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THE DECOMPOSITION SERIES OF A CONVERGENCE SPACE

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INTRODUCTION

Associated with each convergence space is a well-ordered, descending chain of pretopologies defined on the same base set, beginning with the pretopological modification and terminating with the topological modification: this chain is called the *decomposition series* of the space, and the ordinal number of terms in the series is called its *length*. Roughly speaking, the length of this series describes how "non-topological" a convergence space is; for a sequential topological space, the series length of its first countable modification describes how far removed the original space is from being a Frechet space. The decomposition series is considered in the following papers: [1], [4], [7], [8], and [11]. This paper attempts to bring together some results from these and other papers and to initiate the study of this series as a mathematical concept of interest in its own right.

The paper is divided into two sections; the first treats the behavior of the decomposition series relative to such basic constructions as products, quotients, disjoint sums, and subspaces, while the second is concerned with the length of the series. In the second section, we first obtain certain conditions under which the series length will be bounded, and then proceed to show that some important classes of convergence spaces, such as locally compact regular and minimal regular spaces, can have arbitrarily long decomposition series.

1. BASIC CONSTRUCTIONS

The reader is asked to refer to [4] and [6] for the notation, terminology, and other background material on convergence spaces. We will use the abbreviations "u.f." for ultrafilter and "c.s." for convergence structure; "space" will always mean convergence space.

In the discussion of the decomposition series in [4] the definition was given by considering the various iterations of the interior operator. For present purposes, it is more convenient to work with the closure operator, so we shall rephrase our definition accordingly.

Given an ordinal number $\alpha \geqq 1$, let Γ^α be the α-th iteration of the closure operator. Using the natural one-to-one correspondence between closure operators and pretopologies, let $\pi^\alpha q$ be the pretopology on S associated with Γ^α. The neighborhood filter $\mathscr{V}_q^\alpha(x)$ for $\pi^\alpha q$ at x is the set of all subsets A of S such that $x \in \mathrm{Co}\, \Gamma^\alpha \, \mathrm{Co}\, A$, where Co denotes "set complement". In the natural order of the lattice $C(S)$ of all convergence structures on S, $\alpha \leqq \beta$ implies $\pi^\alpha q \geqq \pi^\beta q$; the chain $\{\pi^\alpha q : \alpha \geqq 1\}$ is called the decomposition series of (S, q). The series terminates with the least ordinal α such that Γ^α is idempotent, and this ordinal, call it γ_q, will be called the *length* of the series. The initial term $\pi^1 q$, also written πq, is called the *pretopological modification* of q. The last term $\pi^{\gamma_q} q$ is a topology, the finest topology coarser than q; this topology, which we denote by λq, is called the topological modification of q. Note that $\gamma_q \leqq 1$ iff $\pi q = \lambda q$.

There are two reasonable ways to define the α-th iteration of the closure of a filter $\mathscr{F}$. In addition to $\Gamma_q^\alpha \mathscr{F}$ defined in [6] and [7] (which, in case α is a limit ordinal, is given recursively by $\bigcap\{\Gamma_q^\beta \mathscr{F} : \beta < \alpha\}$), we shall make use in this paper of $\Delta_q^\alpha \mathscr{F}$, defined to be the filter generated by the set $\{\Gamma_q^\alpha F : F \in \mathscr{F}\}$. Note that $\Delta_q^\alpha \mathscr{F} = \Gamma_{\pi^\alpha q} \mathscr{F}$, $\Delta_q^n \mathscr{F} = \Gamma_q^n \mathscr{F}$ for all finite ordinals n, and $\Delta_q^\alpha \mathscr{F} \leqq \Gamma_q^\alpha \mathscr{F}$ for all ordinals α.

A continuous onto function is called a *map*. A map $f : (S, q) \to (T, p)$ is said to be *open* if, whenever $\mathscr{I}$ is an u.f. which converges to y in T and $x \in f^{-1}(y)$, then there is an u.f. $\mathscr{F}$ which maps on $\mathscr{I}$ and converges to x in S. A map $f : (S, q) \to (T, p)$ is said to be *proper* if, whenever $\mathscr{I}$ is an u.f. which converges to y in T and $\mathscr{F}$ is an u.f. on S which maps on $\mathscr{I}$, then there is a point x in $f^{-1}(y)$ such that $\mathscr{F}$ converges to x in S.

