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A REMARK ON PROJECTIVELY CLOSED PURITIES

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The purpose of this remark is to show that the class of projectively closed purities coincides with the class of Γ-purities. As a consequence we obtain the solution of one part of problem 7 from [1], namely that any purity ω defined by a class of „ω-projective" modules is bi-triangular (for the precise formulation see below).

For the convenience of the reader we shall give basic definitions and notations. We shall say that in the category of Λ-modules (here Λ stands for an associative ring with unity) a purity ω is given if there is given a class $\mathfrak{H}_\omega$ of monomorphisms satisfying the following axioms:

P0: Any homomorphism $\varphi : A \to B$ having left inverse $\psi : B \to A$ belongs to $\mathfrak{H}_\omega$,

P1: $\varphi\psi \in \mathfrak{H}_\omega$ whenever $\varphi, \psi \in \mathfrak{H}_\omega$,

P2: if $\psi\varphi \in \mathfrak{H}_\omega$ and ψ is a monomorphism, then $\varphi \in \mathfrak{H}_\omega$,

P3: if in the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \xrightarrow{j} & A & \xrightarrow{\sigma} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \\
0 & \longrightarrow & K & \xrightarrow{i} & B & \xrightarrow{\tau} & D & \longrightarrow & 0
\end{array} \tag{1}
$$

with exact rows and columns φ belongs to $\mathfrak{H}_\omega$, then $\psi \in \mathfrak{H}_\omega$ as well,

P4: if in (1) $i, \psi \in \mathfrak{H}_\omega$, then $\varphi \in \mathfrak{H}_\omega$.

If we introduce the class $\mathfrak{H}_\omega^*$ of all epimorphisms σ for which the canonical embedding of Ker σ belongs to $\mathfrak{H}_\omega$, then P3 and P4 are respectively equivalent to:

P3*: if $\tau\sigma \in \mathfrak{H}_\omega^*$ and σ is an epimorphism, then $\tau \in \mathfrak{H}_\omega$,

P4*: for $\sigma, \tau \in \mathfrak{H}_\omega^*$ there is $\tau\sigma \in \mathfrak{H}_\omega^*$.

The axioms P2 and P3* can be strengthened as follows:

$\mathrm{P}\bar{2}$: for $\psi\varphi \in \mathfrak{H}_\omega$ it is $\varphi \in \mathfrak{H}_\omega$,

$\mathrm{P}\bar{3}^*$: for $\tau\sigma \in \mathfrak{H}_\omega^*$ it is $\tau \in \mathfrak{H}_\omega^*$.

A purity ω is said to be triangular if it satisfies $\mathrm{P}\bar{2}$ instead of P2. Similarly, a purity ω is said to be co-triangular if it satisfies $\mathrm{P}\bar{3}^*$ instead of P3* (and hence P3). A purity ω is bi-triangular if it is both triangular and co-triangular.

Let Γ be an arbitrary class of couples (F, U), where U is a submodule of a free module F. Let us form the class $\mathfrak{H}_\Gamma$ of all monomorphisms $A \xrightarrow{i} B$ such that for any commutative diagram

$$(2) \qquad \begin{array}{ccc} U & \xrightarrow{\chi} & F \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{i} & B \end{array}$$

where $(F, U) \in \Gamma$ and χ is the canonical embedding, there exists a homomorphism $\psi : F \to A$ making the diagram

$$(3) \qquad \begin{array}{ccc} U & \xrightarrow{\chi} & F \\ {\scriptstyle\varphi}\downarrow & \swarrow{\scriptstyle\psi} & \\ A & & \end{array}$$

commutative. It can be shown that the class $\mathfrak{H}_\Gamma$ defines a bi-triangular purity (see (1,23) in [1]), the so called Γ-purity.

A module P is called co-projective with respect to a monomorphism $i : A \to B$ if for any diagram

$$(4) \qquad \begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & & \downarrow{\scriptstyle\vartheta} & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \end{array}$$

with exact row there exists a homomorphism $\psi' : P \to B$ making the diagram

$$(5) \qquad \begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & {\scriptstyle\psi'}\swarrow & \downarrow{\scriptstyle\vartheta} & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \end{array}$$

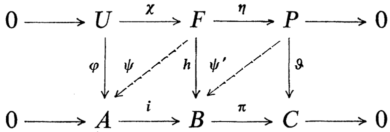
commutative. For a purity ω let us call a module P ω-projective if it is co-projective with respect to any $i \in \mathfrak{H}_\omega$. The class of all ω-projective modules is denoted by $\mathfrak{P}_\omega$. If $\mathfrak{M}$ is an arbitrary class of modules then the class $\mathfrak{H}^{\mathfrak{M}}$ of all monomorphisms i such that any $M \in \mathfrak{M}$ is co-projective with respect to i defines a purity (see (1,20) in [1]), which we denote by $\omega^{\mathfrak{M}}$. The purity $\bar{\omega} = \omega^{\mathfrak{P}_\omega}$ is called the projective closure of ω. Finally, a purity ω is called projectively closed, if $\omega = \bar{\omega}$ (i.e. $\mathfrak{H}_\omega = \mathfrak{H}^{\mathfrak{P}_\omega}$).

