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LEVEL SETS AND NEIGHBORHOODS OF STABLE ATTRACTORS[1])

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I. In [1] the following theorem is established: *Let* (X, π) *be a dynamical system on the locally compact metric space* X. *If* $F \subset X$ *is compact, invariant, and (positively) asymptotically stable, then there is a real valued map (continuous),* $v : A(F) \to R^+$, *from the region of attraction of* F *into the non-negative reals which is uniformly unbounded on* $A(F)$ (i.e., *for each* $\alpha > 0$ *there is a compact subset* $M \subsetneqq A(F)$ *such that* $v(x) \geqq \alpha$ *for each* $x \in A(F) \setminus M$) *and, in addition, satisfies*:

(i) $v(x) = 0$ iff $x \in F$;

(ii) $v(\pi(x, t)) \equiv e^{-t} . v(x)$ for every $(x, t) \in A(F) \times R$.

It follows immediately from (ii) that each level set of v, $K = v^{-1}(k)$, $k > 0$, is a section for the invariant set $A^*(F) \equiv A(F) \setminus F$; i.e., K is closed in $A^*(F)$ and for each $x \in A^*(F)$ there is a unique $t_x \in R$ such that $\pi(x, t_x) \equiv x\pi t_x \in K$. Our remarks in II and III below concern these sets and the sets $Q_k = A(F) \setminus v^{-1}[0, k]$, $k \geqq 0$. Throughout, E^n denotes euclidean n-space, $n \geqq 1$.

II. Recall that $A(F)$ is always open in X and invariant. The following well-known result is also important:

Lemma 1. *If* $L^-(x)$ *denotes the negative limit set of* $x \in X$, *then* $L^-(x) \cap A(F) = \emptyset$ *for each* $x \in A^*(F)$.

Proof. See [1].

From the uniform unboundedness of v follows directly

Lemma 2. $v^{-1}[0, k]$, hence K, is compact for each $k \geqq 0$.

Proof. Choose $k' > k$. Then there is a compact subset $B \subsetneqq A(F)$ such that $v(x) \geqq k'$ whenever $x \in A(F) \setminus B$. $v^{-1}[0, k]$ is a closed subset of $A(F)$ which is contained in B. Therefore, $v^{-1}[0, k]$ is compact.

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In [2] we define a subset N of X to be pre-admissible for F, if N is a neighborhood of F and $N \subset A(F)$. When N is pre-admissible for F, we put $P = A(F) \setminus N$. Using this notation we have:

Theorem 1. *Suppose N is compact, pre-admissible for F and that P is connected. Then Q_k is connected for any $k \geqq 0$.*

Proof. Fix k. Note that P and Q_k are both open in X. Suppose $Q_k = Q_{k_1} \cup Q_{k_2}$ is a separation of Q_k. Let

$$P_1 \equiv \{x \in P \mid \gamma(x) \cap Q_{k_1} \neq \emptyset\}$$

and

$$P_2 \equiv \{x \in P \mid \gamma(x) \cap Q_{k_2} \neq \emptyset\}$$

where $\gamma(x)$ denotes the trajectory through x. We claim that $P = P_1 \cup P_2$ is a separation of P contradicting the hypothesis that P is connected.

First, $P_i \neq \emptyset$, $i = 1, 2$. For let $y \in Q_{k_i}$. Then there is $t \in R$ such that $\omega = y\pi t \in P$. Otherwise, $\gamma(y) \subset N$. Since N is compact, $L^-(y) \neq \emptyset \subset N \subset A(F)$. But $y \in A^*(F)$, contradicting Lemma 1. By definition of P_i, $\omega \in P_i$.

Also, $P = P_1 \cup P_2$. For let $z \in P$. Then there is a unique $t_z \in R$ such that $v(z\pi t_z) = k$. Let $t' > t_z$. Then $z\pi t' \in Q_k$. Therefore $z\pi t' \in Q_{k_1}$ or $z\pi t' \in Q_{k_2}$; i.e., $z \in P_1$ or $z \in P_2$.

Again, $P_1 \cap P_2 = \emptyset$. Otherwise, let $z \in P_1 \cap P_2$. Then there is a unique $t_z \in R$ such that $v(z\pi t_z) = k$. Since $\gamma(z) \cap Q_{k_1} \neq \emptyset$ and $\gamma(z) \cap Q_{k_2} \neq \emptyset$,

$$z\pi(-\infty, t_z) = \{z\pi(-\infty, t_z) \cap Q_{k_1}\} \cup \{z\pi(-\infty, t_z) \cap Q_{k_2}\}$$

is a separation of $z\pi(-\infty, t_z)$. This contradicts the fact that $z\pi(-\infty, t_z)$ is connected.

