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ON TRANSITIVE SUBMANIFOLDS OF $\mathscr{C}^2$ AND $\mathscr{C}^3$

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1. THE MAIN THEOREMS

In $\mathscr{C}^n$, consider the coordinates $(z_1, \ldots, z_n)$, $z_i = x_i + iy_i$. Let $\iota : \mathscr{C}^n \to \mathscr{R}^{2n}$ be the usual identification $\iota(z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n)$. In $\mathscr{R}^{2n}$, we have the well known induced endomorphism $I : \mathscr{R}^{2n} \to \mathscr{R}^{2n}$, $I^2 = -\mathrm{id.}$, given by

$$I\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad I\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}.$$

Denote by Γ the pseudogroup of all local holomorphic diffeomorphisms in $\mathscr{C}^n$ (or $\iota(\Gamma)$ in $\mathscr{R}^{2n}$ resp.), let $\Gamma_s \subset \Gamma$ be the sub-pseudogroup of maps $z'_i = f_i(z_1, \ldots, z_n)$ satisfying

$$\left|\det \frac{\partial(z'_1, \ldots, z'_n)}{\partial(z_1, \ldots, z_n)}\right| = 1\,. \tag{1.1}$$

Let $M^m \subset \mathscr{C}^n$ be a real submanifold; let us write again M^m instead of $\iota(M^m)$. Consider a point $p \in M^m$, the tangent space $T_p = T_p(M^m)$, and define τ_p as $T_p \cap JT_p$.

Lemma 1.1. *Let $v_0 \in \tau_p$. In a neighbourhood $U \subset M^m$ of p, consider a vector field v such that $v_p = v_0$ and $v_q \in \tau_q$ for each $q \in U$. The map $L_p^{(1)} : \tau_p \to T_p/\tau_p$ be given by $L_p^{(1)}(v_0) = \pi_1([v, Jv]_p)$, $\pi_1 : T_p \to T_p/\tau_p$ being the projection; $L_p^{(1)}(v_0)$ depends on v_0 only. Let $\sigma_p \subset T_p$ be the linear hull of the set $\pi_1^{-1}(L_1^{(p)}(\tau_p))$. The map $L_p^{(2)} : \tau_p \to T_p/\sigma_p$ be defined by $L_p^{(2)}(v_0) = \pi_2([v, [v, Jv]]_p)$, $\pi_2 : T_p \to T_p/\sigma_p$ being the projection; $L_p^{(2)}(v_0)$ depends on v_0 only.*

$L_p^{(1)}$ and $L_p^{(2)}$ are the so-called *Levi maps*.

Write $G(M^m) = \{\gamma \in \Gamma; \gamma(M^m) = M^m\}$ and $G_s(M^m) = G(M^m) \cap \Gamma_s$. We propose to prove the following two theorems.

Theorem 1.1. *Consider the case* $n = 2$, $m = 3$, *i.e.*, $M^3 \subset \mathscr{C}^2$. *Suppose* $L_p^{(1)} \not\equiv 0$ *at each point* $p \in M^3$. *If* $G_s(M^3)$ *is transitive on* M^3, *then it is a Lie group with* $\dim G_s(M^3) \leqq 4$. *Consider the following manifolds*

$$(1.2) \qquad N_r^3 : z_1\bar{z}_1 + z_2\bar{z}_2 = r^2 \quad (r > 0)\,,$$

$$(1.3) \qquad N_R^3 : z_1\bar{z}_2 + \bar{z}_1 z_2 = 2R \quad (R > 0)\,,$$

$$(1.4) \qquad N_0^3 : i(z_2 - \bar{z}_2) = (z_1 - \bar{z}_1)^2\,.$$

Let $\dim G_s(M^3) = 4$. *Then there is exactly one manifold among the manifolds* N_r^3, N_R^3, N_0^3 *– denote it by* N^3 *– with the following property: choose* $p \in M^3$, $q \in N^3$, *then there is a neighbourhood* $U \subset M^3$ *of* p *and a* $\gamma \in \Gamma$ *such that* $\gamma(p) = q$, $\gamma(U) \subset \subset N^3$. *The groups* $G_s(N_r^3)$, $G_s(N_R^3)$, $G_s(N_0^3)$ *are given by*

$$(1.5) \qquad z_1' = \alpha z_1 - \beta z_2\,, \quad z_2' = e^{ia}(\bar{\beta} z_1 + \bar{\alpha} z_2)\,,$$

where $\alpha, \beta \in \mathscr{C}$, $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$, $a \in R$;

$$(1.6) \qquad z_1' = e^{if}(az_1 + ibz_2)\,, \quad z_2' = e^{if}(icz_1 + dz_2)\,,$$

where $a, b, c, d, f \in \mathscr{R}$, $ad + bc = 1$;

$$(1.7) \qquad \begin{aligned} z_1' &= e^{ia}z_1 + b + ci\,,\\ z_2' &= 4e^{ia}cz_1 + i(1 - e^{2ia})\, z_1^2 + z_2 + d + 2c^2 i\,, \end{aligned}$$

where $a, b, c, d \in \mathscr{R}$.

Theorem 1.2. *Consider the case* $n = 3$, $m = 4$, *i.e.*, $M^4 \subset \mathscr{C}^3$. *Suppose* $\dim \tau_p = 2$, $L_p^{(1)} \not\equiv 0$, $L_p^{(2)} \not\equiv 0$ *at each point* $p \in M^4$. *If* $G(M^4)$ *is transitive on* M^4, $G(M^4)$ *is a Lie group and* $\dim G(M^4) \leqq 5$. *Let us consider a manifold* M^4 *with* $\dim G(M^4) = 5$ *and the manifold* N^4 *given by*

$$(1.8) \qquad \bar{z}_2 - z_2 = i(\bar{z}_1 - z_1)^2\,, \quad \bar{z}_3 - z_3 = (\bar{z}_1 - z_1)^3\,.$$

If $p \in M^4$, $q \in N^4$ *are arbitrary points, there is a neighbourhood* $U \subset M^4$ *of* p *and a* $\gamma \in \Gamma$ *such that* $\gamma(U) \subset N^4$, $\gamma(p) = q$, *i.e.*, M^4 *and* N^4 *are locally* Γ*-equivalent. The group* $G(N^4)$ *is*

$$(1.9) \qquad \begin{aligned} z_1' &= az_1 + b + ci\,,\\ z_2' &= 4acz_1 + a^2 z_2 + d + 2c^2 i\,,\\ z_3' &= -12ac^2 z_1 - 6a^2 cz_2 + a^3 z_3 + f - 4c^3 i\,, \end{aligned}$$

where $a, b, c, d, f \in \mathscr{R}$.

The first two chapters of this paper are devoted to the equivalence problems. The treatment is based on the theory of partial differential equations due to K. KURANISHI's notes *Lectures on involutive systems of partial differential equations* (Publ. da Soc. Math. de Sao Paulo, 1967) which are unfortunately not very well known. Chapter 3 contains the theory of structures induced on manifolds $M^3 \subset \mathscr{C}^2$ with respect to the pseudogroup Γ_s and the proof of Theorem 1.1; in chapter 4, the manifolds $M^3 \subset \mathscr{C}^2$ with $\dim G_s(M^3) = 3$ are studied. Finally, Theorem 1.2 is proved in the last chapter.

From the literature, I mention just two papers of E. CARTAN devoted to the determination of all manifolds $M^3 \subset \mathscr{C}^2$ with $\dim G(M^3) \geqq 3$ (Annali di Mat., *11*, 1932, 17–90 and Verh. int. math. Kongr. Zürich, t. II, 1932, 54–56). Unfortunately, these papers are written in such a way that I do not fully understand them.

Parts of the results have been obtained during my stays in the USSR (State Univ. at Vilnius) and India (Delhi, Punjab and Bombay Univ. and Tata Inst. of Fund. Research). The paper has been written during my stay at the Humboldt-Univ. in Berlin (GDR). My thanks go to all these institutions.

2. SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

A *fiber manifold* is a triple (M, N, ϱ), where:

(i) M and N are analytic manifolds, $\dim M = n + m$, $\dim N = n$;

(ii) $\varrho : M \to N$ is an analytic map of M onto N;

(iii) to each $y \in M$ there exists its coordinate neighbourhood $U \subset M$ such that

$$\begin{array}{ccc} U & \xrightarrow{\mu_1} & \mathscr{R}^n \times \mathscr{R}^m \\ \downarrow \varrho & & \downarrow pr_1 \\ \varrho(U) & \xrightarrow[\mu_2]{} & \mathscr{R}^n \end{array}$$

is commutative; here, (U, μ_1) and $(\varrho(U), \mu_2)$ are charts and pr_1 is the natural projection. Denote by $J^k = J^k(M, N, \varrho)$ the analytic manifold of all k-jets of local sections of the fiber manifold (M, N, ϱ); let us write $J^0 = M$, $J^{-1} = N$. The triple (J^l, J^k, ϱ_k^l), $l > k$, is again a fiber manifold, ϱ_k^l being the natural projection.

Let $X \in J^k$, $y = \varrho_0^k(X) \in M$, $x = \varrho_{-1}^k(X) \in N$. The space $Q_X(J^k) \subset T_X(J^k)$ be defined by the exact sequence

$$0 \to Q_X(J^k) \to T_X(J^k) \xrightarrow{d\varrho^k_{k-1}} T_{\varrho^k_{k-1}(X)}(J^{k-1}).$$

Let $\xi \in Q_X(J^k)$. Then there exists a neighbourhood $U \subset M$ of y and local sections $f(t) : \varrho(U) \to M$, $t \in (-\varepsilon, \varepsilon)$, such that

$$\xi = \frac{d}{dt} j_x^k(f(t))\bigg|_{t=0}.$$

The section $f(t)$ be chosen in such a manner that $j_x^{k-1}(f(t)) = \varrho_{k-1}^k(X)$. Let U be such that we have local coordinates (x^i, y^α); $i = 1, \ldots, n$; $\alpha = 1, \ldots, m$; in it, (y^α) be the local coordinates in $\varrho(U)$. In $\varrho(U)$, the local section $f(t)$ be given by $y^\alpha = f^\alpha(x^i, t)$. The mapping

$$\tau : Q_X(J^k) \to Q_y(M) \otimes S^k T_x^*(N)$$

be defined by

$$\tau(\xi) = \frac{\partial^{k+1} f^\alpha(x_0^i, 0)}{\partial t\, \partial x^{i_1} \ldots \partial x^{i_k}} \cdot \frac{\partial}{\partial y^\alpha} \otimes \mathrm{d}x^{i_1} \otimes \ldots \otimes \mathrm{d}x^{i_k},$$

here, (x_0^i) are the coordinates of the point x and $S^k V$ is the k-th symmetric tensor product of the space V. The mapping τ does not depend on the choice of coordinates (x^i, y^α). It is an isomorphism which is called the *fundamental identification.*

Let $R^k \subset J^k$ be a submanifold. R^k is said to be *regular* at the point $X \in R^k$ if there is a neighbourhood $V \subset J^k$ of X and functions $f_a : V \to \mathscr{R}$; $a = 1, \ldots, A$; with the following properties:

(i) $A + \dim R^k = \dim J^k$,
(ii) $V \cap R^k = \{Y \in J^k;\ f_a(Y) = 0 \text{ for } a = 1, \ldots, A\}$,
(iii) $\mathrm{d}f_1, \ldots, \mathrm{d}f_A \in T_X^*(J^k)$ are linearly independent.

A submanifold $R^k \subset J^k$ is said to be a *partial differential equation of order k* if it is regular at each of its points. The section $f : U \to M$, $U \subset N$ being an open set, is said to be a *solution* of R^k if $j_x^k(f) \in R^k$ for each $x \in U$.

Be given a function $F : V \to \mathscr{R}$, $V \subset J^k$ being an open set; further, let v be a vector field on $\varrho_{k-1}^k(V) \subset N$. The function $\partial_v F : (\varrho_k^{k+1})^{-1}(V) \to \mathscr{R}$ be defined as follows. Let $X \in (\varrho_k^{k+1})^{-1}(V)$, and let $f : N \to M$ be a local section such that $X = j_{x_0}^{k+1}(f)$, $x_0 = \varrho_{-1}^{k+1}(X)$. Consider the local section $j^k f : N \to J^k$. Then we have the local map $F \circ j^k f : N \to \mathscr{R}$; set $(\partial_v F)(X) = v(F \circ j^k f)|_{x=x_0}$.

The differential equation R^k being given in a neighbourhood $V \subset J^k$ of its point $X \in R^k$ as $\{X \in V;\ f_a(X) = 0 \text{ for } a = 1, \ldots, A\}$, define $pR^k|_{(\varrho_k^{k+1})^{-1}(V)} = \{X \in$ $\in (\varrho_k^{k+1})^{-1}(V) \subset J^{k+1};\ f_a(\varrho_k^{k+1}(X)) = 0,\ (\partial_v f_a)(X) = 0$ for $a = 1, \ldots, A$ and for each vector field $v \in T(\varrho_{-1}^k(V))\}$. It is easy to see that this definition does not depend on the choice of the neighbourhoods V and the functions f_a; thus we have a well defined subset $pR^k \subset J^{k+1}$ which is called the *prolongation* of R^k.

Let R^k be a differential equation, $X \in R^k$; R^k be given – in a neighbourhood of the point X – by means of the functions f_a. Set

$$C_X(R^k) = \{\xi \in Q_X(J^k);\ \xi f_a = 0 \quad \text{for} \quad a = 1, \ldots, A\}.$$

By means of the fundamental identification and the natural mappings, we get

$$C_X(R^k) \subset Q_X(J^k) = Q_y(M) \otimes S^k T_x^*(N) \subset$$
$$\subset Q_y(M) \otimes S^{k-1} T_x^*(N) \otimes T_x^*(N) = Q_{\varrho_{k-1}^k(X)}(J^{k-1}) \otimes T_x^*(N);$$

here, $y = \varrho_0^k(X)$, $x = \varrho_{-1}^k(X)$. Write

$$A = C_X(R^k)\,, \quad F = Q_{\varrho^k_{k-1}(X)}(J^{k-1})\,, \quad E = T_x(N)\,,$$

and define

$$pA = (A \otimes E^*) \cap (F \otimes S^2E^*)\,.$$

Let $e_1, \ldots, e_n$ be a basis of E, let $e^1, \ldots, e^n$ be the dual basis. Let $E^*_{n-i} \subset E^*$ be the subspace spanned by the vectors $e^{i+1}, \ldots, e^n$. Set

(2.1) $$A_{(i)} = A \cap (F \otimes E^*_{n-i})\,, \quad \tau_i = \dim A_{(i)}\,.$$

The basis $e_1, \ldots, e_n$ is called *quasi-regular* with respect to A if

(2.2) $$\dim pA = \tau_0 + \ldots + \tau_{n-1}\,;$$

the space A is said to be *involutive* if there is a basis which is quasi-regular with respect to it.

