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CONNECTED CM-HOMOMORPHISMS INTO $\mathfrak{C}[I]$

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A T_1 topological space X is a $\mathfrak{C}$-space if it has the property that the composition of any two closed relations on X (that is, closed subsets of $X \times X$) is again a closed relation on X. The class of $\mathfrak{C}$-spaces includes all discrete spaces and all compact Hausdorff spaces so that, in particular, the closed unit interval I is a $\mathfrak{C}$-space. The semigroup, under composition, of all closed relations on a $\mathfrak{C}$-space X is denoted by $\mathfrak{C}[X]$. A homomorphism θ from $\mathfrak{C}[X]$ into $\mathfrak{C}[Y]$ is referred to as a CM-homomorphism if it preserves unions, takes symmetric relations into symmetric relations and does not map everything onto a single element. It is known that for a Hausdorff space X, there exists a CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ if and only if X is compact and second countable. The focus in this paper is on those CM-homomorphisms from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ which take connected relations into connected relations. Any homomorphism which has the latter property will be referred to as a connected homomorphism. All the results in this paper are consequences of an attempt to characterize those Hausdorff $\mathfrak{C}$-spaces with the property that there exists a connected CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$. It is first shown that if X is an infinite $\mathfrak{C}$-space, then such a homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ must have a particularly simple form. This information is then applied to show that if X is any Hausdorff $\mathfrak{C}$-space, then there exists a connected CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ if and only if X is a compact subspace of an arc. From this it readily follows that if X is a connected Hausdorff $\mathfrak{C}$-space and there exists a connected CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$, then X is an arc and $\mathfrak{C}[X]$ and $\mathfrak{C}[I]$ are actually isomorphic.

We recall that for any two $\mathfrak{C}$-spaces X and Y, a homomorphism θ from $\mathfrak{C}[X]$ into $\mathfrak{C}[Y]$ is said to be union preserving if whenever $\alpha \in \mathfrak{C}[X]$ and $\alpha = \bigcup\{\beta_a : a \in A\}$ where each $\beta_a \in \mathfrak{C}[X]$, then $\theta(\alpha) = \bigcup\{\theta(\beta_a) : a \in A\}$. The study of union preserving

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endomorphisms which are also symmetry preserving (i.e., take symmetric relations into symmetric relations) was initiated by CLIFFORD and MILLER in [1] where they completely determined the form these endomorphisms must take on $\mathscr{B}_X$ the semigroup of all binary relations on a set X [1, Theorem 3, p. 310]. Then in [8, Theorem (3.3)] the theorem of Clifford and Miller was given a topological setting. That is, the union and symmetry preserving homomorphisms from $\mathfrak{C}[X]$ into $\mathfrak{C}[Y]$ were completely determined where X and Y are any two $\mathfrak{C}$-spaces. It was also shown that all of these homomorphisms must either be injective or must map everything onto a single closed partial equivalence. By a partial equivalence, we mean a relation which is both symmetric and transitive. Those union and symmetry preserving homomorphisms which do not map everything onto a single element or, equivalently, are injective, were referred to in [8, Definition (1.3)] as CM-homomorphisms. Various results about these homomorphisms are proven in [8]. In particular, the result mentioned earlier, that *for a Hausdorff space X, there exists a CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ if and only if X is compact and second countable*, is Theorem (5.2) of that paper.

For some properties of $\mathfrak{C}$-spaces one may consult [6]. The fact that compact Hausdorff spaces are $\mathfrak{C}$-spaces follows from Theorem (2.6) of [7, p. 267]. For a discussion of some of the elementary properties of the semigroup of all binary relations on a set as well as basic terminology concerning semigroups, one should consult [2].

1. SOME PROPERTIES OF CONNECTED CM-HOMOMORPHISMS

The purpose in this section is to determine what form the connected CM-homomorphisms must take which map $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ where X is any $\mathfrak{C}$-space. As we mentioned before, CM-homomorphisms have been completely determined in Theorem (2.3) of [8]. The statement of the following theorem will include this information as well as additional conclusions one can draw by assuming that the homomorphism in question is connected. Before we state this result, we need to recall (e.g., see [9], p. 70) that a function f from a topological space X onto a topological space Y is said to be monotone if it is continuous and $f^{-1}(y)$ is compact and connected for each $y \in Y$. We also mention that a relation α on X is said to be rectangular if $\alpha = \mathscr{D}(\alpha) \times \mathscr{R}(\alpha)$ where $\mathscr{D}(\alpha)$ and $\mathscr{R}(\alpha)$ denote respectively the domain and range of α and are defined by

$$\mathscr{D}(\alpha) = \{x \in X\colon (x, y) \in \alpha \text{ for some } y \in X\}\,,$$

$$\mathscr{R}(\alpha) = \{y \in X\colon (x, y) \in \alpha \text{ for some } x \in X\}\,.$$

Theorem (1.1). *Let X be any $\mathfrak{C}$-space and let θ be a CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$. Then there exist two disjoint partial equivalences π and ζ on I*

($\pi \neq \emptyset$) *and a continuous function* μ *from the domain* E *of* π *onto* X *such that the following conditions are satisfied*:

(1.1.1) $\pi \cup \zeta$ *and* ζ *are both closed subsets of* $I \times I$.