Finally, P_i is open in P, $i = 1, 2$. For let $z \in P_i$ and let $\omega = z\pi\tau \in \gamma(z) \cap Q_{k_i}$. Let $\mathscr{U}$ be an open neighborhood of ω in X which lies in Q_{k_i}. Then there is an open neighborhood $\mathscr{V}$ of z in X and an open interval I in R and containing τ such that $\pi(\mathscr{V} \times I) \subset \mathscr{U}$. Hence, for each $y \in \mathscr{V} \cap P$, $\gamma(y) \cap Q_{k_i} \neq \emptyset$. Then $\mathscr{V} \cap P \subset P_i$ and $\mathscr{V} \cap P$ is open in P and contains z. This completes the proof.

For each $k > 0$ there is a function $p_k : A^*(F) \to R$ given by $p_k(x) \equiv t_x$ such that $x\pi t_x \in K = v^{-1}(k)$. These functions are continuous (See [2]) and give rise to the maps $\alpha_k : A^*(F) \to K$ which are onto and given by $\alpha_k(x) \equiv x\pi p_k(x)$. We use these functions to obtain

Corollary 1. *Under the hypothesis of Theorem 1, each K is connected.*

Proof. Let $K = v^{-1}(k)$. Then Q_k is connected. α_k restricted to Q_k is a map onto K. Hence, K is connected.

We show in [2] that $A(F)$ is connected iff F is connected. This fact helps establish:

Theorem 2. *If $A^*(F)$ is connected, then the boundary of F in X, ∂F, is connected.*

Proof. Consider the flow on $B = A(F) \setminus \text{interior } F$. We have ∂F is compact, invariant, and asymptotically stable with $A(\partial F) = A^*(F) \cup \partial F$. If $A^*(F)$ is connected, $\overline{A^*(F)}$ (in B) is connected. But $\overline{A^*(F)} = A(\partial F)$. Therefore, ∂F is connected.

Theorems 1 and 2 give:

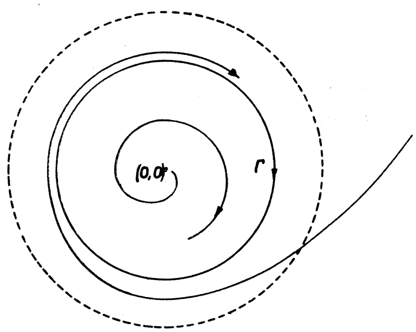
Corollary 2. *Under the hypothesis of Theorem* 1, *∂F is connected.*

Proof. $A^*(F)$ is connected. Apply Theorem 2.

It is interesting to note that Theorem 1 is false if N is not compact and the converse of Theorem 2 is false. For the system (van der Pol)

$$\dot{x} = y,$$
$$\dot{y} = \varepsilon(1 - x^2)\, y - x,$$

with $\varepsilon > 0$ small, there is a unique (non trivial) periodic trajectory, say Γ, toward which every nonconstant trajectory tends as $t \to +\infty$. The phase portrait sketches as:



If N is the set of all points inside the dotted curve except for (0,0), then since $A(\Gamma) = E^2 \setminus \{(0,0)\}$, $A(F) \setminus N$ is connected but Q_k and K are not connected for any $k > 0$. Also, $F = \Gamma = \partial F$ is connected but $A^*(F)$ is separated.

Theorem 3. *Let N be a compact and pre-admissible for F.*

(i) *If P is path connected, Q_k is path connected for any $k \geqq 0$;*

(ii) *If P is simply connected, Q_k is simply connected for any $k \geqq 0$.*

Proof. (i) Fix k. Let $x, y \in Q_k$. Choose $k' > k$ such that $N \subsetneqq v^{-1}[0, k']$. This can be done since v' is continuous, uniformly unbounded on $A(F)$, and N is compact.

Then there are $t_x \leqq 0$, $t_y \leqq 0$ such that $x\pi t_x, x\pi t_y \in Q_k$, and $x\pi t, y\pi t' \in Q_k$ for $t_x \leqq t \leqq 0$, $t_y \leqq t' \leqq 0$.

Now let $\omega : I \to P$ be a path in P connecting $x\pi t_x$ with $y\pi t_y$ where $I \equiv [0,1]$. Then $\omega(I)$ is compact and there is $\hat{\tau} \leqq 0$ such that $p_{k'}(\omega(s)) \geqq \hat{\tau}$ whenever $s \in I$.

Then $x\pi[t_x + \hat{\tau}, 0] \cup \omega(I)\,\pi\hat{\tau} \cup y\pi[t_y + \hat{\tau}, 0]$ is a path in Q_k connecting x with y.

