 3, \ldots)$. Let G be the subgroup of $H\alpha$ consisting of all the integral valued functions. Clearly G is an l-subgroup of $H\alpha$ and hence of $\prod_{i=1}^{\infty} Z_i$ and $G \in \mathscr{S}$. Now we construct a prime subgroup C of G such that G/C is not cyclic.

$$(0, 1/2, 0, 1/2, 0, \ldots) \in H \quad \text{maps onto} \quad x = (0, 1, 0, 2, 0, 3, 0, 4, \ldots)$$

and

$$(0, 0, 0, 1/4, 0, 0, 0, 1/4, 0, \ldots) \in H \quad \text{maps onto}$$

$$y = (0, 0, 0, 1, 0, 0, 0, 2, 0, 0, 0, 3, 0, \ldots) \quad \text{etc}.$$

Next choose the following subsets of $N = 1, 2, 3, \ldots$

$$\{1, 3, 5, 7, 9, 11, \ldots\}$$
$$\{1, 2, 3;\ 5, 6, 7;\ 9, 10, 11; \ldots\}$$
$$\{1, 2, 3, 4, 5, 6, 7;\ 9, 10, 11, 12, 13, 14, 15;\ \ldots\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

These are contained in a dual ultra filter $\mathscr{F}$ of the set of all proper subsets of N. Let C be the set of all functions in G whose support belongs to this ultrafilter. Then (see [10]) C is prime, but $C + x > C + y > \ldots$ and so G/C is not cyclic. For suppose that $x = y \bmod C$, then

$$x - y = (0, 1, 0, 1, 0, 3, 0, 2, 0, 5, 0, 3, 0, 7, 0, 4, \ldots) \in C$$

but this is impossible since

$$(1, 0, 3, 0, 5, 0, 7, \ldots) \in C$$

and this means that C contains a strong order unit of G.

Open questions. 1) If $G \in \mathscr{E}$ and G is a subdirect sum of integers then does $G \in \mathscr{S}$?

2) Does each $G \in \mathscr{E}$ have a representation that satisfies (a) of Lemma A?

3) Find an example of $G \in \mathscr{E}$ that is not contained in an epiarchimedean f-ring with no nilpotent elements.

4) Suppose that G is an l-subgroup of $\Pi_I R_i$ that satisfies (a) of Lemma A. Does the l-subring of ΠR_i generated by G belong to $\mathscr{E}$?

5) Is a vector lattice in $\mathscr{E}$ a-closed?

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