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THE SEMIGROUP OF FULLY INDECOMPOSABLE RELATIONS AND HALL RELATIONS

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An $n \times n$ non-negative matrix is called decomposable if there is a permutation matrix P such that PAP^{-1} is of the form

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}, \tag{1}$$

where B, D are square matrices. Otherwise it is called indecomposable.

The following notion has been introduced by MARCUS and MINC [4] and used in LEVIN [3].

A is called partly decomposable if there are two permutation matrices P_1, P_2 such that P_1AP_2 is of the form (1). Hence A is partly decomposable if it contains an $s \times (n - s)$ zero submatrix for some s, $1 \leqq s \leqq n - 1$.

A matrix which is not partly decomposable is called fully indecomposable. Clearly if A is fully indecomposable, it contains in each row (column) at least two non-zero elements.

In this paper we sharpen some results of [3] by formulating them in terms of binary relations. Next we introduce a class of binary relations called Hall relations and study the relationship between two quite naturally obtained semigroups.

Since all results of this paper depend only on the distribution of zeros and non-zeros in A it is advantageous to work with binary relations instead of non-negative matrices. We follow in this direction the idea contained in author's paper [5], thought the present paper is independent of [5].

Let $\Omega = \{a_1, \ldots, a_n\}$ be a finite set containing $n > 1$ different elements. By a binary relation on Ω we mean a subset ϱ of $\Omega \times \Omega$. To any non-negative matrix $A = (a_{ij})$ we assign the binary relation ϱ_A on Ω by the requirement $(a_i, a_j) \in \varrho_A \Leftrightarrow \Leftrightarrow a_{ij} > 0$. The usual matrix representation of a binary relation ϱ on Ω as a $0-1$ matrix will be denoted by $M(\varrho)$. Hence $M(\varrho_A)$ is obtained from A by writing formally 1 on the places where $a_{ij} > 0$ and zero on the places where $a_{ij} = 0$.

If A, B are non-negative matrices, then $A \to \varrho_A$, $B \to \varrho_B$ imply $AB \to \varrho_A\varrho_B$, where $\varrho_A\varrho_B$ is the usual product of two relations.

The following notations will be used:

$$a_i\varrho = \{x \in \Omega \mid (a_i, x) \in \varrho\}, \quad \varrho a_i = \{y \in \Omega \mid (y, a_i) \in \varrho\},$$

$$pr_1(\varrho) = \bigcup_{j=1}^{n} \varrho a_j, \quad pr_2(\varrho) = \bigcup_{i=1}^{n} a_i\varrho.$$

If N is a subset of Ω, then $N\varrho = \bigcup_{a_i \in N} a_i\varrho$. The universal relation on Ω (i.e. $\Omega \times \Omega$) will be denoted by ω. The empty relation will be denoted by z. The diagonal of $\Omega \times \Omega$ is denoted by Δ. The set all permutation relations on Ω will be denoted by Π.

The set $B = B(\Omega)$ of all binary relations on Ω (including z) forms with respect to the multiplication of relations a semigroup containing 2^{n^2} elements. The semigroup B has (for our purposes) one unpleasant property, namely that if, e.g., $pr_1(\varrho) = N \subset \Omega$, then $pr_1(\varrho^2)$ can be $\neq N$. E.g., if ϱ is the relation on $\Omega = \{a_1, a_2, a_3\}$ with

$$M(\varrho) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{then} \quad M(\varrho^2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $pr_1(\varrho) = \{a_2, a_3\}$, while $pr_1(\varrho^2) = \{a_3\}$.

Having in mind further applications we restrict our considerations to the set $B_0 = B_0(\Omega)$ consisting of all binary relations on Ω with $pr_1(\varrho) = pr_2(\varrho) = \Omega$. It is easy to see that the set B_0 (with respect to the multiplication of relations) is again a semigroup, i.e. if $pr_1(\varrho) = pr_2(\varrho) = pr_1(\sigma) = pr_2(\sigma) = \Omega$, then $pr_1(\varrho\sigma) = pr_2(\varrho\sigma) = \Omega$. Clearly any permutation relation on Ω is contained in B_0.

We now introduce formal definitions:

Definition 0,1. An element $\varrho \in B_0$ is called *decomposable* if there is a $\pi \in \Pi$ such that $M(\pi\varrho\pi^{-1})$ is of the form (1), where B, D are $0-1$ square matrices. Otherwise it is called indecomposable.

