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A CLASS OF CONNECTED SPACES WITH MANY RAMIFICATIONS

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It is easy to verify (cf. 3.1d) that each non-degenerate connected topological space has at least one non-closed connected subset. However, it will be shown in section 4 that some connected (even Hausdorff) spaces have a point x_0 such that every connected subset containing x_0 is closed. The class of these spaces, named **RAM** is the subject of this note. It originated as a class of counterexamples in studying conditions equivalent to (weak) linear orderability of connected topological spaces (see section 5). The main results can be found in section 3, where also a characterization of **RAM** is given in terms of a set equipped with a partial order and a topology satisfying some interrelations (see 3.3).

1. DEFINITIONS AND AUXILIARY PROPOSITIONS

By $A \oplus B$ we denote the disjoint topological sum of two disjoint spaces A and B. $(X, x_0) \in \mathbf{RAM}$ if X is a non-degenerate, connected T_1-space and $x_0 \in X$ is such that every connected subset of X containing x_0 is closed.

Let us first prove that the point x_0 of X is unique, i.e. if $(X, x_0) \in \mathbf{RAM}$ and $(X, x_1) \in \mathbf{RAM}$ then $x_0 = x_1$. Suppose $(X, x_0) \in \mathbf{RAM}$ and $x \in X \setminus \{x_0\}$. Let C be the component of $X \setminus \{x\}$ that contains x_0. By definition of **RAM**, C is closed in X. Thus $X \setminus C$ is not closed, but is connected (cf. 1.2) and contains x. Hence $(X, x) \notin \mathbf{RAM}$. If no confusion is likely, then we will write $X \in \mathbf{RAM}$ instead of $(X, x_0) \in \mathbf{RAM}$. In studying **RAM** the following relation R plays a crucial role:

If $(X, x_0) \in \mathbf{RAM}$ and $x, y \in X$ then we write xRy if either $x = x_0 \neq y$ or x separates X between x_0 and y (i.e. $X \setminus \{x\} = A \oplus B$ and $x_0 \in A$, $y \in B$ for some $A, B \subset X$).

The topological tools of this study are the following three propositions. The first two are well-known, the third can essentially also be found in [3], p. 75.

Proposition 1.1. *If Z and $Y \subset Z$ are connected and $Z \setminus Y = A \oplus B$ then $Y \cup A$ is connected.*

Proposition 1.2. *If Z and $Y \subset Z$ are connected and C is a component of $Z \setminus Y$, then $Z \setminus C$ is connected.*

Proposition 1.3. *If X is a connected T_1-space, with a dense subspace of cardinality $\mathfrak{m}$, then the number of points $x \in X$ for which $X \setminus \{x\}$ has at least three components does not exceed $\mathfrak{m}$.*

Before we prove 1.3 we mention the following immediate consequence of 1.1.

Lemma. *If X is connected, $a, b \in X$ and $X \setminus \{a\} = A_1 \oplus A_2 \oplus A_3$, $X \setminus \{b\} = B_1 \oplus B_2 \oplus B_3$ and $a \in B_1$, $b \in A_1$ then $B_2 \cup B_3 \subset A_1$.*

Proof of 1.3. Let $D \subset X$ be dense and card $D = \mathfrak{m}$, and $A = \{a \in X \mid X \setminus \{a\}$ has $\geqq 3$ components$\}$. For each $a \in A$ we choose three nonempty (necessarily open and disjoint) subsets $A_i^a \subset X$, $i = 1, 2, 3$, such that $X \setminus \{a\} = A_1^a \oplus A_2^a \oplus A_3^a$. Moreover we choose, again arbitrarily, $a_i \in A_i^a \cap D$, $i = 1, 2, 3$. Now we define $\phi : A \to D \times D \times D$ by

$$\phi(a) = (a_1, a_2, a_3) .$$

From the lemma we see that ϕ is $1-1$ (independently of the choice of A_i^a and a_i). Hence card $A \leqq (\text{card } D)^3 = \mathfrak{m}$.