(ii) Again, fix k. Choose $k' > k$ such that $N \subsetneqq v^{-1}[0, k']$ as in (i). Let $\Gamma = \omega(S^1)$ be a closed curve lying in Q_k where S^1 is the unit circle in E^2. Since Γ is compact, there is a $\tau \leqq 0$ such that $p_{k'}(\omega(s)) \geqq \tau$ for each $s \in S^1$ and $\Gamma\pi[\tau, 0] \subset Q_{k'}$.

Since P is simply connected, there is a continuous extension of $\pi^\tau \circ \omega$ ($\pi^\tau : X \to X$ is the homeomorphism given by $\pi^\tau(x) \equiv x\pi\tau$ for each $x \in X$), say $W : D \to P$, from the unit disc in E^2 into P. But $W(D)$ is compact and there is $\tau' \leqq 0$ such that $p_{k'}(z) \geqq \tau'$ for $z \in W(D)$. Then $W(D)\,\pi\tau' \subset Q_{k'} \cup K' \subset Q_k$ where $K' = v^{-1}(k')$. This completes the proof.

It is easy to see, again from the earlier example, that N must be compact in both parts of Theorem 3.

III. We now take $X = E^n$, $n \geqq 1$. $B(0; r)$ always denotes the closed ball in E^n with center at the origin and with positive radius r. Its boundary, denoted by $S(0; r)$, is, of course, topologically S^{n-1}.

Theorem 4. *If $A(F) = E^n$, then each K is a Peano space.*

Proof. Choose $r > 0$ such that $K \subset B(0; r)$. The map α_k restricted to $S(0; r)$ is a closed map onto K. Since $S(0; r)$ is compact, connected, and locally connected so is K.

Suppose $A(F) = E^2$. It follows from [3; Theorem 1] and Theorem 4 that each K is either a simple arc or a simple closed curve. But from [2], K is the same homotopy type as S^{n-1}, hence, as S^1 in this case. Then K must be a simple closed curve.

Corollary 3. *If $A(F) = E^2$, each K is a simple closed curve.*

When $x \in X$ we write $\gamma'(x) \equiv \pi_x'(-\infty, 0)$ where $\pi_x : R \to X$ is the map $\pi_x(t) \equiv x\pi t$ and π_x' is its restriction to $(-\infty, 0)$.

Theorem 5. *Suppose $A(F) = E^n$. For each pair of points $a, b \in K$, $K \setminus \{a \cup b\}$ is connected whenever $n \geqq 3$.*

Proof. Let $K = v^{-1}(k)$. Then α_k restricted to Q_k, $\alpha_k \mid Q_k$, is a map onto K. If $K = K_1 \cup K_2$ is a separation of K, then

$$T_1 = (\alpha_k \mid Q_k)^{-1}\,(K_1)$$

and

$$T_2 = (\alpha_k \mid Q_k)^{-1}\,(K_2)$$

are disjoint open subsets of Q_k. Further, $Q_k \setminus \{\gamma'(a) \cup \gamma'(b)\} = T_1 \cup T_2$. Hence, $Q_k \setminus \{\gamma'(a) \cup \gamma'(b)\}$ is separated. We now need

Lemma 3. $\pi'_a, \pi'_b : (-\infty, 0) \to Q_k$ *are homeomorphisms.*

Proof. We do the proof for π_a. The argument for π_b is exactly the same.

First, π'_a is 1-1. For if $a\pi t = a\pi t'$, $t \neq t'$, then a is a periodic or a rest point. Then $L^-(a) \neq \emptyset \subset A(F)$. But $a \in A^*(f)$ contradicting Lemma 1.

Hence, it suffices to show the limit set of π'_a; i.e., $L(\pi'_a) \equiv \{y \in Q_k|$ for some sequence $t_n \in (-\infty, 0)$ having no convergent subsequence, $\pi'_\alpha(t_n)$ converges to $y\}$, is empty. But this is just $L^-(a)$ and is empty since $A(F) = E^n$ and $a \in A^*(F)$. This proves the lemma.

Now from [4], $\gamma'(a)$ and $\gamma'(b)$ have dimension 1 and clearly each is closed in Q_k. Hence, $\gamma'(a) \cup \gamma'(b)$ has dimension $\leqq 1$. But Q_k is an n-dimensional manifold with $n \geqq 3$ and cannot be disconnected by a subset of dimension $\leqq 1$. This contradiction establishes the theorem.

We could not establish that for $n = 3$, K is a sphere. (If K is a 2-manifold without boundary, then it is a sphere since it is the same homotopy type as S^2.) In light of Theorems 4 and 5 it would be sufficent to show that each simple closed curve on K disconnects K.

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