The first theorem summarizes some known results concerning the behavior of the decomposition series relative to the maps described above; the first assertion is proved in [4], the second and third in [7].

Theorem 1.1. (a) *If $f : (S, q) \to (T, p)$ is a map, then $f : (S, \pi^\alpha q) \to (T, \pi^\alpha p)$ is a map for all α.*

(b) *If $f : (S, q) \to (T, p)$ is an open map, then $f : (S, \pi^\alpha q) \to (T, \pi^\alpha p)$ is an open map for all α.*

(c) *If $f : (S : q) \to (T, p)$ is an open map, or a proper map, then $\gamma_p \leqq \gamma_q$.*

Corollary 1.2. *If q and p are c.s. on S such that $\lambda q = \lambda p$ and $q \leqq p$, then $\gamma_q \leqq \gamma_p$.*

Proof. If $\gamma_q > \gamma_p$, then there is ordinal number α such that $\pi^\alpha p = \lambda p$, and $\pi^\alpha q \neq \lambda q$. But Theorem 1.1 (a) implies that the identity map from $(S, \lambda p)$ to $(S, \pi^\alpha q)$ is continuous, and thus $\lambda p \geqq \pi^\alpha q$, a contradiction.

If (S, q) is a space and T a subset of S, then let q_T be the c.s. on T defined as follows: A filter $\mathscr{F}$ q_T-converges to x in T iff the filter on S generated by $\mathscr{F}$ q-converges to x. Then (T, q_r) is called a *subspace* of (S, q).

Proposition 1.3. *Let (T, p) be a subspace of (S, q), α an ordinal number $\geqq 1$, A a subset of T. Then:*

(a) *$\Gamma_p A = T \cap \Gamma_q A$, and $\Gamma_p^\alpha A \subseteq T \cap \Gamma_q^\alpha A$;*

(b) *If (T, p) is a closed (open) subspace of (S, q), then $\Gamma_p^\alpha(A) = (\Gamma_q^\alpha A) \cap T$.*

Proof. (a) The assertion $\Gamma_p^\alpha A \subseteq (\Gamma_q^\alpha A) \cap T$ is clear for all $\alpha \geqq 1$. Let $x \in (\Gamma_q A) \cap T$; then there is an u.f. $\mathscr{F}$ which contains A and q-converges to $x \in T$. The restriction of $\mathscr{F}$ to T p-converges to x, and $x \in \Gamma_p A$.

(b) Induction on α. Assume that the equality holds for $\sigma < \alpha$. If $\alpha - 1$ exists, and $x \in \Gamma_q^\alpha A$, then there is an u.f. $\mathscr{F}$ on S such that $\Gamma_q^{\alpha-1} A \in \mathscr{F}$, and $\mathscr{F}$ q-converges to $x \in T$. Because T is closed, $\Gamma_q^{\alpha-1} A \subseteq T$; thus $\mathscr{F}$ has a restriction $\mathscr{F}_T$ to T, and $\mathscr{F}_T$ p-converges to x, establishing $(\Gamma_q^\alpha A) \cap T \subseteq \Gamma_p^\alpha A$. If α is a limit ordinal and $x \in \Gamma_q^\alpha A$, then, by the induction hypothesis,

$$x \in T \cap (U\{\Gamma_q^\sigma A : \sigma < \alpha\}) = U\{T \cap \Gamma_q^\sigma A : \sigma < \alpha\} = U\{\Gamma_p^\sigma A : \sigma < \alpha\} = \Gamma_p^\alpha A .$$

In case T is open, a similar induction proof can be employed.

Corollary 1.4. *Let (T, p) be a subspace of (S, q). Then:*

(a) *$(T, \pi(p))$ is a subspace of $(S, \pi(q))$.*

(b) *In general, $(\lambda q)_T \leqq \lambda p$; if T is open or closed, then $(\lambda q)_T = \lambda p$.*

(c) *If T is either open or closed, then $\gamma_p \leqq \gamma_q$.*

In the example that follows, we show that a space with a short decomposition series can have a subspace with an arbitrarily long decomposition series. Examples of T_1 spaces with arbitrarily long series are given in the next section.