2. COMPUTATIONS CONCERNING THE RELATION R

The following proposition does apply to any connected T_1-space X with some fixed point x_0 and a relation R as defined in section 1 : xRy iff either $x = x_0 \neq y$ or x separates X between x_0 and y. Its short proof can be found in several textbooks with various modifications.

2.1. *The relation R is a partial order (i.e. is antisymmetric and transitive). For each $x \in X$ the set $\{y \in X \mid yRx\}$ is linearly ordered.*

From now on we will think of R as a partial order, saying "x is smaller than y" for xRy, etc.

In the following definitions X is a fixed set and R a partial order of X. In understanding the definitions better, it may be helpful to glance at 2.5.

Definitions. We denote the inverse of R by $\tilde{R} : x\tilde{R}y$ iff yRx. By $\boldsymbol{R}$ we mean the relation "R or $=$" : $x\boldsymbol{R}y$ iff xRy or $x = y$. The relation non R is the negation of $R : x$ non Ry iff xRy does not hold. A subset A of X is called *right-saturated* if $\forall a \in A\ \forall x \in X\ a\boldsymbol{R}x \Rightarrow x \in A$. If A is any subset of X then there is a smallest right-saturated subset of X containing A. It is named the *right-saturation* of A and denoted by A^*. Because R is transitive we have:

$$A^* = \{x \in X \mid \exists a \in A\ a\boldsymbol{R}x\} .$$

If $x \in X$ and $A \subset X$ then we say that x *immediately precedes* A if

(i) $\forall a \in A\ xRa$ (hence $x \notin A$) and
(ii) If $y \in X$ and $\forall a \in A\ yRa$, then $y\boldsymbol{R}x$.

If $x \in X$ and $A \subset X$ then we say that x is the *greatest lower bound* of A if

(i) $\forall a \in A\ x\boldsymbol{R}a$ and
(ii) If $y \in X$ and $\forall a \in A\ y\boldsymbol{R}a$, then $y\boldsymbol{R}x$.

If $A = \{x_1, x_2\}$ then we also write $x_1 \wedge x_2$ for the greatest lower bound of A.

From now on, throughout this section, (X, x_0) denotes a fixed member of **RAM**, and R is the relation on X defined before.

2.2. *Let $x \in X \setminus \{x_0\}$ and denote the components of $X \setminus \{x\}$ by C_α, $\alpha \in J$, while $x_0 \in C_{\alpha_0}$. Then C_{α_0} is closed. For each $\alpha \in J \setminus \{\alpha_0\}$, C_α is open and $C_\alpha^- = C_\alpha \cup \{x\}$. Moreover J is infinite.*

The simple proof is left to the reader.

2.3. *For any $x \in X \setminus \{x_0\}$ the set $\{y \mid x \text{ non } \boldsymbol{R}y\}$ is the component of $X \setminus \{x\}$ which contains x_0, and hence this set is closed. So its complement $\{y \mid x\boldsymbol{R}y\}$ is connected and open. This also implies that X has no maximal elements.*

Proof. By definition of R $\{y \mid x \text{ non } \boldsymbol{R}y\}$ is the quasicomponent of x_0 in $X \setminus \{x\}$. If this set was not connected, then it contained some component C_α of $X \setminus \{x\}$ with $x_0 \notin C_\alpha$. By 2.2 C_α is open in X and closed in $X \setminus \{x\}$. Contradiction.

2.4. *Let $A \subset X$ be a linearly ordered subset, with $x_0 \notin A$. Then its right-saturation A^* is open and has an immediate predecessor z. Moreover $A^{*-} = A^* \cup \{z\}$.*

Before we prove 2.4 we first mention the following.