Definition 0,2. An element $\varrho \in B_0$ is called *partly decomposable* if there are two elements $\pi_1, \pi_2 \in \Pi$ such that $M(\pi_1\varrho\pi_2)$ is of the form (1). Otherwise it is called fully indecomposable.

In all what follows the cardinal number of a subset $N \subset \Omega$ will be denoted by $|N|$.

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Theorem 1,1. *A binary relation $\varrho \in B_0$ is fully indecomposable iff for any proper subset $N \subset \Omega$ we have $|N| < |N\varrho|$.*

Proof. 1. Suppose that ϱ is fully indecomposable and $N = \{a_\alpha, a_\beta, \ldots, a_\nu\}$ contains $s < n$ different elements. If $N\varrho = a_\alpha\varrho \cup a_\beta\varrho \cup \ldots \cup a_\nu\varrho$ contained at most s different elements, say $\{a_{l_1}, a_{l_2}, \ldots, a_{l_s}\}$, we would have

$$a_\alpha\varrho \subset \{a_{l_1}, \ldots, a_{l_s}\}\,,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$a_\nu\varrho \subset \{a_{l_1}, \ldots, a_{l_s}\}\,.$$

Hence $M(\varrho)$ would contain s rows (namely those corresponding to $a_\alpha, a_\beta, \ldots, a_\nu$) with $n - s$ zeros in the same columns, i.e. an $s \times (n - s)$ zero submatrix, contrary to the assumption that ϱ is fully indecomposable. Hence $N\varrho$ contains at least $s + 1$ elements $\in \Omega$, i.e. $|N| < |N\varrho|$.

2. Let conversely ϱ be partly decomposable. Then $M(\varrho)$ contains an $s \times (n - s)$ zero submatrix. Suppose that the s rows are contained in the rows indexed by $\{\alpha, \beta, \ldots \ldots, \nu\} \subsetneqq \{1, 2, \ldots, n\}$. Denote $N = \{a_\alpha, a_\beta, \ldots, a_\nu\}$. Then $N\varrho = a_\alpha\varrho \cup a_\beta\varrho \cup \ldots \ldots \cup a_\nu\varrho$ contains at most s different elements $\in \Omega$, so that $|N\varrho| \leqq |N|$. This proves Theorem 1,1.

Before going further recall that a finite semigroup S with zero element z_0 is called nilpotent with the index of nilpotency l if $S^l = z_0$ while $S^{l-1} \neq z_0$. In this terminology we have:

Theorem 1,2. *The set $F = F(\Omega)$ of all fully indecomposable relations $\in B_0$ is a nilpotent semigroup having ω as the zero element. The index of nilpotency of F is equal to the number $n - 1$.*

Proof. 1. ω is clearly the zero element of F since $\omega \in F$, and for any $\varrho \in B_0$ we have $\varrho\omega = \omega\varrho = \omega$.

2. Suppose that $\varrho_1, \varrho_2 \in F$. Take any proper subset $N \subset \Omega$. We then have $|N| < |N\varrho_1|$. If $N\varrho_1$ is a proper subset of Ω, we also have $|N\varrho_1| < |N\varrho_1\varrho_2|$, hence $|N| < |N\varrho_1\varrho_2|$. If $|N\varrho_1| = n$, hence $N\varrho_1 = \Omega$, we also have $N\varrho_1\varrho_2 = \Omega$ (since ϱ_2 has no zero column), hence again $|N| < |N\varrho_1\varrho_2|$. Therefore $\varrho_1\varrho_2 \in F$, i.e. F is a semigroup.

3. Suppose that $\varrho_1, \varrho_2, \ldots, \varrho_{n-1}$ are $n - 1$ elements $\in F$. Let a_i be any element $\in \Omega$. By Theorem 1,1

$$|a_i\varrho_1| \leqq |a_i\varrho_1\varrho_2| \leqq |a_i\varrho_1\varrho_2\varrho_3| \leqq \ldots \leqq |a_i\varrho_1\varrho_2 \ldots \varrho_{n-1}|\,.$$

Hereby the sign of equality in $|a_i\varrho_1 \ldots \varrho_{l-1}| \leqq |a_i\varrho_1 \ldots \varrho_{l-1}\varrho_l|$ holds iff $|a_i\varrho_1 \ldots \ldots \varrho_{l-1}| = n$. Since $a_i\varrho_1$ contains at least two different elements $\in \Omega$, there is an integer $1 \leqq k_i \leqq n - 1$ such that $|a_i\varrho_1 \ldots \varrho_{k_i}| = n$. This implies $a_i\varrho_1 \ldots \varrho_{k_i} = \Omega$. The more $a_i\varrho_1 \ldots \varrho_{n-1} = \Omega$ for any $a_i \in \Omega$. Therefore $\varrho_1\varrho_2 \ldots \varrho_{n-1} = \Omega \times \Omega = \omega$.