(1.1.2) $\mu \circ \pi = E \times X$.

(1.1.3) $\theta(\alpha) = (\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta$ for each $\alpha \in \mathfrak{C}[X]$.

If, in addition, θ *is a connected homomorphism, the following conditions are satisfied*:

(1.1.4) $\mu^{-1}(A) \cup \mathscr{D}(\zeta)$ *is a closed subinterval of* I *for each closed, connected subset* A *of* X.

(1.1.5) *If* X *has more than one point, then* $\mu^{-1}(A)$ *is connected for each closed, connected subset* A *of* X.

(1.1.6) *If* X *has more than two points, then* $\zeta = \emptyset$ *and* μ *is monotone.*

(1.1.7) *If* X *has an infinite number of points, then* π *is rectangular.*

Proof. The fact that conditions (1.1.1), (1.1.2) and (1.1.3) hold is a consequence of Theorem (2.3) of [8] so our task here is to prove that if the CM-homomorphism θ also takes connected relations into connected relations, then the last four conditions hold. First, we show that for any $\alpha \in \mathfrak{C}[X]$, we have

$$\mathscr{D}(\theta(\alpha)) = \mu^{-1}(\mathscr{D}(\alpha)) \cup \mathscr{D}(\zeta)\,. \tag{1.1.8}$$

Suppose $x \in \mathscr{D}(\theta(\alpha))$. Then $(x, y) \in \theta(\alpha)$ for some $y \in I$ which, by (1.1.3) places it in $(\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta$. If $(x, y) \in \zeta$, then $x \in \mathscr{D}(\zeta)$ and if $(x, y) \in \pi \cap \mu^{-1} \circ \alpha \circ \mu$, it follows that $x \in \mu^{-1}(\mathscr{D}(\alpha))$. In either event, $x \in \mu^{-1}(\mathscr{D}(\alpha)) \cup \mathscr{D}(\zeta)$. On the other hand, suppose $x \in \mu^{-1}(\mathscr{D}(\alpha))$. Then $(\mu(x), a) \in \alpha$ for some $a \in X$ and $(x, a) \in \mu \circ \pi$ by condition (1.1.2). Thus, there exists an element $b \in I$ such that $(x, b) \in \pi$ and $(b, a) \in \mu$. We now have $(x, \mu(x)) \in \mu$, $(\mu(x), a) \in \alpha$ and $(a, b) \in \mu^{-1}$ which puts $(x, b) \in \pi \cap \mu^{-1} \circ \circ \alpha \circ \mu$. Hence $(x, b) \in \theta(\alpha)$ by (1.1.3) and it follows that $x \in \mathscr{D}(\theta(\alpha))$. Since it is immediate from (1.1.3) that $\mathscr{D}(\zeta) \subset \mathscr{D}(\theta(\alpha))$, the verification of (1.1.8) is complete.

Now let A be any closed, connected subset of X and put $\alpha = A \times A$. Then α is a connected element of $\mathfrak{C}[X]$ and by (1.1.8), we have $\mu^{-1}(A) \cup \mathscr{D}(\zeta) = \mathscr{D}(\theta(\alpha))$. Since θ is a connected homomorphism, $\theta(\alpha)$ must be a closed, connected subset of $I \times I$ and since $\mathscr{D}(\theta(\alpha))$ is the image of $\theta(\alpha)$ under a projection mapping, it must be a closed connected subset of I. This proves (1.1.4).

Now we verify (1.1.5). If $\zeta = \emptyset$, (1.1.5) follows immediately from (1.1.4) so we assume that $\zeta \neq \emptyset$. Choose any point $p \in X$. Then $\mu^{-1}(p) \cup \mathscr{D}(\zeta)$ is a closed subinterval of I by (1.1.4) and we denote it by $[a, b]$. Furthermore, ζ is the image of the empty relation under θ and must therefore be a closed, connected subset of $I \times I$. It follows that $\mathscr{D}(\zeta)$ is also a closed subinterval of I which we will denote by $[c, d]$.