2.5. Corollary. (a) *Each $x \in X$ has an immediate predecessor z. Moreover $\{y \mid xRy\}^- = \{y \mid xRy\} \cup \{x, z\}$.*

(b) *Each linearly ordered subset A of X with a maximal element (or an upper bound) is well-ordered by $\tilde{\boldsymbol{R}}$.*

(c) *If $A \subset X$ is linearly ordered and $a_n \in A$, a_nRa_{n+1} for $n = 1, 2, 3, \ldots$, then $\{a_n \mid n = 1, 2, 3, \ldots\}$ is cofinal with A (i.e. $\forall a \in A\ \exists n\ aRa_n$).*

(d) *Any nonempty $A \subset X$, for which $x_0 \notin A$ has a unique immediate predecessor a_1 and a unique greatest lower bound a_2. If $a_2 \notin A$, then $a_1 = a_2$. Else a_1 is the immediate predecessor of $a_2 = \min A$.*

Proof of 2.5, using 2.4.

(a) Clearly $\{y \mid xRy\}$ and $\{y \mid x\boldsymbol{R}y\}$ are right-saturated. So we only have to notice that $x \in \{y \mid xRy\}^-$, which follows from 2.2 and 2.3.

(b) Because X has no maximal elements (2.3) the condition that A has a maximal element implies that A has an upper bound, say x. Thus it suffices to show that $\{y \mid y\boldsymbol{R}x\}$ is well-ordered by $\tilde{R}$, for each $x \in X$.

Clearly $\{y \mid y\boldsymbol{R}x\}$ is linearly ordered (see 2.1). Let A be a non-empty subset of $\{y \mid yRx\}$. We will show that A has an $\tilde{R}$-smallest $=$ R-largest element. Assume $x \notin A$. Consider the set $B = \{y \mid \forall a \in A \; aRyRx\}$ and its right-saturation B^* (in X). By 2.4 B^* has an immediate predecessor, say z. It is easy to see that $z \in A$ and z is the R-largest element of A.

(c) Suppose $a_1, a_2, \dots$ is not cofinal with A, i.e. $\exists a \in A \, \forall n \; a_n Ra$. Then $\{a' \in \in A \mid a'Ra\}$ would not be well-ordered by $\tilde{R}$, contradicting (b).

(d) Choose some $a \in A$. Now $\{z \mid z\boldsymbol{R}a\}$ is well-ordered by $\tilde{R}$. Moreover $x_0 Ra$ for each $a \in A$. Let a_2 be the $\tilde{R}$-minimal element such that $a_2 \boldsymbol{R}a$ for all $a \in A$. If $a_2 \notin A$, then take $a_1 = a_2$, else let a_1 be the immediate $\tilde{R}$-successor of a_1 in $\{z \mid z\boldsymbol{R}a\}$. Uniqueness follows simply from the fact that $\{z \mid z\boldsymbol{R}a\}$ is linearly ordered.

Proof of 2.4. Because $A^* = \cup\{\{x \mid a\boldsymbol{R}x\} \mid a \in A\}$, 2.3 shows that A^* is open. Being a proper subset of the connected space X it cannot be closed. Let z be any boundary point of A^*. We will show that z immediately precedes A. Observe that thus, by definition, z is unique.

First we will show that zRa for all $a \in A$ and hence for all $a \in A^*$. Because $z \notin A^*$ we know already that a non $\boldsymbol{R}z$ for all $a \in A$. Now suppose z and some $a' \in A$ are R-incomparable. Because A and for each a the set $\{x \mid xRa\}$ are linearly ordered (2.1) we find that z is not comparable with any $a \in A^*$, and in particular $\{x \mid z\boldsymbol{R}x\} \cap A^* = = \emptyset$. However $\{x \mid z\boldsymbol{R}x\}$ is an open neighborhood of z (2.3) and $z \in A^{*-}$. Contradiction.