Definition 2.1. *The differential equation R^k is called involutive at the point $X \in R^k$ if: (1) there is a neighbourhood $V \subset J^k$ of X such that $pR|_{V_1}$, $V_1 = (\varrho_k^{k+1})^{-1}(V)$, is a submanifold of J^{k+1} and $(pR^k|_{V_1}, R^k|_V, \varrho_k^{k+1})$ is a fiber manifold; (2) the space $C_X(R^k)$ is involutive. R^k is involutive if it is involutive at each point $X \in R^k$.*

Theorem 2.1. *Let the differential equation R^k be involutive at $X_0 \in R^k$. Suppose that in a neighbourhood of the point $x_0 = \varrho_{-1}^k(X_0) \in N$ we have local coordinates $(x^1, \ldots, x^n)$ such that $\partial/\partial x^1|_{x_0}, \ldots, \partial/\partial x^n|_{x_0}$ is a quasi-regular basis. Then there is a neighbourhood $V \subset J^k$ of the point X_0 such that, for each $X \in R^k \cap V$, $\partial/\partial x^1|_x, \ldots$ $\ldots, \partial/\partial x^n|_x$; $x = \varrho_{-1}^k(X)$; is a quasi-regular basis with respect to $C_X(R^k)$.*

Consider again the subspace $A \subset F \otimes E^*$. Let $e \in E$; the linear map $\delta_e : E^* \to \mathcal{R}$ by defined by $\delta_e(e^*) = e^*(e)$ for $e^* \in E^*$. W being a vector space, define the linear map $\delta_e : W \otimes E^* \to W$ by means of $\delta_e(w \otimes e^*) = e^*(e)\, w$. For each vector $e \in E$, we have thus a linear map $\delta_e : pA \to A$, this map being the restriction of $\delta_e : A \otimes$ $\otimes\, E^* \to A$.

Theorem 2.2. *Let $A \subset F \otimes E^*$ and let $e_1, \ldots, e_n$ be a basis of E. Consider the maps*

(2.3) $$\delta_{e_{i+1}} : pA_{(i)} \to A_{(i)}\,; \quad i = 0, \ldots, n-1\,.$$

The basis $e_1, \ldots, e_n$ is quasi-regular if and only if the maps (2.3) are onto.

Be given a fiber manifold (M, N, ϱ) and a submanifold $N_1 \subset N$, $\dim N =$ $= \dim N_1 + 1$. Set $J_1^k = (\varrho_{-1}^k)^{-1}(N_1) \subset J^k(E)$; $(J_1^k, N_1, \varrho_{-1}^k)$ is again a fiber manifold. Consider the maps $\sigma^k : J_1^k \to J^k(M_1, N_1, \varrho)$; $M_1 = \varrho^{-1}(N_1)$; defined as follows. Let $X \in J_1^k$, $\varrho_{-1}^k(X) = x \in N_1$. Then there is a local section $f : N \to M$ such that

$X = j^k_x(f)$. Set $\sigma^k(X) = j^k_x(f|_{N_1})$, $f|_{N_1} : N_1 \to M_1$ being the restriction of f to N_1. Now, $R^k \subset J^k$ being a differential equation, define $S^k = \sigma^k(R^k \cap J^k_1) \subset J^k(M_1, N_1, \varrho)$.

Theorem 2.3. *Let $R^1 \subset J^1(M, N, \varrho)$ be a differential equation of order one, suppose that R^1 is involutive at $X \in R^1$. Let $N_1 \subset N$ be a submanifold such that:* (i) $x = \varrho^1_{-1}(X) \in N_1$, (ii) $\dim N = \dim N_1 + 1$, (iii) *there is a quasi-regular basis $e_1, \ldots, e_n \in T_x(N)$ with respect to $C_X(R^1)$ such that $e_1, \ldots, e_{n-1} \in T_x(N_1)$. Then $S^1 \subset J^1(M_1, N_1, \varrho)$ is a differential equation involutive at X. Let $\sigma_1 : N_1 \to M_1$ be a solution of S^1 defined in a neighbourhood of $x \in N_1$. Then there exists a neighbourhood $U \subset N$ of the point x and a solution $\sigma : U \to M$ of the equation R^1 such that $\sigma|_{N_1 \cap U} = \sigma_1|_{N_1 \cap U}$.*

3. INDUCED STRUCTURES

Let us consider the space $\mathscr{C}^n$ and its coordinates $z_i = x_i + ix_{n+i}$; $i = 1, \ldots, n$. Let $\mathscr{R}^{2n}$ be the real representation of $\mathscr{C}^n$ with the coordinates (x_i, x_{n+i}) endowed with the automorphism $I : \mathscr{R}^{2n} \to \mathscr{R}^{2n}$, $I^2 = -\text{id.}$, given by

$$(3.1) \qquad I \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{n+i}}, \quad I \frac{\partial}{\partial x_{n+i}} = -\frac{\partial}{\partial x_i}; \quad i = 1, \ldots, n.$$

Further, consider the fiber manifold $E = (\mathscr{R}^{2n} \times \mathscr{R}^{2n}, \mathscr{R}^{2n}, \pi_1)$, $\pi_1 : \mathscr{R}^{2n} \times \mathscr{R}^{2n} \to \mathscr{R}^{2n}$ being the natural projection onto the first factor. In $\mathscr{R}^{2n} \times \mathscr{R}^{2n}$, we have the coordinates $(x_i, x_{n+i}, y_i, y_{n+i})$; the coordinates of the prolongation $J^1(E)$ are $(x_i, x_{n+i}, y_i, y_{n+i}, y_{ij}, y_{i,n+j}, y_{n+i,j}, y_{n+i,n+j})$. The holomorphic mappings $\varphi : U \subset \mathscr{R}^{2n} \to \mathscr{R}^{2n}$ are now to be considered as the (local) sections of the fiber manifold E satisfying the Cauchy-Riemann equations R^1

$$(3.2) \qquad y_{ij} - y_{n+i,n+j} = 0, \quad y_{i,n+j} + y_{n+i,j} = 0; \quad i, j = 1, \ldots, n.$$

Now, let $M^m \subset \mathscr{C}^n = \mathscr{R}^{2n}$ be an analytic submanifold, $p \in M^m$ its point. Consider the space

$$\tau_p(M^m) = T_p(M^m) \cap I\, T_p(M^m).$$

This space is always of even dimension; let us restrict ourselves to submanifolds M^m with $\dim \tau_p(M^m) = 2q = \text{const}$. To the submanifold $M^m \subset \mathscr{C}^n$, we associate a G-structure $B_G(M^m)$ as follows. The tangent frame

$$\{v_1, \ldots, v_q, v_{q+1}, \ldots, v_{2q}, v_{2q+1}, \ldots, v_m\}$$

at the point $p \in M^m$ is situated in $B_G(M^m)$ if and only if $v_1, \ldots, v_{2q} \in \tau_p(M^m)$ and $v_{q+\alpha} = Iv_\alpha$ for $\alpha = 1, \ldots, q$. We have the following

Theorem 3.1. *Let $M^m, \tilde{M}^m \subset \mathscr{C}^n = \mathscr{R}^{2n}$ be two analytic submanifolds, let* $\dim \tau_p(M^m) = \dim \tau_p(\tilde{M}^m) = \text{const.}$ for $p \in M^m$, $\tilde{p} \in \tilde{M}^m$. *Be given an analytic map $\varphi : M^m \to \tilde{M}^m$ satisfying $\varphi_* B_G(M^m) = B_G(\tilde{M}^m)$. Let $p_0 \in M^m$ be a fixed point. Then there exists a neighbourhood $U \subset \mathscr{R}^{2n}$ of the point p_0 and a holomorphic mapping $\Phi : U \to \mathscr{R}^{2n}$ such that $\Phi|_{M^m \cap U} = \varphi$.*

This theorem follows directly from Theorem 2.3. It has been proved by B. CENKL and myself for $m = 2n - 1$ and by J. VANŽURA for a general m; both papers are unpublished.

The proof of lemma 1.1 is easy.

Now, let $\varphi : \mathscr{C} \to \mathscr{C}$ be a (local) biholomorphic mapping given by $z' = z'(z)$, i.e., $x' + iy' = f(x, y) + ig(x, y)$. The mapping φ induces a mapping $\varphi^* : \mathscr{R}^2 \to \mathscr{R}^2$ given by $x' = f(x, y)$, $y' = g(x, y)$. We have

$$\Delta \equiv \left|\frac{\mathrm{d}z'}{\mathrm{d}z}\right| = \left|\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f + ig)\right| = \left|\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right| = \left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right\}^{1/2},$$

$$D \equiv \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{vmatrix} = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2,$$

i.e., $D = \Delta^2$. Thus the mapping φ satisfies $\Delta = 1$ if and only if $D = 1$. A similar (and, of course, a more complicated) calculation shows that the same property takes place for biholomorphic mappings $\varphi : \mathscr{C}^2 \to \mathscr{C}^2$. In the associated space $\mathscr{R}^4$, we thus get a volume structure. On each hypersurface $M^3 \subset \mathscr{C}^2 = \mathscr{R}^4$ we naturally obtain, with respect to the pseudogroup Γ_s, a G-structure described in more detail in the next chapter. Its definition is as follows: a frame $\{v_1, v_2, v_3\}$ at the point $m \in M^3$ belongs to $B_G(M^3)$ if and only if $v_1 \in \tau_m$, $v_2 = Iv_1$, $v_3 \in T_m(M^3)$ and the volume $[v_1, v_2, v_3, Iv_3] = 1$.

4. THE INDUCED G-STRUCTURE

Be given a 3-dimensional manifold M together with a G-structure B_G, G being the group of matrices of the form

$$\begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ \gamma & \delta & \varphi \end{pmatrix}, \quad (\alpha^2 + \beta^2)\,\varphi^2 = 1\,. \tag{4.1}$$

$\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ being two frames of B_G at $m \in M$, we have

$$w_1 = \alpha v_1 - \beta v_2\,, \quad w_2 = \beta v_1 + \alpha v_2\,, \quad w_3 = \gamma v_1 + \delta v_2 + \varphi v_3\,; \tag{4.2}$$

the plane $\tau_m \subset T_m(M)$ spanned by the vectors v_1, v_2 is thus invariant as well the endomorphism $I_m : \tau_m \to \tau_m$, $I_m^2 = -\mathrm{id.}$, determined by $I_m v_1 = v_2$, $I_m v_2 = -v_1$.

Consider a point $m_0 \in M$, its neighbourhood $U \subset M$, and two sections $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ of B_G over U. We have (4.2), $\alpha, \ldots, \varphi$ being real-valued functions over U. Let us write

$$(4.3)\quad \begin{aligned} [v_1, v_2] &= a_1 v_1 + a_2 v_2 + a_3 v_3\,, & [w_1, w_2] &= A_1 w_1 + A_2 w_2 + A_3 w_3\,,\\ [v_1, v_3] &= b_1 v_1 + b_2 v_2 + b_3 v_3\,, & [w_1, w_3] &= B_1 w_1 + B_2 w_2 + B_3 w_3\,,\\ [v_2, v_3] &= c_1 v_1 + c_2 v_2 + c_3 v_3\,, & [w_2, w_3] &= C_1 w_1 + C_2 w_2 + C_3 w_3\,. \end{aligned}$$

From the Jacobi identity

$$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0\,,$$

we get

$$(4.4)\quad \begin{aligned} v_1 c_1 - v_2 b_1 + v_3 a_1 + a_1 c_2 + b_1 c_3 - b_3 c_1 - a_2 c_1 &= 0\,,\\ v_1 c_2 - v_2 b_2 + v_3 a_2 + b_2 c_3 + a_2 b_1 - b_3 c_2 - a_1 b_2 &= 0\,,\\ v_1 c_3 - v_2 b_3 + v_3 a_3 + a_3 c_2 + a_3 b_1 - a_1 b_3 - a_2 c_3 &= 0 \end{aligned}$$

and analoguous equations for $A_1, \ldots, C_3$. Let us study the relations existing between $a_1, \ldots, c_3$ and $A_1, \ldots, C_3$. We have

$$\begin{aligned} [w_1, w_2] &= [\alpha v_1 - \beta v_2, \beta v_1 + \alpha v_2] = (\cdot)\, v_1 + (\cdot)\, v_2 + (\alpha^2 + \beta^2)\, a_3 v_3\,,\\ [w_1, w_3] &= [\alpha v_1 - \beta v_2, \gamma v_1 + \delta v_2 + \varphi v_3] = (\cdot)\, v_1 + (\cdot)\, v_2 +\\ &\quad + (\alpha\,.\, v_1\varphi - \beta\,.\, v_2\varphi + \alpha\delta a_3 + \alpha\varphi b_3 + \beta\gamma a_3 - \beta\varphi c_3)\, v_3\,,\\ [w_2, w_3] &= [\beta v_1 + \alpha v_2, \gamma v_1 + \delta v_2 + \varphi v_3] = (\cdot)\, v_1 + (\cdot)\, v_2 +\\ &\quad + (\beta\,.\, v_1\varphi + \alpha\,.\, v_2\varphi + \beta\delta a_3 + \beta\varphi b_3 - \alpha\gamma a_3 + \alpha\varphi c_3)\, v_3\,, \end{aligned}$$

i.e.,

$$(4.5)\quad \begin{aligned} \varphi A_3 &= (\alpha^2 + \beta^2)\, a_3\,,\\ \varphi B_3 &= \alpha\,.\, v_1\varphi - \beta\,.\, v_2\varphi + \alpha\delta a_3 + \alpha\varphi b_3 + \beta\gamma a_3 - \beta\varphi c_3\,,\\ \varphi C_3 &= \beta\,.\, v_1\varphi + \alpha\,.\, v_2\varphi + \beta\delta a_3 + \beta\varphi b_3 - \alpha\gamma a_3 + \alpha\varphi c_3\,. \end{aligned}$$