Example 1.5. Given an ordinal α, let (T, p) be any T_1 space such that $\gamma p \geqq \alpha$. Let A be an infinite set disjoint from T, and let S be the union of A and T. Define q to be the finest c.s. on S such that:

(a) If $\mathscr{F}$ is a free u.f. on A, then $\mathscr{F}$ q-converges to each point in S.

(b) If $\mathscr{F}$ is any free u.f. which contains T, then $\mathscr{F}$ q-converges to y iff $y \in T$ and $\mathscr{F}$ p-converges to y, or else $y \in A$. It is easy to verify that $\gamma_q = 2$, and that (T, p) is a subspace of (S, q).

A map $f : (S, q) \to (T, p)$ is called a *retraction* if (T, p) is a subspace of (S, q), and f reduces to the identity map when restricted to T.

Theorem 1.6. *If $f:(S,q)\to(T,p)$ is a retraction, then $f:(S,\pi^\alpha q)\to(T,\pi^\alpha p)$ is a retraction for all α, and $\gamma_p \leqq \gamma_q$.*

Proof. $f:(S,\pi^\alpha q)\to(T,\pi^\alpha p)$ is continuous by Theorem 1.1; in particular, if (T,p_α) is a subspace of $(S,\pi^\alpha q)$, then $f_T:(T,p_\alpha)\to(T,\pi^\alpha p)$ is continuous, and f_T is the identity on T. Thus $p_\alpha \geqq \pi^\alpha p$. If $A \subseteq S$, $\Gamma_{p_\alpha}A = T\cap\Gamma_q^\alpha A \subseteq \Gamma_p^\alpha A$, since finer spaces yield smaller closures. This result, along with Proposition 1.3, implies $(\Gamma_q^\alpha A)\cap$ $\cap\, T = \Gamma_p^\alpha A$, and so $(T,\pi^\alpha p)$ is a subspace of $(S,\pi^\alpha q)$. From this fact, the assertions of the theorem readily follow.

Let $\{(S_i,q_i): i\in I\}$ be a family of pairwise disjoint spaces. The disjoint sum (S,q) of this family is obtained by taking S to be the union of the S_i's with q defined as follows: A filter $\mathscr{F}$ q-converges to x in S iff $\mathscr{F}$ contains the set S_i to which x belongs, and the restriction of $\mathscr{F}$ to S_i q_i-converges to x.

Theorem 1.7. *If (S,q) is the disjoint sum of the family*

$$\{(S_i,q_i): i\in I\}\,,\quad \textit{then}\quad \gamma_q = \sup\{\gamma_{q_i}: i\in I\}.$$

Proof. Since (S_i,q_i) is a closed subspace of (S,q), $\gamma_q \geqq \{\gamma_{q_i}: i\in I\}$. To eastablish the desired equality, it would suffice to show that, for each $B\subseteq S$, $\gamma_q B = \sup\{\gamma_{q_i}(B\cap$ $\cap\, S_i): i\in I\}$, where $\gamma_q B$ is the least ordinal number β such that $\Gamma_q^\beta B = \Gamma_q^{\beta+1}B$. Let $B_i = B\cap S_i$, and let $\alpha = \sup\{\gamma_{q_i}B_i: i\in I\}$, and let $\beta = \gamma_q B$. It is clear that $\alpha \leqq \beta$; we will assume that $\alpha<\beta$ and obtain a contradiction. If $\alpha>\beta$, then there is x in $\Gamma_q^{\alpha+1}B$ such that $x\notin\Gamma_q^\alpha B$, and hence there is an u.f. $\mathscr{F}$ q-converging to x which contains $\Gamma_q^\alpha B$. Let $x\in S_i$; then the restriction $\mathscr{F}_i$ of $\mathscr{F}$ of S_i q_i-converges to x. By Proposition 1.3(b), $\Gamma_q^\alpha B_i = \Gamma_{q_i}^{\alpha+1}B_i$. Since $(\Gamma_q^\alpha B)\cap S_i = \Gamma_q^\alpha B_i$, $\Gamma_{q_i}^\alpha B_i\in\mathscr{F}_i$, and so $x\in\Gamma_{q_i}^{\alpha+1}B_i$. But $x\notin\Gamma_{q_i}^\alpha B_i$, which implies $\gamma_{q_i}B_i>\alpha$, a contradiction.