Since π and ζ are disjoint partial equivalences, their domains are also disjoint and since $\mathscr{D}(\mu) = E = \mathscr{D}(\pi)$, $\mu^{-1}(x) \cap \mathscr{D}(\zeta) = \emptyset$ for each $x \in X$. Thus, $[c, d]$ is a proper subinterval of $[a, b]$. We first consider the possibility that $a < c \leqq d < b$. By assumption, there exists a point $q \in X$ different from p and the arguments given above yield the fact that $\mu^{-1}(q) \cup \mathscr{D}(\zeta)$ is also a closed subinterval of I which properly contains $\mathscr{D}(\zeta) = [c, d]$. This leads to the conclusion that $\mu^{-1}(q)$ must intersect either $[a, c)$ or $(d, b]$. In either event, the contradiction follows that $\mu^{-1}(p) \cap \cap \mu^{-1}(q) \neq \emptyset$. Thus, either $a = c$ or $d = b$ and there will be no loss in generality in assuming that the latter is true. Consequently, we have

$$\mu^{-1}(p) = [a, c) \quad \text{and} \quad \mathscr{D}(\zeta) = [c, b]. \tag{1.1.9}$$

Since $\mu^{-1}(q) \cup \mathscr{D}(\zeta)$ is a closed interval which properly contains $[c, b]$ and $\mu^{-1}(p) \cap \cap \mu^{-1}(q) = \emptyset$, it follows that

$$\mu^{-1}(q) = (b, e] \quad \text{for some} \quad e \in I. \tag{1.1.10}$$

If X has only the two points p and q, then (1.1.5) follows from (1.1.9) and (1.1.10). Suppose, however, that X has at least one more point t. As with the other two points, $\mu^{-1}(t) \cup \mathscr{D}(\zeta)$ is a closed interval which properly contains $[c, b]$. But this will quickly lead to the contradiction that $\mu^{-1}(t)$ intersects either $\mu^{-1}(p)$ or $\mu^{-1}(q)$. This contradiction arose because we simultaneously assumed that $\zeta \neq \emptyset$ and that X has more than two points. Thus $\zeta = \emptyset$ if X has more than two points and for this case, the conclusion of (1.1.5) follows from (1.1.4). We have actually verified (1.1.6) as well as (1.1.5), for when $\zeta = \emptyset$, it follows from (1.1.4) that $\mu^{-1}(x)$ is a closed subinterval of I for each $x \in X$ and hence that μ is monotone.

We have yet to prove (1.1.7) and to do it, we need the following fact: there exists a mutually disjoint family $\{B_a : a \in A\}$ of nonempty closed subsets of I such that

$$\mu \text{ maps each } B_a \textit{ onto } X \tag{1.1.11}$$

and

$$\pi = \bigcup\{B_a \times B_a : a \in A\}. \tag{1.1.12}$$

To verify this, let a be any point in I and define

$$B_a = \{x \in I : (a, x) \in \pi\}. \tag{1.1.13}$$

Since π is symmetric and transitive, two sets of the form B_a and B_b are either identical or disjoint. Let $J = \{a \in I : B_a \neq \emptyset\}$ and for two elements $a, b \in J$, define a to be equivalent to b if $B_a = B_b$. Choose exactly one element from each equivalence class and denote the resulting set by A. Then $\{B_a : a \in A\}$ is a mutually disjoint family of subsets of I. Moreover, by (1.1.1) and (1.1.6), π is a closed subset of $I \times I$ and it readily follows that each B_a is a closed subset of I. Now choose any B_a, $a \in A$ and let x be any element of X. Then there exists an element $y \in I$ such that $(a, y) \in \pi$. By (1.1.2), the pair (y, x) belongs to $\mu \circ \pi$ so there exists an element $w \in I$ such that $(y, w) \in \pi$ and $(w, x) \in \mu$. Since π is transitive $(a, w) \in \pi$ and since $\mu(w) = x$, we have verified (1.1.11). The validity of (1.1.12) follows from the fact that π is symmetric as well as transitive.