Secondly, assume that for some $y \in X \; zRy$, while yRa holds for all $a \in A$. For $a \in A$ let C_a be the component of a in $X \setminus \{y\}$. Now yRa implies $x_0 \notin C_a$. Since, by 2.3, $\{x \mid a\boldsymbol{R}x\} \subset C_a$, and X has no maximal elements, the family $\{C_a \mid a \in A\}$ has no disjoint members. Hence this family has a connected union, which contains all of A^*. Thus A is fully contained in one component, say C, of $X \setminus \{y\}$, and $x_0 \notin C$. By 2.2 $C^- = C \cup \{y\}$, but $z \notin C$ because y non $\boldsymbol{R}z$ (as zRy). This contradicts $z \in A^- \subset C^-$.

This completes the proof.

2.6. *For any $x \in X \setminus \{x_0\}$ the following families of sets are equal:*

(a) *the components of $\{y \mid xRy\}$,*

(b) *the components of $X \setminus \{x\}$ that do not contain x_0,*

(c) *the right-saturations of maximal linearly ordered subsets of $\{y \mid xRy\}$,*

(d) *the maximal subsets A of $\{y \mid xRy\}$ that satisfy both*

  (i) *A is right-saturated*

  (ii) *for any $a, a' \in A$ also $a \wedge a' \in A$.*

Proof. For (a) = (b) see 2.3. The simple proof of (c) = (d) only uses trivial properties of partial orderings like R.

Next we will show that two different sets of type (c) = (d) are disjoint. Since all sets of type (c) are open (by 2.4) and cover $\{y \mid xRy\}$, this will show that the sets of type (a) = (b) refine the sets of type (c) = (d). Suppose $z \in A_1^* \cap A_2^*$, where A_1, A_2 are maximal linearly ordered subsets of $\{y \mid xRy\}$. So $\exists a_i \in A_i\ a_i\boldsymbol{R}z$ $(i = 1, 2)$. By (d) $a_1 \wedge a_2 \in A_1^* \cap A_2^*$, and because $\{y \mid yRa_i\}$ is linearly ordered, and A_i is maximal, even $a_1 \wedge a_2 \in A_1 \cap A_2$. Now it is obvious that $A_1^* = A_2^* = \{y \mid \exists a .\ . xRa\boldsymbol{R}(a_1 \wedge a_2) \text{ and } a\boldsymbol{R}y\}$.

Finally we have to show that a set A of type (c) = (d) is connected. Suppose not, $A = Y \oplus Z$. Choose $y' \in Y$, $z \in Z$ and suppose $y' \wedge z \in Y$. Put $y = y' \wedge z$. Thus $z \in \{z' \mid y\boldsymbol{R}z'\} \subset A$, but $\{z' \mid y\boldsymbol{R}z'\}$ is connected (2.3). This contradicts that $\{z' \mid y\boldsymbol{R}z'\}$ meets both Y (in y) and Z (in z).

3. PROPERTIES OF **RAM**

Summarizing the results of the previous sections we can easily deduce the following theorems:

3.1. Theorem. *For $(X, x_0) \in$ **RAM** we have the following properties:*

(a) *The x_0 is unique, and is the only point that may be a non-cutpoint. For each $x \in X \setminus \{x_0\}$ the subspace $X \setminus \{x\}$ has infinitely many components.*

(b) *X is not compact.*

(b′) *If X is Hausdorff, then no point of X has arbitrarily small neighborhoods with (countably) compact boundary.*

(c) *If $D \subset X$ is dense then* card D = card X.

(c′) *X cannot be both separable and regular.*

(d) *For any $x \in X$ there exists an open neighborhood O in X such that $(O, x) \in$* **RAM**.

(e) *No point of X has arbitrarily small open connected neighborhoods.*

(f) *There exists a Hausdorff space X in* **RAM**, *such that X is countable.*

(g) *Each connected subset of X has at most one non-cutpoint.*

Proof.

(a) See section 1, and 2.2.

(b) It is well-known that any compact connected T_1-space has at least two non-cutpoints.

(b′) Combine Thm. 6, p. 10 of [2] with (d).

(c) This follows from 2.2 (or 3.1(a)) and 1.3.