Let us restrict ourselves to the case $a_3 \neq 0$, this being equivalent to the non-integrability of the field of the planes τ_m. We get – from (4.5) – the possibility to choose the section $\{w_1, w_2, w_3\}$ in such a way that $A_3 = 1$, $B_3 = C_3 = 0$. Suppose that the section $\{v_1, v_2, v_3\}$ has been already chosen in such a manner that $a_3 = 1$, $b_3 = c_3 = 0$. Then the equation (4.5_1) reduces to $\varphi = \alpha^2 + \beta^2$, and we get $\varphi = 1$, $\alpha^2 + \beta^2 = 1$ from (4.1). The equations $(4.5_{2,3})$ reduce to $0 = \alpha\delta + \beta\gamma$, $0 = \beta\delta - \alpha\gamma$, i.e., $\delta = \gamma = 0$. From this, we get

Lemma 4.1. *The considered G-structure B_G may be reduced to the H-structure B_H the sections $\{v_1, v_2, v_3\}$ of which satisfy*

$$\begin{aligned}
[v_1, v_2] &= a_1 v_1 + a_2 v_2 + v_3 , \\
[v_1, v_3] &= b_1 v_1 + b_2 v_2 , \\
[v_2, v_3] &= c_1 v_1 + c_2 v_2 ,
\end{aligned} \tag{4.6}$$

H being the group of matrices of the form

$$\begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha^2 + \beta^2 = 1 . \tag{4.7}$$

The equations (4.4) reduce to

$$\begin{aligned}
v_2 c_1 - v_2 b_1 + v_3 a_1 + a_1 c_2 - a_2 c_1 &= 0 , \\
v_1 c_2 - v_2 b_2 + v_3 a_2 + a_2 b_1 - a_1 b_2 &= 0 , \\
c_2 + b_1 &= 0 .
\end{aligned} \tag{4.8}$$

Now, let $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ be two sections of the reduction G_H. Then

$$\begin{aligned}
[w_1, w_2] &= [\alpha v_1 - \beta v_2, \beta v_1 + \alpha v_2] = (\alpha . v_1 \beta - \beta . v_1 \alpha + a_1) v_1 + \\
&\quad + (-\beta . v_2 \alpha + \alpha . v_2 \beta + a_2) v_2 + v_3 = \\
&= (\alpha A_1 + \beta A_2) v_1 + (-\beta A_1 + \alpha A_2) v_2 + v_3 , \\
[w_1, w_3] &= [\alpha v_1 - \beta v_2, v_3] = (-v_3 \alpha + \alpha b_1 + \beta c_1) v_1 + \\
&\quad + (-v_3 \beta + \alpha b_2 + \beta c_2) v_2 = \\
&= (\alpha B_1 + \beta B_2) v_1 + (-\beta B_1 + \alpha B_2) v_2 , \\
[w_2, w_3] &= [\beta v_1 + \alpha v_2, v_3] = (-v_3 b + \beta b_1 + \alpha c_1) v_1 + \\
&\quad + (-v_3 \alpha + \beta b_2 + \alpha c_2) v_2 = \\
&= (\alpha C_1 + \beta C_2) v_1 + (-\beta C_1 + \alpha C_2) v_2 .
\end{aligned}$$

From the last two relations, we have

$$\begin{aligned}
\alpha(b_1 - c_2) - \beta(b_2 + c_1) &= \alpha(B_1 - C_2) + \beta(B_2 + C_1) , \\
\beta(b_1 - c_2) + \alpha(b_2 + c_1) &= -\beta(B_1 - C_2) + \alpha(B_2 + C_1) ,
\end{aligned}$$

i.e.,

$$\begin{aligned}
B_1 - C_2 &= (\alpha^2 - \beta^2)(b_1 - c_2) - 2\alpha\beta(b_2 + c_1) , \\
B_2 + C_1 &= 2\alpha\beta(b_1 - c_2) + (\alpha^2 - \beta^2)(b_2 + c_1) .
\end{aligned}$$

Thus there exist sections $\{w_1, w_2, w_3\}$ such that $B_1 - C_2 = 0$. Suppose that the section $\{v_1, v_2, v_3\}$ has been already chosen in such a way that $b_1 - c_2 = 0$; from

(4.8_3), we get $b_1 = c_2 = 0$. Then

$$\alpha\beta(b_2 + c_1) = 0\,, \quad B_2 + C_1 = (\alpha^2 - \beta^2)(b_2 + c_1)\,.$$

Suppose $b_2 + c_1 \neq 0$. Then $\alpha\beta = 0$, i.e., $\alpha = 0$, $\beta = \varepsilon$ or $\beta = 0$, $\alpha = \varepsilon$ resp.; $\varepsilon = \pm 1$.

Lemma 4.2. *The considered G-structure B_G may be reduced to a K-structure B_K, the sections $\{v_1, v_2, v_3\}$ of which fulfill*

$$\begin{aligned} (4.9)\qquad [v_1, v_2] &= a_1 v_1 + a_2 v_2 + v_3\,, \\ [v_1, v_3] &= b_2 v_2\,, & v_1 c_1 + v_3 a_1 - a_2 c_1 &= 0\,, \\ [v_2, v_3] &= c_1 v_1\,, & -v_2 b_2 + v_3 a_2 - a_1 b_2 &= 0\,. \end{aligned}$$

The relation $b_2 + c_1 = 0$ is invariant. If $b_2 + c_1 \neq 0$, K is the group of the matrices of the form

$$(4.10)\qquad \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -\varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \varepsilon = \pm 1\,.$$

This lemma solves the equivalence problem for the G-structures with $b_2 + c_1 \neq 0$; in fact, to each such G-structure we have associated four $\{e\}$-structures, and the equivalence problem has been reduced to the equivalence problem of $\{e\}$-structures.

Now, let us consider the more complicated case $b_2 + c_1 = 0$. Let us write $c_1 = q$, $b_2 = -q$; the equations (4.9) are now

$$\begin{aligned} (4.11)\quad [v_1, v_2] = a_1 v_1 + a_2 v_2 + v_3\,, \quad [v_1, v_3] = -q v_2\,, \quad [v_2, v_3] = q v_1\,; \\ v_1 q + v_3 a_1 - a_2 q = 0\,, \quad v_2 q + v_3 a_2 + a_1 q = 0\,. \end{aligned}$$

Let $\{w_1, w_2, w_3\}$ be another section of the reduced K-structure B_K; suppose

$$w_1 = \alpha v_1 - \beta v_2\,, \quad w_2 = \beta v_1 + \alpha v_2\,, \quad w_3 = v_3\,; \quad \alpha^2 + \beta^2 = 1\,;$$

and

$$[w_1, w_2] = A_1 w_1 + A_2 w_2 + w_3\,, \quad [w_1, w_3] = -Q w_2\,, \quad [w_2, w_3] = Q w_1\,.$$

We find

$$\begin{aligned} (4.12)\qquad \beta a_1 &= v_1\alpha + \alpha\beta A_1 + \beta^2 A_2\,, \\ \beta a_2 &= v_2\alpha - \beta^2 A_1 + \alpha\beta A_2\,, \\ \beta q &= -v_3\alpha + \beta Q\,, \end{aligned}$$

and it is easy to verify

Lemma 4.3. *We have*

$$(4.13)\qquad v_2a_1 - v_1a_2 + a_1^2 + a_2^2 - q = w_2A_1 - w_1A_2 + A_1^2 + A_2^2 - Q\,.$$

Let us write

$$(4.14)\qquad k = v_2a_1 - v_1a_2 + a_1^2 + a_2^2 - q\,, \quad K = w_2A_1 - w_1A_2 + A_1^2 + A_1^2 - Q\,.$$

Clearly,

$$w_1K = \alpha\,.\,v_1k - \beta\,.\,v_2k\,, \quad w_2K = \beta\,.\,v_1k + \alpha\,.\,v_2k\,, \quad w_3K = v_3k\,.$$

Thus we have $(w_1K)^2 + (w_2K)^2 = (v_1k)^2 + (v_2k)^2$. If $v_1k = 0$, $v_2k = 0$, we get $[v_1, v_2]\,k = 0$ and $v_3k = 0$, i.e., $k =$ const. In the case $k \neq$ const., we are in the position to choose the section $\{w_1, w_2, w_3\}$ in such a way that $w_1K = 0$. Suppose, that the section $\{v_1, v_2, v_3\}$ has been already chosen in this way. Then $\beta = 0$ and $\alpha = \varepsilon = \pm 1$, and we obtain

Lemma 4.4. *Be given a G-structure B_G, the reduction of which to the K-structure of Lemma* 4.2 *is such that $b_2 + c_1 = 0$. Let $k \neq$* const. *Then we are able to reduce our G-structure to the L-structure B_L, the sections of which satisfy $v_1k = 0$; L is the group of matrices*

$$(4.15)\qquad \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varepsilon = \pm 1\,.$$

Thus we have reduced our study to the case $k =$ const. Be given a G-structure B_G by means of a section $\{v_1, v_2, v_3\}$ satisfying (4.11) and $v_1k = v_2k = v_3k = 0$. Consider the system of partial differential equations

$$(4.16)\qquad v_1\alpha = \beta a_1\,, \quad v_2\alpha = \beta a_2\,, \quad v_3\alpha = -\beta q - \beta k$$

for the unknown function α, β being given by $\alpha^2 + \mathrm{b}^2 = 1$. It is easy to see that this system is completely integrable. Thus, there exists a section $\{w_1, w_2, w_3\}$ of our G-structure such that $A_1 = A_2 = 0$, $Q = -k$, and we have

Lemma 4.5. *Be given a G-structure B_G, and let its reduction to the K-structure of Lemma* 4.2 *be such that $b_2 + c_1 = 0$ and $k =$* const. *Then there are sections of B_K satisfying*

$$(4.17)\qquad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = kv_2\,, \quad [v_2, v_3] = -kv_1\,.$$

All other sections satisfying (4.17) *are given by $w_1 = \alpha v_1 - \beta v_2$, $w_2 = \beta v_1 + \alpha v_2$, $w_3 = v_3$, where $\alpha^2 + \beta^2 = 1$ and $\alpha =$* const.

On the manifolds M and N, be given G-structures B_G and B_G^1 resp. of the type described in Lemma 4.5; suppose $k = k'$. In a neighbourhood of a point $m_0 \in M$, let us choose a section of B_G satisfying (4.17), similarly, in a neighbourhood of a point $n_0 \in N$, let us choose a section $\{u_1, u_2, u_3\}$ satisfying analoguous equations $[u_1, u_2] = u_3$, $[u_1, u_3] = ku_2$, $[u_2, u_3] = -ku_1$. Consider the manifold $M \times N$ and, in a suitable neighbourhood of the point (m_0, n_0), the vector fields

$$V_1 = u_1^* + \alpha v_1^* - \beta v_2^* , \quad V_2 = u_2^* + \beta v_1^* + \alpha v_2^* , \quad V_3 = u_3^* + v_3^* ;$$

here, $\alpha = \text{const.}$, $\alpha^2 + \beta^2 = 1$ and the vector fields u_i^*, v_i^* are given by the conditions $(\mathrm{d}\pi_1) v_i^* = v_i$, $(\mathrm{d}\pi_2) v_i^* = 0$, $(\mathrm{d}\pi_1) u_i^* = u_i$, $(\mathrm{d}\pi_2) u_i^* = 0$, $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ being the natural projections. It is easy to see that

$$[V_1, V_2] = V_3 , \quad [V_1, V_3] = kV_2 , \quad [V_2, V_3] = -kV_1 .$$

Thus the distribution determined on a neighbourhood of $(m_0, n_0) \in M \times N$ by means of the vector fields V_1, V_2, V_3 is completely integrable, and it has an integral manifold going through the point (m_0, n_0). This integral manifold is then a local diffeomorphism transforming B_G into B'_G.

Finally, let us investigate *transitive G-structures*. One type of these structures is given by Lemma 4.5. Consider the type given by Lemma 4.2 with $b_2 + c_1 \neq 0$. The functions a_1, a_2, b_2, c_1 being constant, we get $a_2 c_1 = a_1 b_2 = 0$ from (4.9). Thus we obtain

Theorem 4.1. *Let B_G be a transitive G-structure on M. Then it is possible (in a suitable neighbourhood of each point $m_0 \in M$) to choose its section $\{v_1, v_2, v_3\}$ in such a way that*

(4.18) $$[v_1, v_2] = av_1 + v_3 , \quad [v_1, v_3] = 0 , \quad [v_2, v_3] = cv_1 ;$$
$$a, c = \text{const.} , \quad c \neq 0 ;$$

or

(4.19) $$[v_1, v_2] = v_3 , \quad [v_1, v_3] = bv_2 , \quad [v_2, v_3] = cv_1 ;$$
$$b, c = \text{const.} , \quad b \neq 0 , \quad c \neq 0 , \quad b + c \neq 0 ;$$

or

(4.20) $$[v_1, v_2] = v_3 , \quad [v_1, v_3] = kv_2 , \quad [v_2, v_3] = -kv_1 ;$$
$$k = \text{const.} ;$$

respectively.