Now we can complete the proof of (1.1.7). Because of (1.1.6), each $\mu^{-1}(x)$, $x \in X$ is a (possibly degenerate) closed subinterval of I which we will denote by $[c_x, d_x]$ where $c_x \leqq d_x$. Then $\{d_x : x \in X\}$ is an infinite subset of I and thus has a limit point p. Now $\{d_x : x \in X\} \subset E = \mathscr{D}(\pi)$ which is closed since π is closed. Thus, $p \in E$ which, by (1.1.12) implies that $p \in B_r$ for some $r \in A$. We assert that

$$\mu^{-1}(\mu(p)) \subset B_r. \tag{1.1.14}$$

To prove this, choose any $t \in \mu^{-1}(\mu(p))$ and choose a subfamily $\{[c_{x_n}, d_{x_n}]\}_{n=1}^{\infty}$ of the family $\{[c_x, d_x] : x \in X\}$ with the property that the sequence $\{d_{x_n}\}_{n=1}^{\infty}$ converges to p. Since the intervals are mutually disjoint, the sequence $\{c_{x_n}\}_{n=1}^{\infty}$ also converges to p. Since $t \in E$, we have $t \in B_w$ for some $w \in A$ and it follows from (1.1.11) that

$$B_w \cap [c_{x_n}, d_{x_n}] = B_w \cap \mu^{-1}(x_n) \neq \emptyset$$

for all n. We choose $e_n \in B_w \cap [c_{x_n}, d_{x_n}]$ for each n and note that the sequence $\{e_n\}_{n=1}^{\infty}$ converges to p. Since B_w is closed this implies that $p \in B_w$ and we have $B_w \cap B_r \neq \emptyset$ which, as we observed previously, implies that $B_w = B_r$. Hence $t \in B_r$ and (1.1.14) is valid. It follows from this and (1.1.12) that $\pi = B_r \times B_r$ for let any any $a \in A$ be given. By (1.1.11), $\mu^{-1}(\mu(p)) \cap B_a \neq \emptyset$. But this, together with (1.1.14) implies that $B_a \cap B_r \neq \emptyset$ and hence that $B_a = B_r$. This verifies (1.1.7) and completes the proof of the theorem.

Before we prove a corollary to this theorem, we need to recall some definitions. Let T be any semigroup with a zero element which we denote by 0. A two-sided ideal J of T is said to be 0-minimal if it properly contains 0 but does not properly contain any other two-sided ideal which properly contains 0. If S and T are two semigroups and both have zero elements, then a homomorphism from S into T which sends 0 into 0 is said to be 0-preserving. For any $\mathfrak{C}$-space X, the empty relation is the zero of $\mathfrak{C}[X]$. We noted in [8, Proposition (2.1)] that each $\mathfrak{C}[X]$ has a unique 0-minimal ideal and that it consists of all closed rectangular relations on X. With this in mind, we prove

Corollary (1.2). *Let X be any infinite Hausdorff $\mathfrak{C}$-space. Then the following statements concerning a mapping θ from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ are equivalent*:

(1.2.1) *θ is a connected CM-homomorphism.*

(1.2.2) *θ is a 0-preserving connected CM-homomorphism which maps the 0-minimal ideal of $\mathfrak{C}[X]$ into the 0-minimal ideal of $\mathfrak{C}[I]$.*

(1.2.3) *There exists a monotone function μ from a nonempty closed subset of I onto X such that $\theta(\alpha) = \mu^{-1} \circ \alpha \circ \mu$ for each $\alpha \in \mathfrak{C}[X]$.*

Proof. First of all, (1.2.1) implies (1.2.3) because of (1.1.1), (1.1.3), (1.1.6) and (1.1.7). Next, we show that (1.2.3) implies (1.2.2). It follows immediately from Theorem (4.5) of [8] that θ is a 0-preserving CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ which maps the 0-minimal ideal of $\mathfrak{C}[X]$ into the 0-minimal ideal of $\mathfrak{C}[I]$. To show that θ is a connected homomorphism, define a function H from $E \times E$ (where E denotes the domain of μ) onto $X \times X$ by $H(a, b) = (\mu(a), \mu(b))$. Then H is continuous and since $E \times E$ is compact and $X \times X$ is Hausdorff, H is also a closed mapping. Furthermore, since μ is monotone, $\mu^{-1}(x)$ is connected for each $x \in X$. Hence, $H^{-1}(x, y) = \mu^{-1}(x) \times \mu^{-1}(y)$ is connected for all $(x, y) \in X \times X$. That is, preimages of points under H are connected and it follows from [5, Theorem 9, p. 131] that $H^{-1}(\alpha)$ is connected for each connected subset α of $X \times X$. Then θ is a connected homomorphism since $\theta(\alpha) = \mu^{-1} \circ \alpha \circ \mu = H^{-1}(\alpha)$ for each $\alpha \in \mathfrak{C}[X]$. This completes the proof since it is immediate that (1.2.2) implies (1.2.1).

We want to consider some examples which will show that the cardinality restrictions imposed on X in Theorem (1.1) are necessary in order obtain the various conclusions. However, we defer these examples until section 3.