Starting with spaces (S,q) and (T,p) let $(R,r)=(S,q)\times(T,p)$ be the product space. Consider the following diagram:

$$\begin{array}{ccccccccc}
(S,q) & \to & (S,\pi q) & \to \dots \to & (S,\pi^\alpha q) & \to \dots \to & (S,\lambda q)\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
(R,r) & \to & (R,\pi r) & \to \dots \to & (R,\pi^\alpha r) & \to \dots \to & (R,\lambda r)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
(T,p) & \to & (T,\pi p) & \to \dots \to & (T,\pi^\alpha p) & \to \dots \to & (T,\lambda p)
\end{array}$$

where the vertical arrows are the respective projection maps, P_1 and P_2, and each horizontal arrow is the identity map. Since the projection maps are open, it follows from Theorem 1.1 that $\gamma_r \geqq \sup\{\gamma_q,\gamma_p\}$.

The pair $(S,q),(T,p)$ is said to be *pretopologically coherent* if $(R,\pi r) = (S,\pi q)\times$ $\times\,(T,\pi p)$. Sufficient conditions for pretopological coherence are given in [2] along with examples of pairs of spaces which are not pretopologically coherent. Let us say that a pair $(S,q),(T,p)$ is *totally coherent* if the entire decomposition series of the two spaces is productive in the sense that $(R, \pi^\alpha r) = (S, \pi^\alpha q) \times (T, \pi^\alpha p)$, for all ordinal numbers α.

Let P be some convergence space property; a space (S, q) is said to be *almost P* if there is a c.s. p on S with property P such that p-convergence and q-convergence coincide on ultrafilters. It was shown in [2] that, given a space (S, q) which is not almost pretopological, there is a topology p on S such that the pair (S, q), (S, p) is not pretopologically coherent. Our next theorem establishes a similar result for total coherence.

Theorem 1.8. (a) *If (S, q) and (T, p) are almost topological spaces, then the pair (S, q), (T, p) is totally coherent.*

(b) *If (S, q) is a space which is not almost topological, then there is a topology p on S such that the pair (S, q), (S, p) is not totally coherent, provided $(S, \lambda q)$ is Hausdorff.*

Proof. (a) Under the given assumptions, total coherence is equivalent to pretopological coherence, and the result follows from Theorem 2, [2].

(b) If (S, q) is not almost pretopological, then by Theorem 2, [2] there is a topology p on S such that the pair (S, q), (S, p) is not pretopologically coherent, and hence not totally coherent. We shall therefore assume that (S, q) is almost pretopological but not almost topological. Thus there is an u.f. $\mathscr{F}$ such that $\mathscr{F} \geqq \mathscr{V}_{\lambda q}(x)$ but $\mathscr{F}$ does not q-converge to x, for some x in S. Since (S, q) is almost pretopological, $\mathscr{F} \ngeqq \mathscr{V}_q(x)$. Let p be the topology on S with neighborhood filters specified as follows: $\mathscr{V}_p(x) = \mathscr{F} \cap \dot{x}$; $\mathscr{V}_p(y) = \dot{y}$ for $y \neq x$. Let $(R, r) = (S, q) \times (S, p)$, and let $A = \{(y, y) \in R : y \neq x\}$. Using the fact that (S, q) is Hausdorff, we see that A is r-closed, and hence λr-closed. But (x, x) is in the closure of A relative to $(S, \lambda q) \times (S, \lambda p)$, and so the pair (S, q), (S, p) is not totally coherent.

2. THE LENGTH OF THE SERIES

We begin this section by estalishing necessary and sufficient conditions for the decomposition series of a convergence space to have length $\leqq \alpha$, where α is an arbitrary ordinal number. First, we need two lemmas.