(c′) A regular, countable space is Lindelöf, thus normal. Hence it admits non-constant real-valued functions and thus it can not be connected.

(d) Let $O = \{y \in X \mid x\boldsymbol{R}y\}$. Then O is connected (see 2.3). Suppose $C \subset O$ is connected and $x \in C$, and $y \in C^- \setminus C$ for some $y \in O$. Let A be the component of $X \setminus \{x\}$ containing y. Now $(X \setminus A) \cup C$ contains x_0 and is connected, because of 1.2 and $x \in (X \setminus A) \cap C$. Thus $(X \setminus A) \cup C$ is closed, contradictory to $y \in$ $\in A \cap C^- \setminus C$. Thus $(O, x) \in$ **RAM**.

(e) Because of (d) it suffices to show that x_0 has no proper connected open neighborhood. This is clear from the definition of RAM.

(f) See section 4.

(g) Let $C \subset X$ be connected, suppose $x \in C$ and $C \setminus \{x\}$ is connected. Then C is not contained in $\{y \mid x \text{ non } Ry\}$, because x is an isolated point of this set (2.3). Because $C \setminus \{x\}$ is connected, C cannot meet both $\{y \mid x \text{ non } \boldsymbol{R}y\}$ and $\{y \mid xRy\}$ (cf. 2.3). So $C \subset \{y \mid x\boldsymbol{R}y\}$. Thus x is the R-smallest element of C, and this makes x unique.

The properties of R and its relation to the topology of X is the subject of the following theorem.

3.2. Theorem. *For* $(X, x_0) \in$ **RAM** *and the relation R defined in section* 1 *the following holds*:

(a) *R is a partial ordering on X and x_0 is the R-smallest point of X.*

(b) *If $x, y \in X$ are not $\boldsymbol{R}$-comparable, then there is no common upper bound, i.e.* non $\exists z \in X$ *such that $x\boldsymbol{R}z$ and $y\boldsymbol{R}z$.*

(b′) *For all $x \in X$ $\{z \in X \mid z\boldsymbol{R}x\}$ is linearly ordered.*

(c) $\forall x \in X\ \exists x' \in X\ xRx'$.

(d) *If $A \subset X$ is linearly ordered, then either*

  (i) *A is well-ordered by $\tilde{R}$ or*

  (ii) *there exist $a_n \in A$ such that $a_n R a_{n+1}$, $n = 1, 2, \ldots$ and each set $\{a_1, a_2, \ldots\}$ with this property is R-cofinal in A.*

(e) *For each $x \in X \setminus \{x_0\}$ there exist infinitely many disjoint maximal linearly ordered subsets of $\{y \mid xRy\}$.*

(f) *For each $x \in X$ the set $\{y \mid x\boldsymbol{R}y\}$ is open.*

(g) *If $A \subset X$ is the right-saturation of a linearly ordered subset and $x \in X$ is an immediate predecessor, then $A^- = A \cup \{x\}$.*

(h) *For each $x \in X$ the set $\{y \mid x \text{ non } Ry\}$ is connected.*

(i) *For each $x \in X$ the components of $X \setminus \{x\}$ are at first the closed set $\{y \in$ $\in X \mid y \text{ non } Ry\}$ and furthermore all (infinitely many, open) right-saturations of maximal linearly ordered subsets of $\{y \in X \mid xRy\}$.*

Before we prove 3.2 we first mention the following "converse":

3.3. Theorem. *Let X be any non-degenerate T_1-space, $x_0 \in X$ and suppose R is a relation on X, such that conditions (a), (b), (f) and (g) of* 3.2 *hold. Then $(X, x_0) \in$ $\in$ **RAM** iff X is connected.*

Let, moreover, R satisfy (h) and let R' be the relation on X defined in section 1 *(and named R there). Then $R = R'$. Hence R also satisfies (c), (d), (e) and (i).*