Now we can start our investigation. For an arbitrary purity ω let us denote by Γ_ω the class of all couples (F, U) where U is a submodule of a free module F such that for any commutative diagram (2) where $i \in \mathfrak{H}_\omega$ and χ is the canonical embedding there exists a homomorphism $\psi : F \to A$ making diagram (3) commutative. The class Γ_ω is non-empty because $(F, F) \in \Gamma_\omega$ for any free module F.

Lemma 1. *Let* $0 \to U \xrightarrow{\chi} F \xrightarrow{\eta} P \to 0$ *be an exact sequence where* U *is a submodule of a free module* F *and* χ *is the canonical embedding. Then* $P \in \mathfrak{P}_\omega$ *if and only if* $(F, U) \in \Gamma_\omega$.

Proof. Let us consider the following diagram

(6)



with exact rows and $i \in \mathfrak{H}_\omega$. At first, let us suppose $P \in \mathfrak{H}_\omega$ and let the diagram (2) be commutative, $i \in \mathfrak{H}_\omega$. In view of $P \cong \operatorname{Coker} \chi$ and $\pi h \chi = \pi i \varphi = 0$ there exists $\vartheta : P \to C$ making the rigt square of (6) (without ψ') commutative. By hypothesis there exists $\psi' : P \to B$ with $\pi \psi' = \vartheta$. Further, $A \cong \operatorname{Ker} \pi$ and $\pi(h - \psi'\eta) = \pi h - \vartheta\eta = 0$ imply the existence of $\psi : F \to A$ with $i\psi = h - \psi'\eta$. Finally, $i\psi\chi = h\chi - \psi'\eta\chi = i\varphi$, hence $\psi\chi = \varphi$, i being a monomorphism. Therefore $(F, U) \in \Gamma_\omega$.

Conversely, let $(F, U) \in \Gamma_\omega$ and let us consider the diagram (4) with $i \in \mathfrak{H}_\omega$. The freeness of F (π epimorphism) implies the existence of h making the right square of (6) (without ψ') commutative. In view of $A \cong \operatorname{Ker} \pi$ and $\pi h \chi = \vartheta\eta\chi = 0$ there exists $\varphi : U \to A$ making the left square of (6) (without ψ) commutative. By hypothesis there exists $\psi : F \to A$ with $\psi\chi = \varphi$. Further, $P \cong \operatorname{Coker} \chi$ and $(h - i\psi)\chi = h\chi - i\varphi = 0$ implies the existence of $\psi' : P \to B$ with $\psi'\eta = h - i\psi$. Finally, $\pi\psi'\eta = \pi h - \pi i \psi = \vartheta\eta$, therefore $\pi\psi' = \vartheta$, η being an epimorphism and hence $P \in \mathfrak{P}_\omega$.

Lemma 2. *If* ω *and* Γ_ω *have the same meaning as above, then* $\mathfrak{H}_\omega \subseteq \mathfrak{H}_{\Gamma_\omega} = \mathfrak{H}_{\bar{\omega}}$.

Proof. The inclusion $\mathfrak{H}_\omega \subseteq \mathfrak{H}_{\Gamma_\omega}$ follows immediately from the definition of Γ_ω, while the equality $\mathfrak{H}_{\Gamma_\omega} = \mathfrak{H}_{\bar{\omega}}$ follows easily from Lemma 1 and its proof.

Theorem 1. *A purity ω is projectively closed if and only if it is a Γ-purity for some class Γ.*

Proof. Any Γ-purity is projectively closed by (1,29) from [1] (see also [3]). Conversely, Lemma 2 gives $\mathfrak{H}_{\bar\omega} = \mathfrak{H}_\omega \subseteq \mathfrak{H}_{\Gamma_\omega} = \mathfrak{H}_{\bar\omega}$, hence $\mathfrak{H}_\omega = \mathfrak{H}_{\Gamma_\omega}$.

Corollary. *Any projectively closed purity is bi-triangular.*

Proof. Follows immediately from Theorem 1 and (1,23) from [1].

Lemma 3. *For any class $\mathfrak{M}$ of modules, the purity $\omega^{\mathfrak{M}}$ is projectively closed.*

Proof. Clearly, $\mathfrak{H}^{\mathfrak{M}} \subseteq \mathfrak{H}^{\mathfrak{P}_\omega \mathfrak{M}}$ while the obvious inclusion $\mathfrak{M} \subseteq \mathfrak{P}_\omega \mathfrak{M}$ implies $\mathfrak{H}^{\mathfrak{P}_\omega \mathfrak{M}} \subseteq \mathfrak{H}^{\mathfrak{M}}$.

Theorem 2. *For an arbitrary class $\mathfrak{M}$ of modules, the purity $\omega^{\mathfrak{M}}$ is bi-triangular.*

Proof. It suffices to use Lemma 3 and Corollary.

References

[1] *А. П. Мишина, Л. А. Скорняков:* Абелевы группы и модули, Москва, 1969.

[2] *S. Mac Lane:* Homology, Berlin—Göttingen—Heidelberg, 1963.

[3] *M. C. R. Butler, G. Horrocks:* Classes of extensions and resolutions, Philos. Trans. Roy. Soc. London A *254* (1961), 155—222.

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