It is easy to verify that the transitive G-structures of all the types of Theorem 4.1 do exist. First of all, consider the G-structure of the type (4.18) on a manifold M. Let $m_0 \in M$, then there is a coordinate neighbourhood about m_0 with local coordinates (x, y, z) such that

(4.21) $$v_1 = \frac{\partial}{\partial x} , \quad v_2 = (ax - cz) \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} , \quad v_3 = \frac{\partial}{\partial z} .$$

To proceed further, a simple check shows us that the vector fields

$$(4.22)\qquad u_1 = \tfrac{1}{2}(1 + 2y - 3x^2)\frac{\partial}{\partial x} + \tfrac{1}{2}(2x + z - 3xy)\frac{\partial}{\partial y} + \tfrac{3}{2}(y - xz)\frac{\partial}{\partial z},$$

$$u_2 = \tfrac{1}{2}(1 - 2y + 3x^2)\frac{\partial}{\partial x} + \tfrac{1}{2}(2x - z + 3xy)\frac{\partial}{\partial y} + \tfrac{3}{2}(y + xz)\frac{\partial}{\partial z},$$

$$u_3 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}$$

on $\mathscr{R}^3$ satisfy

$$(4.23)\qquad [u_1, u_2] = u_3, \quad [u_1, u_3] = u_2, \quad [u_2, u_3] = u_1.$$

In a suitable neighbourhood of the point $(\frac{1}{4}\pi, 0, 0) \in \mathscr{R}^3$, consider the vectors fields

$$(4.24)\qquad w_1 = \sin(y + z)\frac{\partial}{\partial x} + \frac{\cos x}{\sin x}\cos(y + z)\frac{\partial}{\partial y} - \frac{\sin x}{\cos x}\cos(y + z)\frac{\partial}{\partial z},$$

$$w_2 = \cos(y + z)\frac{\partial}{\partial x} - \frac{\cos x}{\sin x}\sin(y + z)\frac{\partial}{\partial y} + \frac{\sin x}{\cos x}\sin(y + z)\frac{\partial}{\partial z},$$

$$w_2 = \frac{\partial}{\partial y} + \frac{\partial}{\partial z};$$

the direct check proves

$$(4.25)\qquad [w_1, w_2] = 2w_3, \quad [w_1, w_3] = -2w_2, \quad [w_2, w_3] = 2w_1.$$

Now, the Lie algebra (4.20) with $k = 0$ is realized by (4.21) with $a = c = 0$. The Lie algebras (4.19) and (4.20) with $k \neq 0$ are of the form

$$(4.26)\qquad [v_1, v_2] = v_3, \quad [v_1, v_3] = Bv_2, \quad [v_2, v_3] = Cv_1; \quad BC \neq 0.$$

The realizations of the Lie algebras (4.26) are as follows:

(4.27)

$$v_1 = \sqrt{B}\,.\,u_1, \quad v_2 = \sqrt{C}\,.\,u_2, \quad v_3 = \sqrt{(BC)}\,.\,u_3 \quad \text{for } B > 0,\ C > 0,$$

$$v_1 = \sqrt{B}\,.\,u_1, \quad v_2 = \sqrt{(-C)}\,.\,u_3, \quad v_3 = \sqrt{(-BC)}\,.\,u_2 \quad \text{for } B > 0,\ C < 0,$$

$$v_1 = \tfrac{1}{2}\sqrt{(-B)}\,.\,w_1, \quad v_2 = \tfrac{1}{2}\sqrt{C}\,.\,w_2, \quad v_3 = \tfrac{1}{2}\sqrt{(-BC)}\,.\,w_3 \quad \text{for } B < 0,\ C > 0,$$

$$v_1 = \sqrt{(-B)}\,.\,u_2, \quad v_2 = \sqrt{(-C)}\,.\,u_1, \quad v_3 = -\sqrt{(BC)}\,.\,u_3 \quad \text{for } B < 0,\ C < 0.$$

Now, we are in the position to prove Theorem 1.1. The equivalence properties have been proved above. Now, let $M \subset \mathscr{C}_s^2$ be a 3-dimensional submanifold. It is clear that $\dim G_s(M) \leqq 4$ and $\dim G_s(M) = 4$ if and only if the induced G-structure B_G over M is of the type (4.20). Thus, it is sufficient to prove

Theorem 4.2. *The hypersurfaces* $N_r^3, N_R^3, N_0^3 \subset \mathscr{C}_s^2$, *i.e., the hypersurfaces* $(z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2)$

$$
\begin{aligned}
(4.28) \qquad & N_r^3 : x_1^2 + y_1^2 + x_2^2 + y_2^2 = r^2 \quad (r > 0), \\
& N_R^3 : \qquad x_1 x_2 + y_1 y_2 = R \quad (R > 0), \\
& N_0^3 : \qquad y_2^2 - 2y_1^2 = 0
\end{aligned}
$$

have the induced G-structure which is reducible to the M-structure B_M (see Lemma 4.5) *of the type* (4.20) *with*

$$
(4.29) \qquad k_r = -\frac{4}{\sqrt[3]{(4r^2)}}, \quad k_R = \frac{1}{\sqrt[3]{R^2}}, \quad k_0 = 0.
$$

Proof. First of all, consider the hypersurface N_r^3. On $\mathscr{R}^4$, consider the vector fields

$$
\begin{aligned}
(4.30) \qquad v_1 &= \frac{1}{\sqrt{(2r)}}\left(y_2 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial y_2}\right), \\
v_2 &= \frac{1}{\sqrt[3]{(2r)}}\left(-x_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial x_2} - y_1 \frac{\partial}{\partial y_2}\right), \\
v_3 &= \frac{2}{\sqrt[3]{(4r^2)}}\left(y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial y_2}\right).
\end{aligned}
$$

These vector fields have the following properties: (i) they satisfy (4.20), k being the k_r given by (4.29_1); (ii) in the points of N_r^3, the considered vector fields are tangent to it; (iii) $Iv_1 = v_2$; (iv) $[v_1, v_2, v_3, Iv_3] = 1$ on N_r^3. For N_R^3 and N_0^3, we have similar results using the vector fields

$$
\begin{aligned}
(4.31) \qquad v_1 &= \frac{1}{2\sqrt[3]{R}}\left(x_1 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_1} - x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2}\right), \\
v_2 &= \frac{1}{2\sqrt[3]{R}}\left(y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial x_2} - x_2 \frac{\varrho}{\partial y_2}\right), \\
v_3 &= \frac{1}{2\sqrt[3]{R^2}}\left(- y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_2}\right)
\end{aligned}
$$

or

$$
(4.32) \qquad v_1 = \frac{1}{\sqrt[3]{2}}\left(\frac{\partial}{\partial x_1} + 2y_1 \frac{\partial}{\partial x_2}\right), \quad v_2 = \frac{1}{\sqrt[3]{2}}\left(\frac{\partial}{\partial y_1} + 2y_1 \frac{\partial}{\partial y_2}\right), \quad v_3 = \sqrt[3]{2}\,\frac{\partial}{\partial x_2}
$$

respectively.

Consider the space $\mathscr{C}^2$ and the pseudogroup Γ_s of its local maps. The relation between the one-parametric local subgroups of the pseudogroup Γ and the holomorphic vector fields on $\mathscr{C}^2$ is well known. Let

$$v = A(z_1, z_2)\frac{\partial}{\partial z_1} + B(z_1, z_2)\frac{\partial}{\partial z_2} \tag{5.1}$$

be a holomorphic vector field on $\mathscr{C}^2$; the corresponding local group G_v consisting of the transformations

$$\varphi_t : \tilde{z}_1 = f(z_1, z_2, t)\,, \quad \tilde{z}_2 = g(z_1, z_2, t)\,, \quad t \in (-\varepsilon, \varepsilon)\,, \tag{5.2}$$

is given by the differential equations

$$\begin{aligned} \frac{\partial f(z_1, z_2, t)}{\partial t} &= A(f(z_1, z_2, t),\ g(z_1, z_2, t))\,, \\ \frac{\partial g(z_1, z_2, t)}{\partial t} &= B(f(z_1, z_2, t),\ g(z_1, z_2, t))\,, \\ f(z_1, z_2, 0) = z_1\,, &\quad g(z_1, z_2, 0) = z_2\,. \end{aligned} \tag{5.3}$$

Theorem 5.1. *Consider the vector field* (5.1) *on* C^2. *Then* $G_v \subset \Gamma_s$ *if and only if*

$$\operatorname{Re}\left(\frac{\partial A(z_1, z_2)}{\partial z_1} + \frac{\partial B(z_1, z_2)}{\partial z_2}\right) = 0\,; \tag{5.4}$$

here, $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$.

Proof. Let us write

$$D(z_1, z_2, t) = \frac{\partial f(z_1, z_2, t)}{\partial z_1}\frac{\partial g(z_1, z_2, t)}{\partial z_2} - \frac{\partial f(z_1, z_2, t)}{\partial z_2}\frac{\partial g(z_1, z_2, t)}{\partial z_1}\,,$$

we have $D(z_1, z_2, 0) = 1$. Then

$$\frac{\partial D}{\partial t} = \frac{\partial^2 f}{\partial z_1\,\partial t}\frac{\partial g}{\partial z_2} + \frac{\partial f}{\partial z_1}\frac{\partial^2 g}{\partial t_2\,\partial t} - \frac{\partial^2 f}{\partial t_2\,\partial t}\frac{\partial g}{\partial z_1} - \frac{\partial f}{\partial z_2}\frac{\partial^2 g}{\partial z_1\,\partial t}$$

from (5.3), we have

$$\frac{\partial^2 f}{\partial z_1\,\partial t} = \frac{\partial A}{\partial z_1}\frac{\partial f}{\partial z_1} + \frac{\partial A}{\partial z_2}\frac{\partial g}{\partial z_1}\,, \quad \frac{\partial^2 f}{\partial z_2\,\partial t} = \frac{\partial A}{\partial z_1}\frac{\partial f}{\partial z_2} + \frac{\partial A}{\partial z_2}\frac{\partial g}{\partial z_2}\,,$$

$$\frac{\partial^2 g}{\partial z_1\,\partial t} = \frac{\partial B}{\partial z_1}\frac{\partial f}{\partial z_1} + \frac{\partial B}{\partial z_2}\frac{\partial g}{\partial z_1}\,, \quad \frac{\partial^2 g}{\partial z_2\,\partial t} = \frac{\partial B}{\partial z_1}\frac{\partial f}{\partial z_2} + \frac{\partial B}{\partial z_2}\frac{\partial g}{\partial z_2}\,.$$

Hence,

$$\frac{\partial D}{\partial t} = \left(\frac{\partial A}{\partial z_1} + \frac{\partial B}{\partial z_2}\right) D$$

and

$$\frac{\partial(D\bar{D})}{\partial t} = \left(\frac{\partial A}{\partial z_1} + \frac{\partial B}{\partial z_2} + \overline{\frac{\partial A}{\partial z_1}} + \overline{\frac{\partial B}{\partial z_2}}\right) D\bar{D}\,.$$

The theorem follows easily.

Now, let us study submanifolds $M^3 \subset \mathscr{C}^2$ for which the Lie algebra of the group $G_s(M^3)$ is given by (1.18), i.e.,

$$(5.5) \qquad [v_1, v_2] = av_1 + v_3\,, \quad [v_1, v_3] = 0\,,$$
$$[v_2, v_3] = cv_1\,; \quad c \neq 0\,.$$

Lemma 5.1. *Let $\mathscr{L}_s$ be the Lie algebra of vector fields (5.1) on $\mathscr{C}^2$ satisfying (5.4). Let $L \subset \mathscr{L}_s$ be the subalgebra (5.5). Then $a = 0$ or $v_3 = A(z_1, z_2)\, v_1$ for some function A.*

Proof. It is always possible to choose the coordinates (z_1, z_2) of $\mathscr{C}^2$ in such a way that (at least locally)

$$(5.6) \qquad v_1 = \frac{\partial}{\partial z_1}\,.$$

Let

$$v_3 = \alpha(z_1, z_2)\frac{\partial}{\partial z_1} + \beta(z_1, z_2)\frac{\partial}{\partial z_2}\,; \quad \frac{\partial \alpha}{\partial z_1} + \frac{\partial \beta}{\partial z_2} = \varkappa i\,, \quad \varkappa \in \mathscr{R}\,.$$

We have $\partial\alpha/\partial z_1 = 0$, $\partial\beta/\partial z_1 = 0$ from (5.5_2). Thus the vector field v_3 may be written as

$$(5.7) \qquad v_3 = \alpha(z_2)\frac{\partial}{\partial z_1} + (\varkappa i z_2 + \lambda)\frac{\partial}{\partial z_2}\,; \quad \varkappa, \lambda \in \mathscr{R}\,.$$

Suppose $\varkappa \neq 0$. Let us choose a solution $\mu(z_2)$ of the differential equation

$$(5.8) \qquad \frac{\partial \mu(z_2)}{\partial z_2} = -\frac{\alpha(z_2)}{\varkappa i z_2 + \lambda}$$

and the transformation of coordinates given by $\zeta_1 = z_1 + \mu(z_2)$, $\zeta_2 = z_2 - i\varkappa^{-1}\lambda$. Of course,

$$\det \frac{\partial(\zeta_1, \zeta_2)}{\partial(z_1, z_2)} = 1\,,$$

and we have

$$(5.9) \qquad v_1 = \frac{\partial}{\partial \zeta_1}, \quad v_3 = \varkappa i \zeta_2 \frac{\partial}{\partial \zeta_2}$$

in the new coordinates. Further, suppose $\varkappa = 0$, $\lambda \neq 0$. In this case, let us consider the transformation of coordinates given by $\zeta_1 = z_1 + \mu(z_2)$, $\zeta_2 = z_2$, $\mu(z_2)$ being again a solution of (5.8). In the new coordinates,

$$(5.10) \qquad v_1 = \frac{\partial}{\partial \zeta_1}, \quad v_3 = \lambda \frac{\partial}{\partial \zeta_2}.$$

Thus, in suitable coordinates, the vector field v_3 may be written as

$$(5.11) \quad v_3 = \varkappa i z_2 \frac{\partial}{\partial z_2} \quad \text{or} \quad v_3 = \lambda \frac{\partial}{\partial z_2} \quad \text{or} \quad v_3 = \alpha(z_2) \frac{\partial}{\partial z_1} \quad \text{respectively}.$$

Suppose that v_3 is the vector field (5.11_1), let

$$(5.12) \quad v_2 = \varrho(z_1, z_2) \frac{\partial}{\partial z_1} + \sigma(z_1, z_2) \frac{\partial}{\partial z_2}; \quad \frac{\partial \varrho}{\partial z_1} + \frac{\partial \sigma}{\partial z_2} = \varkappa' i, \quad \varkappa' \in \mathscr{R}.$$

From $(5.5_{1,3})$, we obtain

$$(5.13) \qquad \frac{\partial \varrho}{\partial z_1} = a, \quad \frac{\partial \sigma}{\partial z_1} = \varkappa i z_2, \quad -\varkappa i z_2 \frac{\partial \varrho}{\partial z_2} = c, \quad \sigma = y \frac{\partial \sigma}{\partial z_2},$$

and we get

$$\frac{\partial \sigma}{\partial z_2} = \varkappa' i - a, \quad \sigma = (\varkappa' i - a) z_2, \quad \text{i.e.,} \quad \frac{\partial \sigma}{\partial z_1} = 0$$

from (5.12_2) and $(5.13_{1,4})$. It follows from (5.13_2) that $\varkappa = 0$, i.e., $v_3 = 0$, this being impossible. Further, suppose that v_3 is given by (5.11_2) and v_2 by (5.12). Then we obtain

$$(5.14) \qquad \frac{\partial \varrho}{\partial z_1} = a, \quad \frac{\partial \sigma}{\partial z_1} = \lambda, \quad -\lambda \frac{\partial \varrho}{\partial z_2} = c, \quad \frac{\partial \sigma}{\partial z_2} = 0$$

from $(5.5_{1,3})$, and $(5.14_{1,4})$ yield $a = \varkappa' i$, i.e., $a = \varkappa' = 0$ because of $a \in \mathscr{R}$. Q.E.D.