4. To show that the index of nilpotency is exactly $n - 1$ it is sufficient to show

that for any $n > 1$ there is a $\varrho \in F$ such that $\varrho^{n-2} \neq \omega$. As remarked in [3] the relation ϱ with

$$M(\varrho) = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

has this property.

Corollary 1,3. *For any $\varrho \in F$, we have $\varrho^{n-1} = \omega$.*

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The result of Theorem 1,1 is very close to a result first proved by PH. HALL (See, e.g., [2].)

Let Ω be as above and let $\{S_1, S_2, \dots, S_n\}$ be a system of n subsets of Ω. We say that the system $\{S_1, S_2, \dots, S_n\}$ admits a system of distinct representatives $\{a_{i_1}, a_{i_2}, \dots, a_{i_n}\}$ if we can choose the a_{i_l} in such a manner that $a_{i_1} \in S_1, \dots, a_{i_n} \in S_n$ and $a_{i_j} \neq a_{i_l}$ for $j \neq l$.

To any system $\{S_1, \dots, S_n\}$ we can construct a relation ϱ on Ω such that $a_1\varrho = S_1, \dots, a_n\varrho = S_n$. Conversely, to any relation ϱ on Ω there is a natural system of subsets of Ω, namely "the rows", which can be studied whether they admit a system of distinct representatives or not.

The following is a natural definition of a class of binary relations on Ω. A binary relation ϱ is called a Hall relation if the sets $a_1\varrho, a_2\varrho, \dots, a_n\varrho$ admit a system of distinct representatives. An equivalent (for our purposes more suitable) definition is the following.

Definition 2,1. A binary relation $\varrho \in B_0$ is called a *Hall relation* if there is a permutation relation $\pi \in \Pi$ such that $\pi \subset \varrho$.

The set of all Hall relations on Ω will be denoted by $H = H(\Omega)$.

Clearly, if $\varrho \in H$, then $\pi_1\varrho\pi_2 \in H$ for any $\pi_1, \pi_2 \in \Pi$.

The following Proposition 2,2 is merely a reformulation of the famous theorem of Ph. Hall. (See, e.g., [2], Theorem 5, 1, 2.)

Proposition 2,2. *A binary relation $\varrho \in B_0$ is a Hall relation iff $|N| \leqq |N\varrho|$ for any non-empty subset N of Ω.*

Theorem 2,3. *The set H of all Hall relations $\in B_0$ is a semigroup. The set F of all fully indecomposable relations $\in B_0$ is a two-sided ideal of H.*

Proof. 1. If $\varrho, \sigma \in H$ and N is a non-empty subset of Ω, then $|N\varrho| \geqq N$, $|N\sigma| \geqq N$ imply $|N\varrho\sigma| \geqq |N\varrho| \geqq |N|$. Hence $\varrho\sigma \in H$. This proves that H is a semigroup.

1a. This follows also immediately from the fact: $\pi_1 \subset \varrho$, $\pi_2 \subset \sigma$ $(\pi_1, \pi_2 \in \Pi)$ imply $\pi_1\pi_2 \subset \varrho\sigma$ and $\pi_1\pi_2 \in \Pi$.

2. Suppose that $\varrho \in H$, $\sigma \in F$ and N is a proper subset of Ω.

i) If $|N| \leqq |N\varrho| < |\Omega|$, we have $|N\varrho\sigma| > |N\varrho| \geqq N$. If $|N| < |N\varrho| = |\Omega|$, we have $|N\varrho\sigma| = |\Omega| > |N|$. In both cases $|N\varrho\sigma| > |N|$, hence $\varrho\sigma \in F$.

ii) Since $|N\sigma| > |N|$, we have $|N\sigma\varrho| \geqq |N\sigma| > N$, hence $|N\sigma\varrho| > |N|$, i.e. $\sigma\varrho \in F$. This proves that F a two-sided ideal of H.

We now give some characterizations of F as a subsemigroup of H.

Definition 2,4. An element $\varrho \in H$ is called *strongly nilpotent* if $\pi_1\varrho\pi_2$ is nilpotent for any $\pi_1, \pi_2 \in \Pi$.