2. SPACES WHOSE SEMIGROUPS ADMIT CONNECTED CM-HOMOMORPHISMS INTO $\mathfrak{C}[I]$

In this section, we use the previous results to characterize those Hausdorff $\mathfrak{C}$-spaces X with the property that there exists a connected CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$.

Theorem (2.1). *Let X be a Hausdorff $\mathfrak{C}$-space. Then there exists a connected CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ if and only if X is a compact subspace of an arc.*

Proof. First, we suppose that there exists a connected CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$. If X is finite, it is immediate that it is a subspace of an arc so we consider the case where X is infinite. By Corollary (1.2), some nonempty closed subset E of I maps onto X with a monotone function μ. Then X is necessarily compact and it follows [3, Corollary 2, p. 105] that it is also second countable. Now we want to show that each component of X is either a point or an arc. Let A be any component of X which consists of more than one point. Since X is a compact metric space, the subspace A is a metric continuum. Since μ is a monotone map which is also closed, it follows from Theorem 9 [5, p. 131] that preimages of connected sets, under μ, are connected. Actually, this also follows from Theorem (1.1) of this paper. In any event, $\mu^{-1}(A)$ is a closed subinterval of I which we denote by $[a, b]$. Choose any $p \in A$

such that $\mu(a) \neq p \neq \mu(b)$. Then $\mu^{-1}(A - \{p\}) = \mu^{-1}(A) - \mu^{-1}(p)$ is a proper subset of $[a, b]$ which contains both a and b and is therefore not connected. Consequently, $A - \{p\}$ is not connected. That is, p is a cut point of A. We have shown that A has at most two non-cut points. Thus, by a well known result (e.g., see [9, Theorem (6.2), p. 54]), A is an arc.

Now let $\{x_n\}_{n=1}^{\infty}$ be any sequence of distinct points in $X - A$ which converges to a point $t \in A$. Choose $w_n \in \mu^{-1}(x_n)$ for each n. Then the sequence $\{w_n\}_{n=1}^{\infty}$ has a subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ which converges to some point $q \in I$. Now q cannot belong to the interior of the closed interval $\mu^{-1}(A)$ since $\mu^{-1}(A)$ does not intersect any $\mu^{-1}(x_n)$. However, q must belong to $\mu^{-1}(A)$ for if it did not, we would be led to the conclusion that $\mu(q) \notin A$ which is a contradiction since $\mu(q) = t$. Thus, q is one on the endpoints of $\mu^{-1}(A) = [a, b]$ and there is no loss of generality if we assume that $q = a$. Then $\mu^{-1}(t)$ is a closed interval which is properly contained in $[a, b]$ and which contains the point a. Thus, there exists a point c such that $\mu^{-1}(A) - \mu^{-1}(t) = (c, b)$. But $\mu^{-1}(A) - - \mu^{-1}(t) = \mu^{-1}(A - \{t\})$ and it follows that $A - \{t\}$ is connected. We have shown that no cut point of A is a limit of a sequence from $X - A$. It now follows from Theorem 1 of [4, p. 876] that X is a compact subspace of an arc.

The proof of the converse follows readily. Suppose X is a compact subspace of an arc. Then there exists a homeomorphism μ from some nonempty closed subset E of I onto X and by Corollary (1.2), the mapping θ from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ which is defined by

$$\theta(\alpha) = \mu^{-1} \circ \alpha \circ \mu$$

for each $\alpha \in \mathfrak{C}[X]$ is a connected CM-homomorphism.

Since a compact connected subspace of an arc is also an arc if it has more than one point and since any two homeomorphic $\mathfrak{C}$-spaces have isomorphic semigroups, the following corollary is an immediate consequence of Theorem (2.1).

Corollary (2.2). *Let X be a connected Hausdorff $\mathfrak{C}$-space with more than one point and suppose there exists a connected CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$. Then X is an arc and $\mathfrak{C}[X]$ and $\mathfrak{C}[I]$ are, in fact, isomorphic.*

3. SOME EXAMPLES

In this section, we consider examples which, among other things, will show that the various cardinality restrictions on X in Theorem (1.1) are necessary if one expects to get the various conclusions which are obtained there. The first part of Theorem (1.1) states that to each CM-homomorphism θ from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$, there correspond two partial equivalences π and ζ and a continuous function μ all satisfying certain conditions, among them, $\theta(\alpha) = (\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta$ for each

$\alpha \in \mathfrak{C}[X]$. As we pointed out in the proof, this fact follows from Theorem (3.3) of [8]. Actually, this is half of Theorem (3.3) (with an arbitrary $\mathfrak{C}$-space Y in place of I). The full statement is that every such homomorphism is obtained in exactly this manner. We will use this fact in constructing the various examples. We should also emphasize that one can show that the relations and the function are unique. That is, two different triples (of two relations and a function) cannot result in the same homomorphism.