Theorem 5.2. *Consider the manifold $N^3 \subset \mathscr{C}^2$ given by*

$$(5.15) \qquad (z_1 - \bar{z}_1)^2 + c^3(z_2 - \bar{z}_2) + 4 = 0, \quad 0 \neq c \in \mathscr{R}.$$

Its group $G_s(N^3)$ is

$$(5.16) \qquad z_1' = m z_1 - c^3 n z_2 + p, \quad z_2' = n z_1 + m z_2 + q,$$

where $m, n, p, q \in \mathscr{R}$, $m^2 + c^3 n^2 = 1$.

Let M^3 be a manifold such that the Lie algebra of $G_s(M^3)$ is (5.5). Then $a = 0$, and the manifolds M^3 and N^3 are (locally) Γ-equivalent.

Proof. It is easy to see that (5.16) preserves N^3. Using the usual coordinates $z_i = x_i + iy_i$, N^3 as a submanifold of $\mathscr{R}^4$ is given by

$$y_1^2 + c^3 y_2^2 = 1 . \tag{5.17}$$

On $\mathscr{R}^4$, consider the vector fields

$$\begin{aligned} v_1 &= c^2 y_2 \frac{\partial}{\partial x_1} - \frac{1}{c} y_1 \frac{\partial}{\partial x_2}, \\ v_2 &= c^2 y_2 \frac{\partial}{\partial y_1} - \frac{1}{c} y_1 \frac{\partial}{\partial y_2}, \\ v_3 &= c y_1 \frac{\partial}{\partial x_1} + c y_2 \frac{\partial}{\partial x_2}. \end{aligned} \tag{5.18}$$

We see easily that the vector fields (5.18) have the following properties: (i) $Iv_1 = v_2$; (ii) they satisfy (5.5) with $a = 0$; (iii) restricted to the points of N^3, they are tangent to it; (iv) we have $[v_1, v_2, v_3, Iv_3] = 1$ at the points of N^3. Thus N^3 is the model for manifolds M^3 of the type (5.5) with $a = 0$. Now suppose that M^3 admits the group $G_s(M^3)$, the lie algebra of which is (5.5) with $a \neq 0$. The manifold M^3 may be constructed as follows. First of all, realize the Lie algebra $L(5.5)$ as a subalgebra $L \subset \mathscr{L}_s$. The vector fields v_1, v_2, v_3 being considered as vector fields on $\mathscr{R}^4$, they span an integrable 3-dimensional distribution Δ. Now, M^3 is an integral manifold of Δ. According to Lemma 5.1, we may choose the coordinates (z_1, z_2) in $\mathscr{C}^2$ in such way that

$$v_1 = \frac{\partial}{\partial z_1}, \quad v_3 = \alpha(z_1, z_2) \frac{\partial}{\partial z_1} = (F + iG) \frac{\partial}{\partial z_1}.$$

These vector fields regarded as vector fields on $\mathscr{R}^4$ are

$$v_1 = \frac{\partial}{\partial x_1}, \quad v_3 = F \frac{\partial}{\partial x_1} + G \frac{\partial}{\partial y_1}. \tag{5.19}$$

Let

$$v_2 = A \frac{\partial}{\partial x_1} + B \frac{\partial}{\partial y_1} + C \frac{\partial}{\partial x_2} + D \frac{\partial}{\partial y_2}. \tag{5.20}$$

The distribution Δ is determined by the vector fields (5.19) and (5.20). Let M^3 be its integral manifold. The plane τ_m being obviously spanned by the vectors $\partial/\partial x_1$, $\partial/\partial y_1$ at each point $m \in M^3$, the distribution $\{\tau_m\}$ is integrable. This is a contradiction as we have excluded such manifolds from our considerations. Q.E.D.

Let us now study manifolds M^3 such that the Lie algebra of $C_s(M^3)$ is of the type (1.19). First of all, let us prove several lemmas.

Lemma 5.2. *Consider the Lie algebra* L^+

$$[v_1, v_2] = v_2\,, \quad [v_1, v_3] = -v_3\,, \quad [v_2, v_3] = -2v_1 \tag{5.21}$$

over $\mathscr{R}$. *The algebra* L^+ *is decomposed in a 1-parametric set of hyperboloids*

$$H_k = \{xv_1 + yv_2 + zv_3;\; x^2 - 4yz = k\}\,, \quad k \in \mathscr{R}\,, \tag{5.22}$$

with the following property: Let $v \in H_k$, $v' \in H_{k'}$; $k = k'$ *if and only there is an automorphism* $\mathscr{A} : L^+ \to L^+$ *satisfying* $\mathscr{A}v = v'$. *The vector*

$$v_+ = \tfrac{1}{2}(v_2 - v_3)\sqrt{k} \quad \text{or} \quad v_- = \tfrac{1}{2}(v_2 + v_3)\sqrt{(-k)} \quad \text{or} \tag{5.23}$$
$$v_0 = v_2 \quad \textit{respectively}$$

is situated in H_k *for* $k > 0$ *or* $k < 0$ *or* $k = 0$ *respectively.*

Proof. L^+ has the following automorphism:

$$\begin{aligned}
\mathscr{A}_1 v_1 &= av_1 + bv_2 + cv_3\,, \\
\mathscr{A}_1 v_2 &= A\left(v_1 + \frac{b}{a-1}\,v_2 + \frac{c}{a+1}\,v_3\right), \\
\mathscr{A}_1 v_3 &= B\left(v_1 + \frac{b}{a+1}\,v_2 + \frac{c}{a-1}\,v_3\right), \\
& a^2 \neq 1\,, \quad 4bc = a^2 - 1 = AB\,;
\end{aligned} \tag{5.24}$$

$$\begin{aligned}
\mathscr{A}_2 v_1 &= v_1 + av_2\,, \quad \mathscr{A}_2 v_2 = bv_2\,, \\
\mathscr{A}_2 v_3 &= A(2av_1 + a^2 v_2 + v_3)\,, \quad Ab = 1\,;
\end{aligned} \tag{5.25}$$

$$\begin{aligned}
\mathscr{A}_3 v_1 &= v_1 + av_3\,, \quad \mathscr{A}_3 v_2 = A(2av_1 + v_2 + a^2 v_3)\,, \\
\mathscr{A}_3 v_3 &= bv_3\,, \quad Ab = 1\,;
\end{aligned} \tag{5.26}$$

$$\begin{aligned}
\mathscr{A}_4 v_1 &= -v_1 + av_2\,, \quad \mathscr{A}_4 v_2 = A(-2av_1 + a^2 v_2 + v_3)\,, \\
\mathscr{A}_4 v_3 &= bv_2\,, \quad Ab = 1\,;
\end{aligned} \tag{5.27}$$

$$\begin{aligned}
\mathscr{A}_5 v_1 &= -v_1 + av_3\,, \quad \mathscr{A}_5 v_2 = bv_3\,, \\
\mathscr{A}_5 v_3 &= A(-2av_1 + v_2 + a^2 v_3)\,, \quad Ab = 1\,.
\end{aligned} \tag{5.28}$$

The vector $v = xv_1 + yv_2 + zv_3$ be called interior (or exterior resp.) if $x^2 - 4yz > 0$ (or $x^2 - 4yz > 0$ respectively); the set of interior (exterior) vectors be denoted by H^+ (H^- respectively). (1) Consider the vector $v = \alpha v_1 + \beta v_2 + \gamma v_3 \notin H_0$. ($1_1$)

Let $\gamma \neq 0$. If $v \in H^+$, we have $4\beta\gamma - \alpha^2 > 0$; choosing $\mathscr{A}_2$ (5.25) with

$$A = \tfrac{1}{2}\sqrt{(4\beta\gamma - \alpha^2)}\,, \quad a = -\frac{\alpha}{\sqrt{(4b\gamma - \alpha^2)}}\,, \quad b = \frac{1}{A}\,,$$

we obtain $\mathscr{A}_2 v = \frac{1}{2}(v_2 + v_3)\sqrt{(4\beta\gamma - \alpha^2)}$. If $v \in H^-$, we have $\alpha^2 - 4\beta\gamma > 0$; choosing $\mathscr{A}_2$ (5.25) with

$$A = -\frac{1}{2\gamma}\sqrt{(\alpha^2 - 4b\gamma)}\,, \quad a = \frac{\alpha}{\sqrt{(\alpha^2 - 4\beta\gamma)}}\,, \quad b = \frac{1}{A}\,,$$

we obtain $\mathscr{A}_2 v = \frac{1}{2}(v_2 - v_3)\sqrt{(\alpha^2 - 4\beta\gamma)}$. (1_2) Let $\gamma = 0$, $\beta \neq 0$. Then $v \in H^-$; choosing $\mathscr{A}_3$ (5.26) with $A = \frac{1}{2}|\alpha|$, $a = -\beta^{-1}\operatorname{sgn}\alpha$, $b = A^{-1}$, we obtain $\mathscr{A}_3 v = \frac{1}{2}|\alpha| \cdot (v_2 - v_3)$. (1_3) Let $\beta = \gamma = 0$. Then $v \in H^-$; choosing $\mathscr{A}_1$ (5.24) with $a = 0$, $b = \frac{1}{2}\operatorname{sgn}\alpha$, $c = -\frac{1}{2}\operatorname{sgn}\alpha$, $A = 1$, $B = -1$, we obtain $\mathscr{A}_1 v = \frac{1}{2}|\alpha| \cdot (v_2 - v_3)$. (2) Suppose $v \in H_0$, i.e., $4\beta\gamma - \alpha^2 = 0$. (2_1) Let $\beta \neq 0$. Choosing $\mathscr{A}_3$ (5.26) with $A = \beta^{-1}$, $a = -\frac{1}{2}\alpha$, $b = \beta$, we have $\mathscr{A}_3 v = v_2$. (2_2) Let $\beta = 0$. Then $\alpha = 0$; choosing $\mathscr{A}_4$ (5.27) with $A = \gamma$, $b = \gamma^{-1}$, $a = 0$, we obtain $\mathscr{A}_4 v = v_2$. Q.E.D.

Lemma 5.3. *Let $(v, \tilde{v})$ be a couple of vectors of the Lie algebra L^+. For a suitable automorphism $\mathscr{A} : L^+ \to L^+$, the couple $w = \mathscr{A}v$, $\tilde{w} = \mathscr{A}\tilde{v}$ becomes one of the following couples:*

$$\begin{array}{llll}
(5.29) & w = k(v_2 - v_3)\,, & \tilde{w} = l_1 v_2 + l_2 v_3\,, & k(l_1 + l_2) \neq 0\,; \\
& w = k(v_2 - v_3)\,, & \tilde{w} = l_1 v_1 + l_2(v_2 - v_3)\,, & kl_1 \neq 0\,; \\
& w = k(v_2 - v_3)\,, & \tilde{w} = l(v_1 \pm v_2)\,, & kl \neq 0\,; \\
& w = k(v_2 - v_3)\,, & \tilde{w} = 2v_1 + v_2 + v_3\,, & k \neq 0\,; \\
& w = k(v_2 - v_3)\,, & \tilde{w} = l(v_2 - v_3)\,, & k \neq 0\,; \\
& w = k(v_2 + v_3)\,, & \tilde{w} = l_1 v_2 + l_2 v_3\,, & k \neq 0\,; \\
& w = kv_2\,, & \tilde{w} = l_1 v_2 + l_2 v_3\,, & kl_2 \neq 0\,; \\
& w = kv_2\,, & \tilde{w} = lv_1\,, & kl \neq 0\,; \\
& w = kv_2\,, & \tilde{w} = lv_2\,, & k \neq 0\,; \\
& w = 0\,, & \tilde{w} = k(v_2 - v_3)\,; & \\
& w = 0\,, & \tilde{w} = k(v_2 + v_3)\,; & \\
& w = 0\,, & \tilde{w} = kv_2\,. &
\end{array}$$