Theorem 2,5. *The semigroup F is exactly the set of all strongly nilpotent elements $\in H$.*

Proof. 1. If $\varrho \in F$, then since F is a two-sided ideal of H, we have $\pi_1\varrho\pi_2 \in F$ for any $\pi_1, \pi_2 \in \Pi \subset H$. Hence $\pi_1\varrho\pi_2$ is nilpotent for any $\pi_1, \pi_2 \in \Pi$.

2. Suppose that ϱ is such that $\pi_1\varrho\pi_2$ is nilpotent for any $\pi_1, \pi_2 \in \Pi$. Hence $\pi_1\varrho\pi_2$ is indecomposable for any $\pi_1, \pi_2 \in \Pi$. This means that ϱ is fully indecomposable, therefore $\varrho \in F$.

Example 2,6. The relation ϱ_1 on $\Omega = \{a_1, \ldots, a_n\}$, $n > 2$, with

$$M(\varrho_1) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 0 \end{pmatrix}$$

is nilpotent, it is contained in H, but it is not strongly nilpotent, since by changing suitably the columns we get the relation ϱ_1' with

$$M(\varrho_1') = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 1 \end{pmatrix},$$

which is clearly not nilpotent.

Theorem 2,5 may be formulated as follows.

Theorem 2,7. *F is the maximal nilpotent ideal of H.*

Proof. It is easy to see that a finite semigroup contains a unique maximal nilpotent ideal. (See, e.g., [1].) Suppose that U is a nilpotent ideal of H and $F \subsetneqq U$. Then there is a nilpotent element $\sigma \in U - F$. Since $\sigma \in U$, we also have $\pi_1 \sigma \pi_2 \in U$ for any $\pi_1, \pi_2 \in \Pi$. Now since σ is not strongly nilpotent there is a couple $\pi', \pi'' \in \Pi$ such that $\pi' \sigma \pi'' \in U$ is decomposable. But $\pi' \sigma \pi''$ is certainly not nilpotent, a contradiction with the supposition that U is nilpotent.

There is also an other semigroup characterization of F.

Recall that an ideal Q of a semigroup S is called prime ideal if $AB \subset Q$ implies either $A \subset Q$ or $B \subset Q$, A, B being two-sided ideals of S.

Proposition 2,8. *The semigroup F is precisely the intersection of all prime ideals of H.*

Proof. 1. We first prove that F is contained in any prime ideal of H. Let Q be a prime ideal of H. Since $F^{n-1} = \omega$ and $\omega \in Q$, $F \,.\, F^{n-2} \subset Q$ implies either $F \subset Q$, in which case our statement is proved, or $F^{n-2} \subset Q$. This implies $F \,.\, F^{n-3} \subset Q$, hence again either $F \subset Q$ or $F^{n-3} \subset Q$. Repeating this argument we find $F \subset Q$.

2. Our assertion will be proved if we are able to prove that to any $\mu \in H - F$ there is a prime ideal Q_μ such that $\mu \notin Q_\mu$.

Note first that if $\mu \in H - F$, then $\pi_1 \mu \pi_2 \in H - F$ for any $\pi_1, \pi_2 \in \Pi$. For, if there were $\pi_3 \mu \pi_4 \in F$ for some $\pi_3, \pi_4 \in \Pi$, this would imply $\pi_3^{-1}(\pi_3 \mu \pi_4)\, \pi_4^{-1} = \mu \in F$, contrary to the choice of μ.

Now since $\mu \notin F$, there are $\pi', \pi'' \in \Pi$ such that the relation $\sigma = \pi' \mu \pi''$ is decomposable. Hence no member of the sequence

$$\sigma, \sigma^2, \sigma^3, \dots \tag{2}$$

is contained in F.

Let Q_μ be the largest ideal of H which does not meet any element of the sequence (2). Q_μ is not empty since it contains F. We state that Q_μ is a prime ideal of H. Suppose for an indirect proof that there are two ideals of H, $A \not\subset Q_\mu$, $B \not\subset Q_\mu$ such that $AB \subset Q_\mu$. Since $Q_\mu \subsetneqq Q_\mu \cup A$ and $Q_\mu \subsetneqq Q_\mu \cup B$, there are some powers σ^t and σ^v such that $\sigma^t \in Q_\mu \cup A$, $\sigma^v \in Q_\mu \cup B$, i.e. $\sigma^t \in A$, $\sigma^v \in B$. But then we would have $\sigma^{t+v} \in Q_\mu$, contrary to the construction of Q_μ.