Before we discuss these examples, we introduce some notation. For any two points x, y of a $\mathfrak{C}$-space X, the relation $\{(x, y)\}$ belongs to $\mathfrak{C}[X]$ since X is T_1. We will denote this relation by $\langle x, y\rangle$. In the event $x = y$, we will simplify matters further by using the notation $\langle x\rangle$. The empty relation also belongs to each $\mathfrak{C}[X]$ and will be denoted by 0.

Example (3.1). Let $X = \{p\}$ be the one-point space. Define relations π and ζ on I by

$$\pi = \{(x, x) \in I \times I : x \neq \tfrac{1}{2}\}$$

and

$$\zeta = \langle \tfrac{1}{2} \rangle .$$

Let μ be the function which maps all of $E = I - \{\frac{1}{2}\}$ onto the point p. Then π and ζ are disjoint partial equivalences, $\pi \cup \zeta$ and ζ are both closed, $\mu \circ \pi = E \times X$ and μ is continuous. Thus, the mapping θ defined by

$$\theta(\alpha) = (\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta$$

for each $\alpha \in \mathfrak{C}[X]$ is a CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$. It is also a connected homomorphism since one can verify that

$$\theta\langle p\rangle = \{(x, x) : x \in I\}$$

and, of course

$$\theta(0) = \langle \tfrac{1}{2} \rangle .$$

Since $\mu^{-1}(p)$ is not connected, this example shows that the conclusion of (1.1.5) need not hold if X has only one point. In addition, $\zeta \neq \emptyset$ and π is not rectangular.

Example (3.2). Let $X = \{p, q\}$ be the two-point discrete space. Define a subset A_x of I by $A_x = \{x, 1 - x\}$ and define relations π and ζ on I by

$$\pi = \bigcup\{A_x \times A_x : 0 \leqq x < \tfrac{1}{2}\}, \quad \zeta = \langle \tfrac{1}{2} \rangle .$$

Since the union of any mutually disjoint family of rectangular relations is a partial equivalence, both π and ζ are partial equivalences and, of course, are disjoint. The relation ζ is evidently closed and although π is not, we show that $\pi \cup \zeta$ is closed. We do this by showing that if (a, b) is any limit point of π which is not in π, then $(a, b) = (\frac{1}{2}, \frac{1}{2})$. Let (a, b) be such a point and let $\{(v_n, w_n)\}_{n=1}^{\infty}$ be a sequence of elements in π which converges to (a, b). Then for each n, we have $(v_n, w_n) \in A_x \times A_x$ for some x such that $0 \leqq x < \frac{1}{2}$. It follows that w_n is either v_n or $1 - v_n$ for each n. Two possible cases arise.

Case 1. There is a subsequence $\{(v_{n_k}, w_{n_k})\}_{k=1}^{\infty}$ of $\{(v_n, w_n)\}_{n-1}^{\infty}$ such that $w_{n_k} = v_{n_k}$ for all k.

Case 2. There is a subsequence $\{(v_{n_k}, w_{n_k})\}_{k=1}^{\infty}$ of $\{(v_n, w_n)\}_{n=1}^{\infty}$ such that $w_{n_k} = 1 - v_{n_k}$ for all k.

In Case 1, $\lim (v_{n_k}, v_{n_k}) = \lim (v_{n_k}, w_{n_k}) = (a, b)$ which implies $a = b$. Thus (a,b) is on the diagonal and since $(a, b) \notin \pi$, we must have $(a, b) = (\frac{1}{2}, \frac{1}{2})$. In Case 2, $\lim (v_{n_k}, 1 - v_{n_k}) = \lim (v_{n_k}, w_{n_k}) = (a, b)$ and it follows that $b = 1 - a$. Again, since $(a, b) \notin \pi$, we conclude that $(a, b) = (\frac{1}{2}, \frac{1}{2})$.