Lemma 2.1. *Let (S, q) be a convergence space, $A \subseteq S$, $\mathscr{F}$ an u.f. on S. If $\Gamma_q^\alpha A \in \mathscr{F}$, then there is an u.f. $\mathscr{I}$ containing A such that $\mathscr{F} \geqq \Delta_q^\alpha \mathscr{I}$.*

Proof. Induction on α. Let $\alpha = 1$. For each $y \in \Gamma_q A - A$, let $\mathscr{H}_y$ be an u.f. containing A which converges to y. For each subset B of A, let $B^* = \{y \in \Gamma_q A - A : B \in \mathscr{H}_y\}$, and let $B^\wedge = B \cup B^*$. Since $(B_1 \cup B_2)^\wedge = B_1^\wedge \cup B_2^\wedge$ and $(B_1 \cap B_2)^\wedge = B_1^\wedge \cap B_2^\wedge$, the set $\{B \subseteq A : B^\wedge \in \mathscr{F}\}$ is a filter base which generates an u.f. $\mathscr{I}$, and it is easy to see that $\mathscr{I}$ satisfies the condition $\mathscr{F} \geqq \Gamma_q \mathscr{I}$.

Next, assume that the result is valid for $\beta < \alpha$. First consider the case where $\alpha - 1$ exists. If $\Gamma_q^{\alpha-1} A \in \mathscr{F}$, then the result follows from the induction hypothesis. Otherwise, $\Gamma_q^\alpha A - \Gamma_q^{\alpha-1} A \in \mathscr{F}$. In this case, let $B = \Gamma_q^{\alpha-1} A$, and repeat the argument just given for $\alpha = 1$ to obtain an u.f. $\mathscr{H}$ such that $B \in \mathscr{H}$ and $\mathscr{F} \geqq \Gamma_q \mathscr{H}$. Then apply the induction hypothesis to obtain an u.f. $\mathscr{I}$ such that $A \in \mathscr{I}$ and $\mathscr{H} \geqq \Delta_q^{\alpha-1} \mathscr{I}$. Then $\mathscr{F} \geqq \Delta_q^\alpha \mathscr{I}$, as desired.

Finally, assume that α is a limit ordinal. Each y in $\Gamma_q^\alpha A$ is in $\Gamma_q^\beta A$ for some $\beta > \alpha$, and so there is an u.f. $\mathscr{K}_y$ on $\Gamma_q^\beta A$ which converges to y, and by the induction hypothesis, there is an u.f. $\mathscr{H}_y$ on A such that $\mathscr{K}_y \geqq \Delta_q^\beta \mathscr{H}_y$. For each subset B of A, let $B^* = \{y \in \Gamma_q^\alpha A - A : B \in \mathscr{H}_y\}$, and let $B^\wedge = B \cup B^*$. As in the argument for $\alpha = 1$, one can show that the set $\{B \subseteq A : B^\wedge \in \mathscr{F}\}$ generates an u.f. $\mathscr{I}$ with the desired properties.

Lemma 2.2. *If $\mathscr{I}$ is an u.f. on a space (S, q), then $\dot{x} \geqq \Delta_q^\alpha \mathscr{I}$ iff $\mathscr{I} \geqq \mathscr{V}_q^\alpha(x)$.*

Proof. Let $\dot{x} \geqq \Gamma_q \mathscr{I}$; then each G in $\mathscr{I}$ belongs to an u.f. $\mathscr{F}_G$ which contains G and q-converges to x. Thus $\mathscr{I} \geqq \bigcap\{\mathscr{F}_G : G \in \mathscr{I}\} \geqq \mathscr{V}_q(x)$. If $\mathscr{I} \geqq \mathscr{V}_q(x)$, then each set in $\mathscr{I}$ belongs to an u.f. which q-converges to x, and so $\dot{x} \geqq \Gamma_q \mathscr{I}$. To obtain the result for α, repeat the argument with q replaced by $\pi^\alpha q$.