Proof of 3.3. Suppose that $x_0 \in C \subset X$ and C is connected. We will show that for each $x \in X \setminus C$ the set $\{y \mid x\boldsymbol{R}y\}$ (which is open by 3.2(f)) does not meet C, thus proving that $C = C^-$ and $(X, x_0) \in$ **RAM**. Suppose $z \in \{y \mid x\boldsymbol{R}y\} \cap C$. Let A be a maximal linearly ordered subset of $\{y \mid x\boldsymbol{R}y\}$ containing z, and A^* its right-saturation. Thus

$$A^* = \{y \in X \mid \exists z' \in X \; xRz'\boldsymbol{R}z \text{ and } z'\boldsymbol{R}y\}\,.$$

Clearly x immediately R-precedes A^*. Hence by (g) A^* is closed in $X \setminus \{x\}$. By (f) A^* is also open. So $C = (C \setminus A^*) \oplus (C \cap A^*)$ and $x_0 \in C \setminus A^*$ while $z \in C \cap A^*$, a contradiction.

We have just shown that if xRy then x separates x_0 from y. I.e. $xRy \Rightarrow xR'y$. If R satisfies 4.2(h), then we immediately obtain x non $\boldsymbol{R}y \Rightarrow x$ non $\boldsymbol{R}'y$, so $R = R'$.

Proof of 3.2. For (a), (b) and (b') see 2.1. Note that (b) and (b′) are equivalent. For (c) and (d) see 2.5. Property (e) follows from 2.6 and 2.2 if we note that the right-saturations of disjoint maximal linearly ordered subsets of $\{y \mid xRy\}$ are disjoint. In 2.3 we prove (f) and (h), and (g) follows from 2.4. Finally (i) is proved in 2.6.

4. EXAMPLES

4.1. First we will construct a countable Hausdorff space in **RAM**. The simplest ordered set $(X, <)$ that satisfies the conditions (a)–(d) of theorem 3.2 can be described as follows:

$$X = \cup \{\mathbf{N}^n \mid n \in \mathbf{N}\} \cup \{0\}$$

where $\mathbf{N} = \{1, 2, 3, \ldots\}$. The ordering is defined by

$$(n_1, \ldots, n_k) < (n'_1, \ldots, n'_{k'}) \quad \text{if} \quad k' > k \quad \text{and} \quad n_i = n'_i \quad \text{for} \quad i = 1, \ldots, k\,,$$

and moreover $0 < (n_1, \ldots, n_k)$ for every sequence $(n_1, \ldots, n_k)$.

If we take for X the weakest topology such that 3.2(f) and 3.2(g) are satisfied, then it is easy to check that $(X, 0) \in$ **RAM**. However X will not be T_2, so we have to make the topology finer (= larger).

For $x \in X$ we define:

$$\text{length } x = \begin{cases} 2 & \text{if } x = 0 \\ k + 2 & \text{if } x \in \mathbf{N}^k \end{cases}$$

$$\phi(x) = \begin{cases} 0 & \text{if } x \in \mathbf{N} \cup \{0\} \\ (n_1, \ldots, n_{k-1}) & \text{if } x = (n_1, \ldots, n_k) \end{cases}$$

$$\max x = \begin{cases} 0 & \text{if } x = 0 \\ \max(n_1, \ldots, n_k) & \text{if } x = (n_1, \ldots, n_k) \end{cases}$$

As a subbase for the topology we take all sets

(i) $\{z \mid x \leqq z\}$ for each $x \in X$

(ii) $\{z \mid x \nleqq z \text{ and } z \neq \phi(x)\}$ for each $x \in X$

(iii) $\{z \mid \text{the only primes dividing length } z \text{ are } p_1, \ldots, p_n\}$ for any finite set of primes $\{p_1, \ldots, p_n\}$.

We will show that X is a Hausdorff-space, and $(X, 0) \in$ **RAM** in several steps (4.2–4.5).

4.2. *X is a Hausdorff-space.*

Proof. Let $u, v \in X$. We distinguish between

(a) $u < v$ and even $u < \phi(v)$.
(b) Neither $u < v$ nor $v < u$.
(c) $u = \phi(v)$.