Proof. The automorphisms $\mathscr{A} : L^+ \to L^+$ satisfying

$$(5.30) \qquad \mathscr{A}(v_2 - v_3) = v_2 - v_3 \quad \text{or} \quad \mathscr{A}(v_2 + v_3) = v_2 + v_3 \quad \text{or} \quad \mathscr{A}v_2 = v_2 \quad \text{respectively}$$

are

$$(5.31)\qquad \begin{aligned} \mathscr{A}v_1 &= av_1 + \alpha v_2 + \alpha v_3\,,\\ \mathscr{A}v_2 &= 2\alpha v_1 + \tfrac{1}{2}(a+1)\,v_2 + \tfrac{1}{2}(a-1)\,v_3\,,\\ \mathscr{A}v_3 &= 2\alpha v_1 + \tfrac{1}{2}(a-1)\,v_2 + \tfrac{1}{2}(a+1)\,v_3\,,\quad a^2 - 1 = 4\alpha^2\,, \end{aligned}$$

or

$$(5.32)\qquad \begin{aligned} \mathscr{A}v_1 &= av_1 + \alpha v_2 - \alpha v_3\,,\\ \mathscr{A}v_2 &= -2\alpha v_1 + \tfrac{1}{2}(a+1)\,v_2 - \tfrac{1}{2}(a-1)\,v_3\,,\\ \mathscr{A}v_3 &= 2\alpha v_1 - \tfrac{1}{2}(a-1)\,v_2 + \tfrac{1}{2}(a+1)\,v_3\,,\quad 1 - a^2 = 4\alpha^2\,, \end{aligned}$$

or

$$(5.33)\qquad \begin{aligned} &\mathscr{A}v_1 = v_1 + av_2\,,\quad \mathscr{A}v_2 = v_2\,,\\ &\mathscr{A}v_3 = 2av_1 + a^2v_2 + v_3 \end{aligned}$$

respectively. Be given a vector $u = \varrho_1v_1 + \varrho_2v_2 + \varrho_3v_3 \in L^+$. (i) There is an automorphism $\mathscr{A}$ (5.31) such that

$$(5.34)\qquad \mathscr{A}u = \sigma_1v_2 + \sigma_2v_3 \qquad \text{for}\quad (\varrho_2+\varrho_3)^2 - \varrho_1^2 > 0$$

or

$$(5.35)\qquad \mathscr{A}u = \sigma_3v_1 + \sigma_4(v_2 - v_3) \quad \text{for}\quad (\varrho_2+\varrho_3)^2 - \varrho_1^2 < 0$$

or

$$(5.36)\qquad \mathscr{A}u = \sigma_5(v_1 \pm v_2) \qquad \text{for}\quad (\varrho_2+\varrho_3)^2 = \varrho_1^2\,,\quad \varrho_1(\varrho_2^2 - \varrho_3^2) \neq 0$$

or

$$(5.37)\qquad \mathscr{A}u = 2v_1 + v_2 + v_3 \quad \text{for}\quad \varrho_2 = \varrho_3\,,\quad \varrho_1^2 = 4\varrho_2^2\,,\quad \varrho_2 \neq 0$$

respectively. Indeed, use the automorphism $\mathscr{A}$ (5.31) with

$$a = -(\varrho_2+\varrho_3)\{(\varrho_2+\varrho_3)^2 - \varrho_1^2\}^{-1/2}\,,\quad \alpha = \tfrac{1}{2}\varrho_1\{(\varrho_2+\varrho_3)^2 - \varrho_1^2\}^{-1/2}$$

or

$$a = \varrho_1\{\varrho_1^2 - (\varrho_2+\varrho_3)^2\}^{-1/2}\,,\quad \alpha = -\tfrac{1}{2}(\varrho_2+\varrho_3)\{\varrho_1^2 - (\varrho_2+\varrho_3)^2\}^{-1/2}$$

or

$$a = (\varrho_2^2 + \varrho_3^2)(\varrho_2^2 - \varrho_3^2)^{-1}\,,\quad \alpha = -\varrho_1^{-1}\varrho_2\varrho_3(\varrho_2 - \varrho_3)^{-1}$$

or

$$a = \tfrac{1}{2}\varrho_2^{-1}(\varrho_2^2 + 1)\,,\quad \alpha = \tfrac{1}{4}\varrho_2^{-1}(\varrho_2^2 - 1)$$

respectively. (ii) There exists an automorphism $\mathscr{A}$ (5.32) such that

$$(5.38)\qquad \mathscr{A}u = \sigma_6v_2 + \sigma_7v_3 \quad \text{for}\quad \varrho_1^2 + (\varrho_2 - \varrho_3)^2 \neq 0\,;$$

it is sufficient to take

$$a = (\varrho_2 - \varrho_3)\{\varrho_1^2 + (\varrho_2-\varrho_3)^2\}^{-1/2}\,,\quad \alpha = \tfrac{1}{2}\varrho_1\{\varrho_1^2 + (\varrho_2-\varrho_3)^2\}^{-1/2}\,.$$

For $\varrho_1 = 0$, $\varrho_3 = \varrho_2$, we have $\mathscr{A}u = u$ for each automorphism $\mathscr{A}$ (5.32). (iii) There exists an automorphism $\mathscr{A}$ (5.33) such that

$$\mathscr{A}u = \sigma_8 v_2 + \sigma_9 v_3 \quad \text{for} \quad \varrho_3 \neq 0 \tag{5.39}$$

or

$$\mathscr{A}u = \sigma_{10} v_1 \qquad \text{for} \quad \varrho_3 = 0\,, \quad \varrho_1 \neq 0 \tag{5.40}$$

respectively; it is sufficient to take

$$a = -\tfrac{1}{2}\varrho_1\varrho_3^{-1} \quad \text{or} \quad a = -\varrho_2\varrho_1^{-1} \quad \text{respectively}\,.$$

The lemma follows from what has been and from the preseding lemma.

Theorem 5.3. *Let $\mathscr{L}_s$ be the Lie algebra of vector fields* (5.1) *satisfying* (5.4) *on $\mathscr{C}^2$. Then, in a neighbourhood of a fixed point $m \in \mathscr{C}^2$, we may choose holomorphic coordinates (z_1, z_2) such that $m = (0, 1)$,*

$$v_1 = z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}\,, \quad v_3 = \frac{\partial}{\partial z_1} \tag{5.41}$$

and v_2 is one of the following vector fields:

$$v_2 = \left(z_1^2 + \frac{p + iq}{z_2^2}\right)\frac{\partial}{\partial z_1} - 2z_1z_2\frac{\partial}{\partial z_2}\,, \tag{5.42}$$

$$v_2 = \left(z_1^2 + \frac{p + iq}{z_2^2}\right)\frac{\partial}{\partial z_1} + \left(-2z_1z_2 - \frac{qr}{p} \pm qi\right)\frac{\partial}{\partial z_2}\,, \quad pq \neq 0\,,$$

$$v_2 = \left(z_1^2 + \frac{s - i}{(s + i)\,z_2^2}\right)\frac{\partial}{\partial z_1} + \left(-2z_1z_2 + \frac{2}{s^2 + 1} + i\,\frac{2s}{s^2 + 1}\right)\frac{\partial}{\partial z_2}\,, \quad s \neq 0\,,$$

$$v_2 = \left(z_1^2 + \frac{1}{z_2^2}\right)\frac{\partial}{\partial z_1} + (-2z_1z_2 + p + iq)\frac{\partial}{\partial z_2}\,,$$

$$v_2 = \left(z_1^2 - \frac{1}{z_2^2}\right)\frac{\partial}{\partial z_1} - 2z_1z_2\frac{\partial}{\partial z_2}\,,$$

$$v_2 = z_1^2\frac{\partial}{\partial z_1} + (-2z_1z_2 + ip)\frac{\partial}{\partial z_2}\,; \quad p, q, r, s \in \mathscr{R}\,.$$

Proof. In a neighbourhood of the point $m \in \mathscr{C}^2$, let us choose coordinates in such a way that (5.41_2). Now, let

$$v_1 = a\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial z_2}\,, \quad \frac{\partial a}{\partial z_1} + \frac{\partial b}{\partial z_2} = \varkappa i\,, \quad \varkappa \in \mathscr{R}\,. \tag{5.43}$$

From (5.21_2), we get the existence of functions $\varphi(z_2), \psi(z_2)$ such that $a = z_1 + \varphi(z_2)$, $b = \psi(z_2)$; the condition (5.43_2) assures the existence of a constant $\lambda \in \mathscr{C}$ such that

$$v_1 = (z_1 + \varphi(z_2)) \frac{\partial}{\partial z_1} + \{(\varkappa i - 1) z_2 + \lambda\} \frac{\partial}{\partial z_2}. \tag{5.44}$$

Let us consider a change of coordinates

$$\zeta_1 = z_1 + \Phi(z_2), \quad \zeta_2 = z_2 + \frac{\lambda}{\varkappa i - 1},$$

$\Phi(z_2)$ being a solution of the differential equation

$$\{(\varkappa i - 1) z_2 + \lambda\} \Phi'(z_2) = \Phi(z_2) \varphi(z_2).$$

We get by a direct calculation

$$v_3 = \frac{\partial}{\partial \zeta_1}, \quad v_1 = \zeta_1 \frac{\partial}{\partial \zeta_1} + (\varkappa i - 1) \zeta_2 \frac{\partial}{\partial \zeta_2};$$

writing $\zeta_i = z_i$, we get

$$v_1 = z_1 \frac{\partial}{\partial z_1} + (\varkappa i - 1) z_2 \frac{\partial}{\partial z_2}. \tag{5.45}$$

Let

$$v_2 = \alpha \frac{\partial}{\partial z_1} + \beta \frac{\partial}{\partial z_2}, \quad \frac{\partial \alpha}{\partial z_1} + \frac{\partial \beta}{\partial z_2} = \varkappa' i, \quad \varkappa' \in \mathscr{R}. \tag{5.46}$$

From (5.21_3), we get

$$\frac{\partial \alpha}{\partial z_1} = 2z_1, \quad \frac{\partial \beta}{\partial z_1} = 2(\varkappa i - 1) z_2,$$

i.e., the existence of functions $f(z_2)$, $g(z_2)$ such that

$$\alpha = z_1^2 + f(z_2), \quad \beta = 2(\varkappa i - 1) z_1 z_2 + g(z_2).$$

From (5.46_2), we get $2\varkappa i z_1 + g'(z_2) = \varkappa i$, i.e.,

$$\varkappa = 0. \tag{5.47}$$

Thus, $g(z_2) = \varkappa' i z_2 + d$; from (5.21_1),

$$z_2 f'(z_2) = -2 f(z_2), \quad \varkappa' = 0, \tag{5.48}$$

i.e., we have (5.40) and

$$v_2 = \left(z_1^2 + \frac{c}{z_2^2}\right) \frac{\partial}{\partial z_1} + (-2z_1 z_2 + d) \frac{\partial}{\partial z_2}; \quad c, d \in \mathscr{C}. \tag{5.49}$$

Thus we have proved that we may choose (at least locally) coordinates in such a way that the vector fields v_1, v_2, v_3 satisfying (5.21) are given by (5.40) and (5.49). But there is the possibility to choose another basis of L^+ satisfying (5.21). We have

$$v_2 = \left(2z_1 - \frac{d}{z_2}\right) v_1 + \left(-z_1^2 + \frac{c}{z_2} + d\frac{z_1}{z_2}\right) v_3 \,;$$

in the point $z_1 = 0$, $z_2 = 1$,

$$v_2 + dv_1 - cv_3 = 0 \,. \tag{5.50}$$

The choice of the new basis in L^+ is now to be done in such a way that (5.50) has the canonical form $w + i\tilde{w} = 0$, w and $\tilde{w}$ being given by (5.29). Q.E.D.

Lemma 5.4. *Let L^+ be the Lie algebra (5.21). Let $L \supset L^+$ be a Lie algebra with* $\dim L = 4$. *Then there is a vector $v_4 \in L - L^+$ and numbers $a_2, a_3, b_2 \in \mathscr{R}$ such that L is given by (5.21) and*

$$\begin{aligned} [v_1, v_4] &= \phantom{2a_3v_1 + {}} a_2v_2 + a_3v_3 \,, \\ [v_2, v_4] &= 2a_3v_1 + b_2v_2 \,, \\ [v_3, v_4] &= 2a_2v_1 \phantom{{}+ b_2v_2} - b_2v_3 \,. \end{aligned} \tag{5.51}$$

Proof. We may write

$$\begin{aligned} [v_1, v_4] &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 \,, \\ [v_2, v_4] &= b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4 \,, \\ [v_3, v_4] &= c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 \,. \end{aligned} \tag{5.52}$$

Consider the Jacobi identities

$$\begin{aligned} [v_1, [v_2, v_4]] + [v_2, [v_4, v_1]] + [v_4, [v_1, v_2]] &= 0 \,, \\ [v_1, [v_3, v_4]] + [v_3, [v_4, v_1]] + [v_4, [v_1, v_3]] &= 0 \,, \\ [v_2, [v_3, v_4]] + [v_3, [v_4, v_2]] + [v_4, [v_2, v_3]] &= 0 \,. \end{aligned} \tag{5.53}$$

From (5.52) and (5.53), we get (5.51). Q.E.D.