Theorem 2.3. *The following statements about a space (S, q) are equivalent.*

(a) $\gamma_q \leqq \alpha$;

(b) *If $\mathscr{I}$ is an u.f. on S and $\dot{x} \geqq \Delta_q^{\alpha+1} \mathscr{I}$, then $\dot{x} \geqq \Delta_q^\alpha \mathscr{I}$;*

(c) *If $\mathscr{I}$ is an u.f. on S and x an adherent point of $\Delta_q^\alpha \mathscr{I}$, then $\mathscr{I} \geqq \mathscr{V}_q^\alpha(x)$.*

Proof. As in [6], we will use $\alpha_q \mathscr{F}$ to denote the set of adherent points of $\mathscr{F}$.

(a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c). Let $\mathscr{I}$ be an u.f. with x in $\alpha_q \Delta_q^\alpha \mathscr{I}$. Then there is an u.f. $\mathscr{F}$ converging to x such that $\mathscr{F} \geqq \Delta_q^\alpha \mathscr{I}$. Thus $\dot{x} \geqq \Gamma_q \mathscr{F} \geqq \Delta_q^{\alpha+1} \mathscr{I}$, and by (b), $\dot{x} \geqq \Delta_q^\alpha \mathscr{I}$. The result then follows from Lemma 2.2.

(c) $\Rightarrow$ (a). We will show that $\Gamma_q^\alpha = \Gamma_q^{\alpha+1}$. Let $x \in \Gamma_q^{\alpha+1} A$ for some $A \subseteq S$. Then there is an u.f. $\mathscr{F}$ containing $\Gamma_q^\alpha A$ such that $\mathscr{F}$ converges to x. By Lemma 2.1, there is an u.f. $\mathscr{I}$ containing A such that $\mathscr{F} \geqq \Delta_q^\alpha \mathscr{I}$. Thus, $x \in \alpha_q \Delta_q^\alpha \mathscr{I}$, and by (c), $\mathscr{I} \geqq$ $\geqq \mathscr{V}_q^\alpha(x)$. Thus $\mathscr{I}$ $\pi^\alpha q$-converges to x, and $x \in \Gamma_q^\alpha A$.

There follow three corollaries which are of interest in the theory of regular convergence spaces. A space is *regular* if it is Hausdorff and has the property: $\Gamma \mathscr{F}$ converges to x whenever $\mathscr{F}$ converges to x.

Corollary 2.4. *If (S, q) is a compact regular space, then $\gamma_q \leqq 1$.*

Proof. Let $\mathscr{I}$ be an u.f. on S, x an adherent point of $\Gamma \mathscr{I}$. $\mathscr{I}$ must converge to a unique point y since the space is compact and Haudsorff, and regularity guarantees that $x = y$. Thus $\mathscr{I} \geqq \mathscr{V}_q(x)$, and the conclusion follows from Theorem 2.3.

A minimal regular space (S, q) is one which is regular and has the property that no strictly coarser regular c.s. exists on the same underlying set. In Theorem 2.4 of [6], it is shown that the following property characterizes minimal regular spaces: $\mathscr{F}$ q-converges to x iff $\{x\} = \alpha_q \Gamma_q^\omega \mathscr{F}$, where ω is the first infinite ordinal number.

Corollary 2.5. *A minimal regular space (S, q) is compact iff $(S, \lambda q)$ is Hausdorff.*

Proof. If (S, q) is compact, then by Corollary 2.4 πq is a compact topological space, and so $\pi q = \lambda q$. To check that πq is Hausdorff, suppose that there is an u.f. $\mathscr{F}$ finer than $\mathscr{V}_q(x) \vee \mathscr{V}_q(y)$, for x, y in S. Then the u.f.'s $\dot{x}$ and $\dot{y}$ are both finer than $\Gamma_q \mathscr{F}$. But $\Gamma_q \mathscr{F}$ must converge to a unique point since (S, q) is compact regular, so $x = y$.