The domain E of π is the set $I - \{\frac{1}{2}\}$. Define a function μ from E onto X by

$$\mu(x) = p \quad \text{for} \quad 0 \leqq x < \tfrac{1}{2}, \qquad \mu(x) = q \quad \text{for} \quad \tfrac{1}{2} < x \leqq 1 .$$

Then μ is continuous and one easily shows that $\mu \circ \pi = E \times X$. Thus, the mapping θ defined by

$$\theta(\alpha) = (\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta$$

is a CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$. In fact, θ is also a connected homomorphism because $\theta(0) = \langle\frac{1}{2}\rangle$ and it is a straightforward matter to verify the following statements:

$$\begin{aligned}
\theta\langle p\rangle \quad &= \{(x, x) \in I \times I : 0 \leqq x \leqq \tfrac{1}{2}\}, \\
\theta\langle q\rangle \quad &= \{(x, x) \in I \times I : \tfrac{1}{2} \leqq x \leqq 1\}, \\
\theta\langle p, q\rangle &= \{(x, y) \in I \times I : 0 \leqq x \leqq \tfrac{1}{2} \text{ and } y = 1 - x\}, \\
\theta\langle q, p\rangle &= \{(x, y) \in I \times I : \tfrac{1}{2} \leqq x \leqq 1 \text{ and } y = 1 - x\}.
\end{aligned}$$

This example serves to illustrate the fact that the conclusion of (1.1.6) need not hold if X consists of two points. The relation ζ is not empty and μ is not monotone because although $\mu^{-1}(p)$ and $\mu^{-1}(q)$ are both connected, they are not compact.

Example (3.3). Let $X = \{p_k\}_{k=1}^{N}$ denote the discrete space consisting of N points and let $\{[a_k, b_k]\}_{k=1}^{N}$ denote any family of mutually disjoint closed subintervals of I. For each k such that $1 \leqq k \leqq N$, choose a homeomorphism h_k from I onto $[a_k, b_k]$ and for each $x \in I$, define a subset A_x of I by $A_x = \{h_k(x)\}_{k=1}^{N}$. Since the h_k are all injective the family $\{A_x : x \in I\}$ is mutually disjoint. Thus,

$$\pi = \bigcup\{A_x \times A_x : x \in I\}$$

is a partial equivalence. Moreover, π is closed for let $\{(v_n, w_n)\}_{n=1}^{\infty}$ be any sequence in π which converges to a point $(a, b) \in I \times I$. Then for each n, there exists an $x_n \in I$

such that $(v_n, w_n) \in A_{x_n} \times A_{x_n}$. Since there are only finitely many of the homeomorphisms h_k, there exists a subsequence $\{(v_{n_i}, w_{n_i})\}_{i=1}^{N}$ of $\{(v_n, w_n)\}_{n=1}^{\infty}$ and two of the homeomorphisms h_k and h_t such that $v_{n_i} = h_k(x_{n_i})$ and $w_{n_i} = h_t(x_{n_i})$ for all n_i. Then some subsequence $\{x_{n_{i_j}}\}_{j=1}^{\infty}$ of $\{x_{n_i}\}_{i=1}^{\infty}$ converges to a point $y \in I$ and it follows that $\lim v_{n_{i_j}} = \lim h_k(x_{n_{i_j}}) = h_k(y)$ and $\lim w_{n_{i_j}} = \lim h_t(x_{n_{i_j}}) = h_t(y)$. Consequently, $(a, b) = (h_k(y), h_t(y)) \in A_y \times A_y \subset \pi$. This proves that π is closed.

Let E denote the domain of π. Then $E = \bigcup[a_k, b_k]_{k=1}^{N}$ and we define a function μ from E onto X by

$$\mu(x) = p_i \quad \text{for} \quad x \in [a_i, b_i]\,.$$

The function μ is continuous and it follows easily that $\mu \circ \pi = E \times X$. Thus, the function θ defined by

$$\theta(\alpha) = \pi \cap \mu^{-1} \circ \alpha \circ \mu$$

is a CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$. Moreover, it is a straightforward matter to verify that

$$\theta\langle p_i, p_j\rangle = \{(h_i(x), h_j(x)) : x \in I\}\,.$$

Since the mapping which sends x into $(h_i(x), h_j(x))$ is a homeomorphism from I into $I \times I$, it follows that $\theta\langle p_i, p_j\rangle$ is connected for all i, j. Thus, θ is a connected CM-homomorphism.

This shows that each finite discrete space X admits a connected CM-homomorphism from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ with the property that the associated partial equivalence π is not rectangular. Consequently, in order to obtain the conclusion of (1.1.7) one must require that X contain an infinite number of elements.

The essential conclusion that one can draw from Theorem (1.1) is that whenever X is infinite, any connected CM-homomorphism θ from $\mathfrak{C}[X]$ into $\mathfrak{C}[I]$ has a particularly simple form. Specifically, ζ is empty, π is rectangular and μ is monotone. Our final example shows that by replacing the second semigroup with something other than $\mathfrak{C}[I]$, situations arise which are as far removed from the latter as possible. We give an example of a connected CM-homomorphism from $\mathfrak{C}[I]$ into $\mathfrak{C}[J]$ (where $J = I \times I$) with the property that the associated partial equivalence ζ is not empty, π is not rectangular and μ is not monotone. Indeed, the preimages of points are not even connected or compact.