Lemma 5.5. *Consider the Lie algebra $L^+ \subset \mathscr{L}_s$ given by the vector fields (5.41) and (5.49). The Lie algebra L satisfying $L^+ \subset L \subset L_s$ and* $\dim L = 4$ *exists if and only if $c = d = 0$.*

Proof. Let L exist, let

$$v_4 = A\frac{\partial}{\partial z_1} + B\frac{\partial}{\partial z_2}\,, \quad \frac{\partial A}{\partial z_1} + \frac{\partial B}{\partial z_2} = \varkappa i\,, \quad \varkappa \in \mathscr{R}\,. \tag{5.54}$$

From (5.51_3),

$$\frac{\partial A}{\partial z_1} = 2a_2 z_1 - b_2\,, \quad \frac{\partial B}{\partial z_1} = -2a_2 z_2\,,$$

from (5.54_2),

$$\frac{\partial B}{\partial z_2} = \varkappa i - 2a_2 z_1 + b_2\,.$$

Thus, there is a function $\varphi(z_2)$ and a constant $\lambda \in \mathscr{C}$ such that

$$A = a_2 z_1^2 - b_2 z_1 + \varphi(z_2)\,, \quad B = -2a_2 z_1 z_2 + \varkappa i z_2 + b z_2 + \lambda\,.$$

From (5.51_1), we get

$$z_1(2a_2 z_1 - b_2) - z_2 \frac{\partial A}{\partial z_2} - A = a_2 z_1^2 + a_2 c \frac{1}{z_2^2} + a_3\,,$$

$$-z_2 \frac{\partial B}{\partial z_2} + B = a_2 d\,,$$

i.e., there exists a number $\mu \in \mathscr{C}$ such that

$$A = a_2 z_1^2 - b_2 z_1 + a_2 c \frac{1}{z_2^2} + a_3 + \mu \frac{1}{z_2}\,,$$

$$B = -2a_2 z_1 z_2 + \varkappa i z_2 + b_2 z_2 + a_2 d\,.$$

Finally, we obtain from (5.51_2)

$$\left(z_1^2 + \frac{c}{z_2^2}\right)(2a_2 z_1 - b_2) + (-2z_1 z_2 + d)\frac{\partial A}{\partial z_2} - 2Az_1 + 2Bc \frac{1}{z_2^2} =$$

$$= 2a_3 z_1 + b_2 \left(z_1^2 + \frac{c}{z_2^2}\right),$$

$$-2a_2 z_1^2 z_2 - 2a_2 c \frac{1}{z_2} + d \frac{\partial B}{\partial z_2} + 2z_1 \left(B - z_2 \frac{\partial B}{\partial z_2}\right) + 2Az_2 =$$

$$= -2a_3 z_2 - 2b_2 z_1 z_2 + b_2 d\,,$$

i.e.,

(5.55) $$a_3 = \mu = \varkappa d = c\varkappa \approx 0\,.$$

Suppose $a_3 = \mu = \varkappa = 0$. Then $v_4 = a_2 v_2 - b_2 v_1$ and dim $L = 3$. Hence $a_3 = \mu = c = d = 0$ and

(5.56) $$v_4 = a_2 v_2 - b_2 v_1 + i\varkappa z_2 \frac{\partial}{\partial z_2}\,.$$

Of course, we have to suppose $\varkappa \neq 0$. Q.E.D.

Theorem 5.4. *Let $M^3 \subset \mathscr{C}^2$ be a hypersurface, let the Lie algebra of the group $G_s(M^3)$ be of the type*

$$[u_1, u_2] = u_3, \quad [u_1, u_3] = bu_2, \quad [u_2, u_3] = cu_1, \tag{5.57}$$
$$c \neq 0, \quad b + c \neq 0, \quad b > 0 \quad \text{or} \quad b < 0, \quad c < 0.$$

Then M^3 is an orbit of the group G generated by the fields (5.41), (5.42); *the field*

$$u_2 = z_1^2 \frac{\partial}{\partial z_1} - 2z_1 z_2 \frac{\partial}{\partial z_2}$$

is to be excluded.

Proof. If $b > 0$, let us choose a new basis

$$v_1 = \frac{1}{\sqrt{b}} u_1, \quad v_2 = \frac{1}{c} u_2 + \frac{1}{c\sqrt{b}} u_3, \quad v_3 = u_2 - \frac{1}{\sqrt{b}} u_3,$$

if $b < 0$, $c < 0$, consider the basis

$$v_1 = -\frac{1}{\sqrt{(bc)}} u_3, \quad v_2 = -\frac{1}{b} u_1 - \frac{1}{\sqrt{(bc)}} u_2, \quad v_3 = -\sqrt{\left(\frac{b}{c}\right)} u_2.$$

Then (5.21) is satisfied and the theorem follows from Theorem 5.3 and Lemma 5.5.

6. TRANSITIVE SUBMANIFOLDS $M^4 \subset \mathscr{C}^3$

In $\mathscr{C}^3$, consider the complex coordinates $z_i = x_i + iy_i$; $i = 1, 2, 3$. The space $\mathscr{C}^3$ be identified with $\mathscr{R}^6$ in the usual way. Thus, $(\partial/\partial x_i, \partial/\partial y_i)$ is the basis of $\mathscr{R}^6$ and the known endomorphism $I : \mathscr{R}^6 \to \mathscr{R}^6$, $I^2 = -\text{id.}$, is given by

$$I \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \quad I \frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}; \quad i = 1, 2, 3.$$

In $\mathscr{C}^3 = \mathscr{R}^6$, consider a real submanifold M^4. Write $\tau_m = T_m(M^4) \cap IT_m(M^4)$, $T_m(M^4)$ being the tangent space of M^4 at $m \in M^4$. *Suppose* $\dim \tau_m = 2$ *for each* $m \in M^4$. In the principal fiber bundle $R(M^4)$ of the frames over M^4, let us choose (locally) a section $\sigma = (v_1, v_2, v_3, v_4)$ in such a way that $v_1(m) \in \tau_m$ and $Iv_1 = v_2$. The section $\tilde{\sigma} = (w_1, w_2, w_3, w_4)$ having the same property, we have

$$\begin{aligned} v_1 &= \alpha w_1 - \beta w_2, \quad v_2 = \beta w_1 + \alpha w_2, \\ v_3 &= \gamma w_1 + \delta w_2 + \varphi w_3 + \psi w_4, \\ v_4 &= Aw_1 + Bw_2 + Cw_3 + Dw_4 \end{aligned} \tag{6.1}$$

with $(\alpha^2 + \beta^2)(\varphi D - \psi C) \neq 0$.

We may write

$$
(6.2)\qquad
\begin{aligned}
[v_1, v_2] &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 ,\\
[v_1, v_3] &= b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 ,\\
[v_1, v_4] &= c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 ,\\
[v_2, v_3] &= d_1 v_1 + d_2 v_2 + d_3 v_3 + d_4 v_4 ,\\
[v_2, v_4] &= e_1 v_1 + e_2 v_2 + e_3 v_3 + e_4 v_4 ,\\
[v_3, v_4] &= f_1 v_1 + f_2 v_2 + f_3 v_3 + f_4 v_4 ,
\end{aligned}
$$

the functions $a_1, \ldots, f_4$ satisfying the Jacobi identities

$$
(6.3)\qquad
\begin{aligned}
&[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0 ,\\
&[v_1, [v_2, v_4]] + [v_2, [v_4, v_1]] + [v_4, [v_1, v_2]] = 0 ,\\
&[v_1, [v_3, v_4]] + [v_3, [v_4, v_1]] + [v_4, [v_1, v_3]] = 0 ,\\
&[v_2, [v_3, v_4]] + [v_3, [v_4, v_2]] + [v_4, [v_2, v_3]] = 0 .
\end{aligned}
$$

Let $p \in M^4$ be a fixed point, $v_0 = A_0\, v_1(p) + B_0\, v_2(p) \in \tau_p$ a given vector. Let us choose a vector field $v = Av_1 + Bv_2$ such that $v(m) \in \tau_m$ for each $m \in M^4$, and suppose $v(p) = v_0$. Then $Iv = -Bv_1 + Av_2$, and we have

$$
\begin{aligned}
[v, Iv] &= [Av_1 + Bv_2,\ -Bv_1 + Av_2] = \\
&= (\cdot)\, v_1 + (\cdot)\, v_2 + (A^2 + B^2)(a_3 v_3 + a_4 v_4) .
\end{aligned}
$$

If $L_p^{(1)} \not\equiv 0$, we do not have $a_3 = a_4 = 0$. Thus, we are in the position to choose σ in such a way that $a_4 = 0$, $a_3 \neq 0$. The space σ_p (for which definition see § 1) is spanned by the vectors $v_1(p)$, $v_2(p)$, $v_3(p)$. Further, we have

$$
[v, [v, Iv]] = (\cdot)\, v_1 + (\cdot)\, v_2 + (\cdot)\, v_3 + (Ab_4 + Bd_4)(A^2 + B^2)\, a_3 v_4 .
$$

If $L_p^{(2)} \not\equiv 0$, we do not have $b_4 = d_4 = 0$. For the vector $v' = d_4 v_1 - b_4 v_2$, $L_p^{(2)}(v') = 0$. Let us choose the section σ in such a way that $L_p^{(2)}(v_2) = 0$, i.e., $d_4 = 0$, $b_4 \neq 0$.

Thus, we consider – over M^4 – only sections $\sigma = (v_1, v_2, v_3, v_4)$ satisfying (6.2) with $a_4 = d_4 = 0$, $a_3 \neq 0$, $b_4 \neq 0$. For any other section $\tilde{\sigma} = (w_1, w_2, w_3, w_4)$ with the same property, we have (6.1) with $\beta = 0$, $\psi = 0$ and, of course,

$$
(6.3)\qquad
\begin{aligned}
[w_1, w_2] &= \tilde{a}_1 w_1 + \tilde{a}_2 w_2 + \tilde{a}_3 w_3 , \ldots ,\\
[w_3, w_4] &= \tilde{f}_1 w_1 + \tilde{f}_2 w_2 + \tilde{f}_3 w_3 + \tilde{f}_4 w_4 .
\end{aligned}
$$

Now,

$$
\begin{aligned}
[v_1, v_2] &= [\alpha w_1, \alpha w_2] = (\cdot)\, w_1 + (\cdot)\, w_2 + \alpha^2 \tilde{a}_3 w_3 = \\
&= (\cdot)\, w_1 + (\cdot)\, w_2 + \varphi a_3 w_3 ,
\end{aligned}
$$

$$[v_1, v_3] = [\alpha w_1, \gamma w_1 + \delta w_2 + \varphi w_3] = (\cdot)\, w_1 + (\cdot)\, w_2 + (\cdot)\, w_3 + \alpha\varphi \tilde{b}_4 w_4 =$$
$$= (\cdot)\, w_1 + (\cdot)\, w_2 + (\cdot)\, w_3 + D b_4 w_4 ,$$

and we have $\alpha^2 \tilde{a}_3 = \varphi a_3$, $\alpha\varphi\tilde{b}_4 = D b_4$. The section σ may be chosen in such a way that $a_3 = 1$, $b_4 = 1$; $\tilde{a}_3 = \tilde{b}_4 = 1$ implies $\varphi = \alpha^2$, $D = \alpha^3$.

Over M^4, we thus consider sections σ satisfying

(6.4)
$$\begin{aligned}
[v_1, v_2] &= a_1 v_1 + a_2 v_2 + v_3 ,\\
[v_1, v_3] &= b_1 v_1 + b_2 v_2 + b_3 v_3 + v_4 ,\\
[v_1, v_4] &= c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 ,\\
[v_2, v_3] &= d_1 v_1 + d_2 v_2 + d_3 v_3 ,\\
[v_2, v_4] &= e_1 v_1 + e_2 v_2 + e_3 v_3 + e_4 v_4 ,\\
[v_3, v_4] &= f_1 v_1 + f_2 v_2 + f_3 v_3 + f_4 v_4 .
\end{aligned}$$

For another section $\tilde{\sigma}$ with the same properties, we have

(6.5)
$$\begin{aligned}
v_1 &= \alpha w_1 ,\\
v_2 &= \qquad \alpha w_2 ,\\
v_3 &= \gamma \omega_1 + \delta w_2 + \alpha^2 w_3 ,\\
v_4 &= A w_1 + B w_2 + C w_3 + \alpha^3 w_4 , \quad \alpha \neq 0 .
\end{aligned}$$

Now,

$$\begin{aligned}
[v_1, v_2] &= [\alpha w_1, \alpha w_2] = \alpha(\alpha \tilde{a}_1 - w_2 \alpha)\, w_1 + \alpha(\alpha \tilde{a}_2 + w_1 \alpha)\, w_2 + \alpha^2 w_3 =\\
&= (\alpha a_1 + \gamma)\, w_1 + (\alpha a_2 + \delta)\, w_2 + \alpha^2 w_3 ,
\end{aligned}$$

$$\begin{aligned}
[v_1, v_3] &= [\alpha w_1, \gamma w_1 + \delta w_2 + \alpha^2 w_3] =\\
&= (\cdot)\, w_1 + (\cdot)\, w_2 + (2\alpha w_1 \alpha + \delta + \alpha^2 \tilde{b}_3)\, w_3 + \alpha^3 w_4 =\\
&= (\cdot)\, w_1 + (\cdot)\, w_2 + (\alpha^2 b_3 + C)\, w_3 + \alpha^3 w_4 ,
\end{aligned}$$

$$\begin{aligned}
[v_2, v_3] &= [\alpha w_2, \gamma w_1 + \delta w_2 + \alpha^2 w_3] =\\
&= (\cdot)\, w_1 + (\cdot)\, w_2 + (2\alpha w_2 \alpha - \gamma + \alpha^2 \tilde{d}_3)\, w_3 =\\
&= (\cdot)\, w_1 + (\cdot)\, w_2 + \alpha^2 d_3 w_3 ,
\end{aligned}$$

$$\begin{aligned}
[v_1, v_4] &= [\alpha w_1,\ A w_1 + B w_2 + \alpha^3 w_4] =\\
&= (\cdot)\, w_1 + (\cdot)\, w_2 + (\cdot)\, w_3 + \alpha(3\alpha^2 w_1 \alpha + C + \alpha^3 \tilde{c}_4)\, w_4 =\\
&= (\cdot)\, w_1 + (\cdot)\, w_2 + (\cdot)\, w_3 + \alpha^3 c_4 w_4 ,
\end{aligned}$$

and we obtain

$$\alpha w_1 \alpha + \alpha^2 \tilde{a}_2 = \alpha a_2 + \delta\,,$$
$$-\alpha w_2 \alpha + \alpha^2 \tilde{a}_1 = \alpha a_1 + \gamma\,,$$
$$2\alpha^2 w_1 \alpha + \alpha\delta + \alpha^3 \tilde{b}_3 = \alpha^2 b_3 + C\,,$$
$$2\alpha w_2 \alpha - \gamma + \alpha^2 \tilde{d}_3 = \alpha d_3\,,$$
$$3\alpha^2 w_1 \alpha + C + \alpha^3 \tilde{c}_3 = \alpha^2 c_4$$

and

$$3\alpha\delta + \alpha^3(\tilde{b}_3 - 2\tilde{a}_2) = \alpha^2(b_3 - 2a_2) + C\,,$$
$$-3\gamma + \alpha^2(\tilde{d}_3 + 2\tilde{a}_1) = \alpha(d_3 + 2a_1)\,,$$
$$C + \alpha^3(\tilde{c}_4 - 3\tilde{a}_2) = \alpha^2(c_4 - 3a_2) - 3\alpha\delta\,.$$