Conversely, assume that λq is Hausdorff, and suppose that there is an u.f. $\mathscr{F}$ which fails to converge in S. Let $\mathscr{I} = \bigcap\{\Gamma_q^n \mathscr{F} : n = 1, 2, \ldots\}$. If $\alpha_q \mathscr{I} = \emptyset$, then, for any x, $\alpha_q(\mathscr{I} \cap \dot{x}) = \{x\}$, and it follows that $\mathscr{F} \cap \dot{x}$ must q-converge to x, a contradiction. Thus there are at least two points, say x and y, in $\alpha_q \mathscr{I}$, and thus x, y are in $\alpha_q \Delta_q^\omega \mathscr{F}$. But Lemma 2.2 implies that $\mathscr{F}$ is finer than both $\mathscr{V}_q^{\omega+1}(x)$ and $\mathscr{V}_q^{\omega+1}(y)$, which contradicts the assumption that λq is Hausdorff.

Corollary 2.4 was previously established in [10] using the Stone-Čech compactification of a convergence space described in [9].

A space (S, q) is said to be: (1) *first countable* if each filter which q-converges to x contains a countable-base filter which q-converges to x; (2) *Frechet* if, for each $A \subseteq S$, $\Gamma_q A$ is the set of all limits of countable-base filters which contain A; (3) *sequential* if a subset A of S is closed whenever it contains all limits of countable-base filters which contain A. The relationship between these three concepts in the realm of topological spaces is pointed out in [3], and a partial extension of these notions to a convergence space setting is given in [5].

Starting with an arbitrary space (S, q), let q^0 be the c.s. on S defined as follows: $\mathscr{F}$ q^0-converges to x iff $\mathscr{F}$ is finer than a countable-base filter which q-converges to x. Since q^0 is easily seen to be the coarsest first countable c.s. finer than q, we shall refer to (S, q^0) as the *first countable modification* of (S, q). The next proposition was partially proved in [5]; the remainder of the proof is straightforward.

Proposition 2.6. *The following statements about a pretopological (topological) space (S, q) are equivalent.*

(a) (S, q) *is Frechet (sequential).*

(b) (S, q) *is the pretopological (topological) modification of a first countable space.*

(c) $q = \pi q^0$ $(q = \lambda q^0)$.

If (S, q) is a sequential topological space, then γ_{q^0} is the ordinal number which is identified in [1] as $\sigma(S, q)$; it is noted in [1] that this ordinal is $\leqq \omega_1$, the least uncountable ordinal. This result also follows from a theorem by Novák [8] which we rephrase as follows.

Theorem 2.7. *Let (S, q) be a convergence space, α the least ordinal of regular cardinality ϱ. Assume that, for each subset A of S, there is for each $x \in \Gamma_q A$ a subset B of A of cardinality $< \varrho$ such that $x \in \Gamma_q B$. Then $\pi^\alpha q = \lambda q$.*

Corollary 2.8. *If (S, q) is a Frechet space, then $\gamma_q \leqq \omega_1$.*

On the other hand, no upper bound can be placed on the length of the decomposition series of a sequential space, as can be seen from the example in [8] which follows Theorem 1.

There are first countable (hence, Frechet) spaces for which the series length is exactly ω_1. One example is the first countable modification of the space S_ω constructed in [1]; a second example is obtained later in this section.

Proposition 2.9. *If (S, q) is a space such that $\mathscr{V}_q(x)$ has the countable intersection property for each x in S, then $\gamma_q \leqq \omega$.*

Proof. If the conclusion were false, then there would be a subset A of S and a point x in $\Gamma_q^{\omega+1} A - \Gamma_q^\omega A$. For each integer $n \geqq 1$, choose V_n in $\mathscr{V}_q(x)$ such that $V_n \cap \Gamma_q^n A = \emptyset$, and let $V = \bigcap V_n \in V_q(x)$. Then $V \cap \Gamma_q^\omega A = \emptyset$, a contradiction.

Since a compact regular space (S, q) has $\gamma_q \leqq 1$, one might suppose that spaces which closely approximate such spaces might also have series of bounded length. For compact Hausdorff, locally compact regular, and minimal regular spaces this is not the case, as we shall now proceed to show.