Example (3.4). For each $x \in I$, define a subset A_x of $I \times I$ by

$$A_x = \{(x, y) \in I \times I : y \in I\}\,.$$

Let

$$\pi = \bigcup\{A_x \times A_x : x \in I - \{\tfrac{1}{2}\}\}\,,$$

and let

$$\zeta = A_{1/2} \times A_{1/2}\,.$$

It is evident that π and ζ are disjoint partial equivalences on J. Furthermore, ζ is homeomorphic to $I \times I$ and

$$\pi \cup \zeta = \{A_x \times A_x : x \in I\}$$

is homeomorphic to $I \times I \times I$. Thus, ζ and $\pi \cup \zeta$ are both closed subsets of $J \times J$. The domain E of π is $I \times I - A_{1/2}$ and we define a function μ from E onto I by

$$\mu(x, y) = y \quad \text{for all} \quad (x, y) \in E .$$

Then μ is continuous and also has the property that

$$\mu \circ \pi = E \times I .$$

To verify the latter, any element of $E \times I$ must have the form $((a, b), x)$ where $a, b, x \in I$ and $a \neq \frac{1}{2}$. Then

$$((a, b), (a, x)) \in A_a \times A_a \subset \pi$$

and

$$((a, x), x) \in \mu .$$

This implies that $((a, b), x) \in \mu \circ \pi$ and hence that $E \times I \subset \mu \circ \pi$. Of course, the reverse inclusion always holds. It follows from all this that the mapping θ which is defined by

$$\theta(\alpha) = (\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta$$

for all $\alpha \in \mathfrak{C}[I]$ is a CM-homomorphism from $\mathfrak{C}[I]$ into $\mathfrak{C}[J]$. The mapping θ is also a connected homomorphism and one way to see this is as follows: for any $\alpha \in \mathfrak{C}[I]$, we have

$$\mu^{-1} \circ \alpha \circ \mu = \{(x, a, y, b) : x, y \in I - \{\tfrac{1}{2}\} \text{ and } (a, b) \in \alpha\} .$$

Now an element of I^4 belongs to π if and only if its first and third coordinates agree. It follows from this that

$$\pi \cap \mu^{-1} \circ \alpha \circ \mu = \{(x, a, x, b) : x \in I - \{\tfrac{1}{2}\} \text{ and } (a, b) \in \alpha\} .$$

Therefore, $\theta(\alpha)$ is the following union:

$$\{(x, a, x, b) : x \in I - \{\tfrac{1}{2}\} \quad \text{and} \quad (a, b) \in \alpha\} \cup \{(\tfrac{1}{2}, x, \tfrac{1}{2}, y) : x, y \in I\}$$

which is homeomorphic to

$$\{(x, a, b) : x \in I - \{\tfrac{1}{2}\} \quad \text{and} \quad (a, b) \in \alpha\} \cup \{(\tfrac{1}{2}, x, y) : x, y \in I\} .$$

But the latter is

$$(A \times \alpha) \cup (B \times \alpha) \cup C$$

where $A = [0, \frac{1}{2})$, $B = (\frac{1}{2}, 1]$ and $C = \{(\frac{1}{2}, x, y) : x, y \in I\}$. Thus, in order to show that θ is a connected homomorphism, it is sufficient to show that $(A \times \alpha) \cup \cup (B \times \alpha) \cup C$ is connected whenever α is connected. So, let α be connected and note first that $A \times \alpha$, $B \times \alpha$ and C are all connected. Then the closure $\mathrm{cl}(A \times \alpha)$ of $A \times \alpha$ is connected and intersects C. Thus, $\mathrm{cl}(A \times \alpha) \cup C$ is connected. But $\mathrm{cl}(A \times \alpha) \cup C = (A \times \alpha) \cup C$. In a similar manner, $(B \times \alpha) \cup C$ is connected and it follows that $(A \times \alpha) \cup (B \times \alpha) \cup C$ is connected.

As we mentioned in the discussion which precedes this example, this situation is about as far removed as possible from the situation where one maps $\mathfrak{C}[X]$, X infinite, into $\mathfrak{C}[I]$ with a connected CM-homomorphism. In this example, ζ is not empty, π is not rectangular and μ is not monotone. In fact preimages of points under μ are neither compact nor connected since for each $y \in I$,

$$\mu^{-1}(y) = \{(x, y) : x \in I - \{\tfrac{1}{2}\}\}\,.$$

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