(a) In this case $\{y \mid v \nleqq y \text{ and } y \neq \phi(v)\}$ and $\{z \mid v \leqq z\}$ are disjoint neighborhoods of u and v.

(b) Now $\{z \mid u \leqq z\}$ and $\{z \mid v \leqq z\}$ are disjoint neighborhoods of u and v.

(c) Let $\{p_1, \ldots, p_n\}$ be the set of all primes dividing length u, and $\{q_1, \ldots, q_m\}$ the same for v. Because length $v = (\text{length } u) + 1$ we find $\{p_1, \ldots, p_n\} \cap \cap \{q_1, \ldots, q_m\} = \emptyset$ and so we can find disjoint neighborhoods for u and v in the subbase of type (iii).

4.3. *Any connected $C \subset X$ that contains* 0 *is closed.*

Proof. If $u \in X \setminus C$ then it is easy to see that $C \cap \{y \mid u \leqq y\} = \emptyset$ (cf the proof of 3.3 or apply 3.3, using 4.4 and 4.5).

4.4. *For each $u \in X$ the points u and $\phi(u)$ have no disjoint closed neighborhoods.*

Proof. Let $u = (a_1, \ldots, a_l)$. For each $x \in X$, and each finite family $\{x_1, \ldots, x_n\}$ such that $x_i \nleqq x$ and $x \neq \phi(x_i)$ $(i = 1, \ldots, n)$ we define the following neighborhood of x:

$$U_{x_1,\ldots,x_n}(x) = \{z \mid x \leqq z\} \cap \{z \mid \text{each prime dividing length } z \text{ also divides length } x\} \cap$$
$$\cap \bigcap_{i=1}^{n} \{z \mid x_i \nleqq z \text{ and } z \neq \phi(x_i)\} .$$

It should be clear that if $n, x_1, \ldots, x_n$ vary we obtain a neighborhood basis for x. (We may even vary only over those x_i for which $x < \phi(x_i)$).

Now suppose $U_{x_1,\ldots,x_n}(\phi(u))$ and $U_{x_{n+1},\ldots,x_m}(u)$ are two arbitrary basic neighborhoods of $\phi(u)$ and u.

Put

$$N = \max\{\max x_i \mid i = 1, \ldots, m\} + 1$$
$$L = (\text{length } u) \,.\, (\text{length } \phi(u)) - 2$$
$$v = (a_1, \ldots, a_l, N, N, \ldots, N) \in \mathbf{N}^L .$$

We will show that

$$v \in (U_{x_1,\ldots,x_n}\phi(u))^- \cap (U_{x_{n+1},\ldots,x_m}(u))^- .$$

Let $U_{m+1,\ldots}(v)$ be an arbitrary neighborhood of v. Put

$$N' = \max\{\max x_i \mid i = 1, \ldots, n, \ldots, m, m+1, \ldots\} + 1 .$$

Let p, q be two primenumbers, such that p divides length $\phi(u)$ and q divides length u. Choose a prime r such that $p^r > L$ and $q^r > L$. Then

$$\underbrace{(\underbrace{a_1, \ldots, a_l, N, \ldots, N}_{L \text{ numbers}}, N', \ldots, N')}_{p^r \text{ numbers}} \in (U_{x_1,\ldots,x_n}(\phi(u)) \cap U_{x_{m+1},\ldots}(v)$$

and

$$\underbrace{(\underbrace{a_1, \ldots, a_l, N, \ldots, N}_{L \text{ numbers}}, N', \ldots, N')}_{q^r \text{ numbers}} \in (U_{x_{n+1},\ldots,x_m}(u)) \cap U_{x_{m+1},\ldots}(v)$$

4.5. *X is connected.*

Proof. Suppose $X = A \oplus B$, $0 \in A$ and $y \in B$ has minimal length. Then $\phi(y) \in A$, contradictory to 4.4.