Example 2.10. Let S be a set of cardinality α, where α is a regular infinite cardinal number. Let σ be an ordinal number whose cardinality is $\leqq \alpha$. Partition S into sets $\{S^\varrho : 0 \leqq \varrho \leqq \sigma\}$, where each set S^ϱ has cardinality α. Let A be a well ordered index set of cardinality α which contains a greatest element, and partition each S^ϱ into sets $\{S^\varrho_\beta : \beta \in A\}$, where each set S^ϱ_β has cardinality α. Also, let the elements of S^ϱ be indexed as follows: $S^\varrho = \{x^\varrho_\beta : \beta \in A\}$. For each $\beta \in A$, let $T_\beta = \{x^\varrho_\beta : 0 \leqq \varrho \leqq \sigma\}$. Let r be the finest c.s. on S subject to the following conditions:

(r1) Each free u.f. which contains S^ϱ_β r-converges to $x^{\varrho+1}_\beta$, for $\varrho < \sigma$.

(r2) Each free u.f. which contains S^σ r-converges to a fixed point z in S^σ.

(r3) For each β, let T_β be well-ordered as follows: $x^\lambda_\beta \leqq x^\mu_\beta$ iff $\lambda \leqq \mu$. A filter which contains T_β r-converges to y in T_β iff the restriction of $\mathscr{F}$ to T_β converges to y in the order topology on T_β. Clearly (S, r) is locally compact. (S, r) is also regular in the following strong sense: if $\mathscr{F}$ r-converges to x, then $\Gamma_r \mathscr{F} = \mathscr{F} \cap \dot{x}$.

Next, let y be an arbitrary point in S distinct from z, and let p be the finest c.s. on S which satisfies the following conditions:

(p1) If $\mathscr{F}$ r-converges to x, then $\mathscr{F}$ p-converges to x.

(p2) If $\mathscr{F}$ is a free u.f. which fails to r-converge, then $\mathscr{F}$ p-converges to y.

The first condition implies $p \leqq r$, the second that p is compact and Hausdorff.

Finally, let q be the finest c.s. on S subject to the following condition:

(q1) $\mathscr{F}$ q-converges to x iff $\mathscr{F}$ p-converges to x and $\{x\} = \alpha_p \Gamma_p^\omega \mathscr{F}$.

By these constructions, $p \leqq q \leqq r$. Note that these three convergence structures differ only in convergence to the point y. The λ-th closure of S^0 relative to any of the three spaces is either S^λ or $S^\lambda \cup \{y\}$; the σ-th closure of S^0 in all three cases is S. Thus each of the three spaces has a decomposition series of length at least σ.

Theorem 2.11. *For each of the following three classes of spaces, there is no upper bound on the length of the decomposition series: compact Hausdorff, locally compact regular, minimal regular.*

Proof. It remains only to show that the space (S, q) of the preceding example is minimal regular. This proof is somewhat lengthy, so we will just outline the main steps. Recall the characterization of minimal regularity given previously in this section.

(a) If $\mathscr{F}$ r-converges to x, then $\Gamma_r^\varrho \mathscr{F} = \Gamma_p^\varrho \mathscr{F} = \Gamma_q^\varrho \mathscr{F}$ for all ordinal numbers ϱ.

(b) If $\mathscr{F}$ is an u.f. and $\mathscr{F}$ q-converges to x, then $\{x\} = \alpha_q \Gamma_q^\omega \mathscr{F}$.

(c) If $\mathscr{F}$ is an u.f. and $\{x\} = \alpha_q \Gamma_q^\omega \mathscr{F}$, then $\mathscr{F}$ q-converges to x. This is the hardest part of proof. A key step is the following: if $\mathscr{F}$ p-converges to z and $\mathscr{F}$ fails to r-converge to z, where $z \notin \alpha_q \Gamma_q^\omega \mathscr{F}$, then $\mathscr{F}$ contains a set F which has a finite intersection with each r-compact subset of S.

(d) (S, q) is minimal regular.

If $\alpha = \aleph_0$ and σ is a countable ordinal, then (S, r) is a first countable space with a decomposition series of length at least σ. By taking a disjoint sum of such spaces, for $\sigma < \omega_1$, we obtain (by Theorem 1.7) a first countable space of series length ω_1.

Question. *Is there a bound on the length of the decomposition series for a locally compact minimal regular space?*

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