Thus we are in the position to choose a section σ in such a way that

$$b_3 = 2a_2\,, \quad d_3 = -2a_1\,, \quad c_4 = 3a_2\,. \tag{6.6}$$

$\tilde{\sigma}$ being another section satisfying (6.6), we have (6.5) with

$$\gamma = \delta = C = 0\,. \tag{6.7}$$

Now,

$$\begin{aligned}[v_1, v_3] = [\alpha w_1, \alpha^2 w_3] &= \alpha^2(-w_3\alpha + \alpha\tilde{b}_1)\, w_1 + \alpha^3 \tilde{b}_2 w_2 + \\ &\quad + 2\alpha^2(w_1\alpha + \alpha\tilde{a}_2)\, w_3 + \alpha^3 w_4 = \\ &= (b_1 + A)\, w_1 + (\alpha b_2 + B)\, w_2 + 2\alpha^2 a_2 w_3 + \alpha^3 w_4\,,\end{aligned}$$

$$\begin{aligned}[v_2, v_3] = [\alpha w_2, \alpha^2 w_3] &= \alpha^3 \tilde{d}_1 w_1 + \alpha^2(-w_3\alpha + \alpha\tilde{d}_2)\, w_2 + 2\alpha^2(w_2\alpha - \alpha\tilde{a}_1)\, w_3 = \\ &= \alpha d_1 w_1 + \alpha d_2 w_2 - 2\alpha^2 a_1 w_3\,;\end{aligned}$$

from these relations, we get

$$\alpha^3 \tilde{b}_2 = \alpha b_2 + B$$

and

$$\alpha^3(\tilde{b}_1 - \tilde{d}_2) = \alpha(b_1 - d_2) + A\,.$$

The section σ may be chosen in such a way that

$$b_2 = 0\,, \quad d_2 = b_1\,; \tag{6.8}$$

$\tilde{\sigma}$ being another section with the same properties, we have (6.5) with (6.7) and

$$A = B = 0\,. \tag{6.9}$$

Lemma 6.1. *Let $M^4 \subset \mathscr{C}^3$ be a submanifold with* $\dim \tau_p = 2$, $L_p^{(1)} \not\equiv 0$, $L_p^{(2)} \not\equiv 0$ *for each $p \in M^4$. Then there is a section $\sigma = (v_1, v_2, v_3, v_4)$ of $R(M^4)$ such that $v_2 = Iv_1$ and*

$$\begin{aligned}
(6.10)\qquad [v_1, v_2] &= a_1v_1 + a_2v_2 + v_3\,,\\
[v_1, v_3] &= b_1v_1 \qquad\quad + 2a_2v_3 + v_4\,,\\
[v_1, v_4] &= c_1v_1 + c_2v_2 + c_3v_3 + 3a_2v_4\,,\\
[v_2, v_3] &= d_1v_1 + b_1v_2 - 2a_1v_3\,,\\
[v_2, v_4] &= e_1v_1 + e_2v_2 + e_3v_3 + e_4v_4\,,\\
[v_3, v_4] &= f_1v_1 + f_2v_2 + f_3v_3 + f_4v_4\,.
\end{aligned}$$

$\tilde{\sigma} = (w_1, w_2, w_3, w_4)$ being another section with the same properties, we have

$$(6.11)\qquad v_1 = \alpha w_1\,, \quad v_2 = \alpha w_2\,, \quad v_3 = \alpha^2 w_3\,, \quad v_4 = \alpha^3 w_4\,; \quad \alpha \neq 0\,.$$

Now,

$$\begin{aligned}
[v_1, v_2] = [\alpha w_1, \alpha w_2] &= \alpha(\alpha\tilde{a}_1 - w_2\alpha)\, w_1 + \alpha(\alpha\tilde{a}_2 + w_1\alpha)\, w_2 + \alpha^2 w_3 =\\
&= \alpha a_1 w_1 + \alpha a_2 w_2 + \alpha^2 w_3\,,
\end{aligned}$$

i.e.,

$$(6.12)\qquad w_2\alpha + \alpha\tilde{a}_1 = a_1\,, \quad w_1\alpha + \alpha\tilde{a}_2 = a_2\,;$$

$$\begin{aligned}
[v_1, v_3] = [\alpha w_1, \alpha^2 w_3] &= \alpha^2(-w_3\alpha + \alpha\tilde{b}_1)\, w_1 + 2\alpha^2(\alpha\tilde{a}_2 + w_1\alpha)\, w_3 + \alpha^3 w_4 =\\
&= \alpha b_1 w_1 + 2\alpha^2 a_2 w_3 + \alpha^3 w_4\,,
\end{aligned}$$

i.e.,

$$(6.13)\qquad -\alpha w_3\alpha + \alpha^2\tilde{b}_1 = b_1\,;$$

$$\begin{aligned}
[v_1, v_4] = [\alpha w_1, \alpha^3 w_4] &= \alpha^3(-w_4\alpha + \alpha\tilde{c}_1)\, w_1 + \alpha^4\tilde{c}_2 w_2 + \alpha^4\tilde{c}_3 w_3 +\\
&+ 3\alpha^3(w_1\alpha + \alpha\tilde{a}_2)\, w_4 = \alpha c_1 w_1 + \alpha c_2 w_2 + \alpha^2 c_3 w_3 + 3\alpha^3 a_2 w_4\,,
\end{aligned}$$

i.e.,

$$(6.14)\qquad \alpha^3\tilde{c}_2 = c_2\,, \quad \alpha^2\tilde{c}_3 = c_3\,,$$

$$(6.15)\qquad -\alpha^2 w_4\alpha + \alpha^3\tilde{c}_1 = c_1\,;$$

$$\begin{aligned}
[v_2, v_3] = [\alpha w_2, \alpha^2 w_3] &= \alpha^3\tilde{d}_1 w_1 + \alpha^2(-w_3\alpha + \alpha\tilde{b}_1)\, w_2 + 2\alpha^2(w_2\alpha - \alpha\tilde{a}_1)\, w_3 =\\
&= \alpha d_1 w_1 + \alpha b_1 w_2 - 2\alpha^2 a_1 w_3\,,
\end{aligned}$$

i.e.,

$$(6.16)\qquad \alpha^2\tilde{d}_1 = d_1\,;$$

$$\begin{aligned}
[v_2, v_4] = [\alpha w_2, \alpha^3 w_4] &= \alpha^4\tilde{e}_1 w_1 + \alpha^3(-w_4\alpha + \alpha\tilde{e}_2)\, w_2 + \alpha^4\tilde{e}_3 w_3 +\\
&+ \alpha^3(3w_2\alpha + \alpha\tilde{e}_4)\, w_4 = \alpha e_1 w_1 + \alpha e_2 w_2 + \alpha^2 e_3 w_3 + \alpha^3 e_4 w_4\,,
\end{aligned}$$

i.e.,

$$\alpha^3\tilde{e}_1 = e_1\,, \quad \alpha^2\tilde{e}_3 = e_3\,, \tag{6.17}$$
$$\alpha^3(\tilde{c}_1 - \tilde{e}_2) = c_1 - e_2\,, \quad \alpha(3\tilde{a}_1 + \tilde{e}_4) = 3a_1 + e_4$$

by means of (6.15) and (6.12);

$$[v_3, v_4] = [\alpha^2 w_3, \alpha^3 w_4] = \alpha^5\tilde{f}_1 w_1 + \alpha^5\tilde{f}_2 w_2 + \alpha^4(-2w_4\alpha + \alpha\tilde{f}_3)\, w_3 +$$
$$+ \alpha^4(3w_3\alpha + \alpha\tilde{f}_3)\, w_4 = \alpha f_1 w_1 + \alpha f_2 w_2 + \alpha^2 f_3 w_3 + \alpha^3 f_4 w_4\,,$$

i.e.,

$$\alpha^4\tilde{f}_1 = f_1\,, \quad \alpha^4\tilde{f}_2 = f_2\,, \tag{6.18}$$
$$\alpha^3(\tilde{f}_3 - 2\tilde{c}_1) = f_3 - 2c_1\,, \quad \alpha^2(\tilde{f}_4 + 3\tilde{b}_1) = f_4 + 3b_1$$

by means of (6.15) and (6.13).

In what follows, let us restrict ourselves to manifolds M^4 with $\dim G(M^4) > 4$. Consider the equation (6.14_1). If $c_2 \neq 0$, we are able to specialize the section σ in such a way that $c_2 = 1$. We see at once that there is exactly one section σ satisfying (6.10) with $c_2 = 1$; in fact, we have $\alpha = 1$ from $c_2 = \tilde{c}_2 = 1$. This section is clearly preserved by $G(M^4)$, hence $\dim G(M^4) \leqq 4$. Thus $\dim G(M^4) > 4$ implies $c_2 = 0$. From similar reasons, we get

Lemma 6.2. *Let $M^4 \subset \mathscr{C}^3$ be a submanifold with $\dim \tau_p = 2$, $L_p^{(1)} \neq 0$, $L_p^{(2)} \not\equiv 0$ for each $p \in M^4$, suppose $\dim G(M^4) > 4$. Then there exists a section $\sigma = (v_1, v_2, v_3, v_4)$ such that $v_2 = Iv_1$ and*

$$\begin{aligned}
[v_1, v_2] &= a_1 v_1 + a_2 v_2 + v_3\,, \\
[v_1, v_3] &= b_1 v_1 + 2a_2 v_3 + v_4\,, \\
[v_1, v_4] &= c_1 v_1 + 3a_2 v_4\,, \\
[v_2, v_3] &= b_1 v_2 - 2a_1 v_3\,, \\
[v_2, v_4] &= c_1 v_2 - 3a_1 v_4\,, \\
[v_3, v_4] &= 2c_1 v_3 - 3b_1 v_4\,.
\end{aligned} \tag{6.19}$$

For another section $\tilde{\sigma}$ with the same properties, we have

$$v_1 = \alpha w_1\,, \quad v_2 = \alpha w_2\,, \quad v_3 = \alpha^2 w_3\,, \quad v_4 = \alpha^3 w_4\,; \tag{6.20}$$

further,

$$(6.21)\qquad \begin{aligned} w_1\alpha + \alpha\tilde{a}_2 &= \ \ a_2\,, & \alpha w_3\alpha - \alpha^2\tilde{b}_1 &= -b_1\,,\\ w_2\alpha - \alpha\tilde{a}_1 &= -a_1\,, & \alpha^2 w_4\alpha - \alpha^3\tilde{c}_1 &= -c_1\,. \end{aligned}$$

The functions a_1, a_2, b_1, c_1 satisfy

$$(6.22)\qquad \begin{aligned} v_3a_1 - v_2b_1 - \ a_1b_1 &= 0\,, & v_3a_2 + v_1b_1 - \ a_2b_1 - c_1 &= 0\,,\\ v_1a_1 + v_2a_2 - \quad b_1 &= 0\,, & v_4a_1 - v_2c_1 - 2a_1c_1 &= 0\,,\\ v_4a_2 + v_1c_1 - 2a_2c_1 &= 0\,, & v_4b_1 - v_3c_1 - b_1c_1 &= 0\,. \end{aligned}$$

The equations (6.22) follow directly from (6.3). Consider now the system of partial differential equations

$$(6.23)\qquad v_1\alpha = \alpha a_2\,,\quad v_2\alpha = -\alpha a_1\,,\quad v_3\alpha = -\alpha b_1\,,\quad v_4\alpha = -\alpha c_1$$

for α. Its integrability conditions are exactly (6.22), i.e., the system is completely integrable and its solution α is determined by the value $\alpha(m_0)$ at a fixed point $m_0 \in M^4$. From this and from (6.21), we obtain

Theorem 6.1. *Let $M^4 \subset \mathscr{C}^3$ be a submanifold with* $\dim \tau_p = 2$, $L_p^{(1)} \not\equiv 0$, $L_p^{(2)} \not\equiv 0$ *for each $p \in M^4$,* $\dim G(M^4) > 4$. *Then there is a section $\sigma = (v_1, v_2, v_3, v_4)$ such that $v_2 = Iv_1$ and*

$$(6.24)\qquad \begin{gathered} [v_1, v_2] = v_3\,,\quad [v_1, v_3] = v_4\,,\\ [v_1, v_4] = [v_2, v_3] = [v_2, v_4] = [v_3, v_4] = 0\,. \end{gathered}$$

Any other section $\tilde{\sigma}$ of the same type is given by

$$(6.25)\qquad v_1 = \alpha w_1\,,\quad v_2 = \alpha w_2\,,\quad v_3 = \alpha^2 w_3\,,\quad v_4 = \alpha^3 w_4\,;\qquad 0 \neq \alpha = \text{const.}$$

Hence, $\dim G(M^4) = 5$.

It is obvious that two manifolds of the type described in this Theorem are (locally) Γ-equivalent.

Consider the manifold $N^4 \subset \mathscr{C}^3$ given by

$$(6.26)\qquad \bar{z}_2 - z_2 = i(\bar{z}_1 - z_1)^2\,,\quad \bar{z}_3 - z_3 = (\bar{z}_1 - z_1)^3\,.$$

Considering it as a submanifold of $\mathscr{R}^6$, its equations are

$$(6.27)\qquad y_2 = 2y_1^2\,,\quad y_3 = -4y_1^3\,;$$

here, $z_i = x_i + iy_i$. On $\mathscr{R}^6$, consider the vector fields

$$
(6.28) \qquad
\begin{aligned}
v_1 &= \frac{\partial}{\partial y_1} + 4y_1 \frac{\partial}{\partial y_2} - 6y_2 \frac{\partial}{\partial y_3}, \\
v_2 &= -\frac{\partial}{\partial x_1} - 4y_1 \frac{\partial}{\partial x_2} + 6y_2 \frac{\partial}{\partial x_3}, \\
v_3 &= -4 \frac{\partial}{\partial x_2} + 24y_1 \frac{\partial}{\partial x_3}, \\
v_4 &= 24 \frac{\partial}{\partial x_3}.
\end{aligned}
$$

It is easy to see that $v_2 = Iv_1$ and we have (6.23). Further, the vectors (6.27) are tangent to N^4 at its points. Thus we have proved Theorem 1.2.

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