This completes the construction of the Hausdorff example. The following constructions are of a different kind, as they start of with any $X \in \mathbf{RAM}$, modifying this in order to obtain certain properties.

4.6. *For each* $(X, x_0) \in \mathbf{RAM}$ *there exists a* $X' \subset X$, *such that* $(X', x_0) \in \mathbf{RAM}$ *and* $X' \setminus \{x_0\}$ *is connected.*

Proof. Let X' be the union of $\{x_0\}$ and any component C of $X \setminus \{x_0\}$. Obviously we only have to show that X' is connected, i.e. $x_0 \in C^-$ in X. Now $X \setminus C$ is connected (by 1.2), contains x_0 and hence is closed. As X is connected C cannot be also closed in X, so $C^- = C \cup \{x_0\}$.

4.7. *There exists a* $(X, x_0) \in \mathbf{RAM}$ *such that* x_0 *has arbitrarily small connected (but not open) neighborhoods.*

Proof. Choose $(Y, y_0) \in \mathbf{RAM}$ and $y \in Y \setminus \{y_0\}$ arbitrarily. Let y' be the immediate predecessor of y, and put $Z = \{z \in Y \mid y'\mathbf{R}z\}$. By 2.3 $(Z, y') \in \mathbf{RAM}$. Now let $X' = Z \times \{1, 2, 3, \ldots\}$ be the countable topological sum of disjoint copies of Z. We define an equivalence relation $\sim$ on X' by identifying $(y', n) \sim (y, n + 1)$. We define X as the thus obtained quotient space union one point, x_0, "at infinity":

$$X = (X'/\sim) \cup \{x_0\},$$

where the n^{th} basic neighborhood of x_0 is defined as x_0 union all equivalence classes of $X'/\sim$ that do not correspond to points $(x, k) \in X'$ with k less than n.

It is easy to see that (X, x_0) satisfies the requirements.

Generalizing the above proof, and applying it to the case where (Z, y') is the space described in 4.1 it is easy to prove the following.

4.8. *For every infinite ordinal* α *their exists an* $X \in \mathbf{RAM}$ *which has a linearly ordered subset of* $\tilde{R}$*-ordertype* α, *but none of larger ordertype.*

5. RELATION TO ORDERABLE SPACES, GENERALIZATION AND MAIN CONJECTURE

Definitions. We say that a space X is *weakly lineary orderable* if there exists a linear order $<$ on X, of which the order topology is weaker than the given topology of X. In the special case that both topologies coincide we say that X is *(strictly) linearly orderable.*

A connected space is said to have *property* H (cf. [2]) or *property* V_1 (cf. [1]) respectively if every connected subset has at most two, respectively at most one non-cutpoint. Here $p \in X$ is cutpoint of X if $X \setminus \{p\}$ is not connected.

It is easy to prove that a connected weakly linearly orderable space is strictly linearly orderable iff it is locally connected. In [2] the following theorem can be found:

A connected T_2*-space* X *is weakly linearly orderable iff it satisfies* H *and for each* $p \in X$ $X \setminus \{p\}$ *has at most two components.*

It was asked, [2] p. 270, whether property H alone is equivalent to weak linear orderability. Now it is easily seen from properties 3.1g and 3.1a that each $X \in \mathbf{RAM}$ satisfies H (and even V_1), whilst no $X \in \mathbf{RAM}$ is weakly linearly orderable.

In [1] the above results have been extended and generalized to the class of V_1-spaces. This class is closely related to **RAM**, as can be seen from the following characterization ([1], prop. 8, p. 8):

A non-degenerate space Y is a V_1-space iff for some $(X, x_0) \in \mathbf{RAM}$ either $X = Y$ or $Y = X \setminus \{x_0\}$ and this set is connected.

Our main conjecture is that no $X \in \mathbf{RAM}$ can be completely regular. We conjecture even that each continuous real-valued function on $X \in \mathbf{RAM}$ is constant.

References

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