Now μ is not contained in the ideal Q_μ, since otherwise $\pi' \mu \pi'' = \sigma$ would be contained in Q_μ, contrary to the choice of σ. This completes the proof our statement.

3

In addition to the semigroups F and H we introduce two further subsets of B_0.

The set of all indecomposable relations $\in B_0$ will be denoted by $I = I(\Omega)$. A relation $\varrho \in B_0$ is called primitive if there is an integer $k = k(\varrho)$ such that $\varrho^k = \omega$. A primitive relation is indecomposable. The set of all primitive relations $\in B_0$ will be denoted by $P = P(\Omega)$. We have

$$B_0 \supset I \supset P \supset F\,.$$

Note that P is exactly the set of all nilpotent elements of the whole semigroup B_0.

Remark 1. For $n > 2$ neither I nor P are semigroups. To show this take for ϱ the relation ϱ_1 defined in Example 2,6 and for σ the relation on $\Omega = \{a_1, \ldots, a_n\}$, $n > 2$, given by

$$M(\sigma) = \begin{pmatrix} 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Then

$$M(\varrho\sigma) = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 1 \\ 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

This shows that $\varrho\sigma$ is decomposable. Hence-though both σ and ϱ are contained in P – their product is outside of I.

We now extend Definition 2,4. An element $\varrho \in P$ is called strongly nilpotent if for any $\pi_1, \pi_2 \in \Pi$ we have $\pi_1\varrho\pi_2 \in P$.

[Note that in Theorem 2,5 we have dealt only with elements $\in H \cap P$.]

If an element $\varrho \in P$ is strongly nilpotent, then $\pi_1\varrho\pi_2$ is nilpotent, hence indecomposable, for any $\pi_2, \pi_2 \in \Pi$. This means that ϱ is fully indecomposable and therefore $\varrho \in F$. Conversely if $\varrho \in F$, then ϱ is fully indecomposable, i.e. $\pi_1\varrho\pi_2$ is indecomposable for any $\pi_1, \pi_2 \in \Pi$ and since $\pi_1\varrho\pi_2 \in F$, $\pi_1\varrho\pi_2$ is nilpotent.

We have proved:

Supplement to Theorem 2,5. *The semigroup F is exactly the set of all strongly nilpotent elements $\in P$.*

Example 3,1. The relation σ from Remark 1 is contained in P (since $\sigma^2 = \omega$). It is not contained in $H \cap P$, hence certainly not strongly nilpotent.

In some cases it is possible to decide directly whether $\varrho \in P$ is contained in F.

Proposition 3,2. *If $\varrho \in P$ and $\Delta \subset \varrho$, then $\varrho \in F$.*

Proof. By supposition $a_i \in a_i\varrho$ (for $i = 1, 2, \ldots, n$). Hence $N \subset N\varrho$ for every subset N of Ω. We show that for a proper subset N of Ω we cannot have $N = N\varrho$. Since ϱ is primitive, there is an integer $k = k(\varrho)$ such that $\varrho^k = \omega$. Now $N = N\varrho$ implies $N = N\varrho = N\varrho^2 = \ldots = N\varrho^k = N\omega = \Omega$, a contradiction with $N \neq \Omega$. Therefore for any proper subset N of Ω we have $N \subsetneqq N\varrho$ so that $|N| < |N\varrho|$. By Theorem 1,1 $\varrho \in F$.

An analogous result is the following.

Proposition 3,3. *If $\varrho \in P$ and $M(\varrho)$ contains at least $(n - 1)$ one's in the main diagonal, then $\varrho \in H$.*

Proof. Suppose without loss of generality that $\{(a_1, a_1), \ldots, (a_{n-1}, a_{n-1})\} \subset \varrho$. If $N \subset \{a_1, \ldots, a_{n-1}\}$, then $N \subset N\varrho$ so that $|N| \leqq |N\varrho|$.

Suppose that $N = N_1 \cup a_n$, where $N_1 \subset \{a_1, \ldots, a_{n-1}\}$. Then $N\varrho = N_1\varrho \cup a_n\varrho$. Now $N_1 \subset N_1\varrho$ and $N_1 \neq N_1\varrho$, since (analogously as above) $N_1 = N_1\varrho$ would imply $N_1 = N_1\varrho^k = N_1\omega = \Omega$, contrary to the assumption. Hence $|N_1\varrho| \geqq |N_1| + 1$. Therefore $|N\varrho| = |N_1\varrho \cup a_n\varrho| \geqq |N_1\varrho| \geqq |N_1| + 1 = |N|$. Since in all cases $|N| \leqq$ $\leqq |N\varrho|$, we have $\varrho \in H$, q.e.d.

Example 3,4. The relation ϱ with

$$M(\varrho) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

is primitive. By proposition 3,3 it is contained in H but it is not contained in F.

Proposition 3,3 is closely connected with a result given by Šidák [6]. Let $\varrho \in P$ and denote by $\|\varrho\|$ the number of non-zero elements in $M(\varrho)$. The sequence

$$\|\varrho\|, \|\varrho^2\|, \|\varrho^3\|, \ldots \tag{3}$$

need not be non-decreasing. It is proved in [6]: If $M(\varrho)$ contains at least $n - 1$ non-zero elements in the main diagonal, then the sequence (3) is non-decreasing.

This result is a consequence of Proposition 3,3 and of the following general statement:

Proposition 3,5. *If $\varrho \in H$, then the sequence (3) is non-decreasing. More precisely: There is an integer $l = l(\varrho)$ such that*

$$\|\varrho\| < \|\varrho^2\| < \ldots < \|\varrho^{l-1}\| < \|\varrho^l\| = \|\varrho^{l+1}\| = \ldots$$

Proof. If $\varrho \in H$, there is a $\pi \in \Pi$ such that $\pi \subset \varrho$, hence $\Delta \subset \pi^{-1}\varrho$. Denote (for typographical reasons) $\pi^{-1} = \pi_1$. Then $\Delta \subset \pi_1\varrho$ implies

$$\Delta \subset \pi_1\varrho \subset \pi_1^2\varrho^2 \subset \ldots \subset \pi_1^l\varrho^l \subset \ldots \tag{4}$$

Let l be the least integer such that $\pi_1^l \varrho^l = \pi_1^{l+1} \varrho^{l+1}$. We then have $\pi_1^{l-1} \varrho^{l-1} \subsetneqq \pi_1^l \varrho^l$, while $\pi_1^l \varrho^l = \pi_1^v \varrho^v$ for any integer $v \geqq l + 1$. Since $\|\pi_0 \varrho^u\| = \|\varrho^u\|$ for any $\pi_0 \in \Pi$, it follows from (4)

$$\|\varrho\| < \|\varrho^2\| < \ldots < \|\varrho^l\| = \|\varrho^{l+1}\| = \ldots$$

Remark 2. If $\varrho \in P$, then $\|\varrho^l\|$ is equal to the number n^2. It follows from Theorem 4,2 below that if ϱ is an indecomposable Hall relation with index of imprimitivity d, then $\|\varrho^l\|$ is equal to n^2/d.

Example 3,6. The condition $\varrho \in H$ is not necessary in order that the sequence (3) should be non-decreasing. Let, e.g., σ be the relation considered in Remark 1. Here $\sigma \notin H$, $\|\sigma\| = 2n - 1$, while $\|\sigma^2\| = \|\sigma^3\| = \ldots = n^2$.

Remark 3. We have seen that $F \subset H \cap P$. For any $n \geqq 2$ this inclusion is proper. For, the relation ϱ_1 considered in Example 2,6 is contained in $H \cap P$ but is not contained in F.

If ϱ is primitive and not fully indecomposable (i.e. $\varrho \in P - F$), then there are $\pi', \pi'' \in \Pi$ such that

$$M(\pi' \varrho \pi'') = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}.$$

If A, C are not fully indecomposable (of the corresponding order) we may proceed in the same manner and we finally obtain that there are $\pi_1, \pi_2 \in \Pi$ such that

$$M(\pi_1 \varrho \pi_2) = \begin{pmatrix} B_{11} & 0 & \ldots & 0 \\ B_{21} & B_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ B_{s1} & B_{s2} & \ldots & B_{ss} \end{pmatrix}, \tag{5}$$

where any of the B_{ii} is either a fully indecomposable $0-1$ matrix or B_{ii} is a zero matrix of order 1. (A non-zero matrix of order 1 is considered as fully indecomposable.)

The next Proposition characterizes all relations $\in H \cap P$.

Proposition 3,7. *If $\varrho \in B_0$ is primitive, then ϱ is a Hall relation iff there does not exist a zero matrix of order* 1 *in the main diagonal of the decomposition* (5).

We conclude this section with the following remark.

In Theorem 2,7 we have seen that F is the maximal nilpotent ideal of H. There is an analogous pertinent question: What can be said about the maximal nilpotent ideal of B_0 (which is neccessarily contained in P).

The answer is simple:

Theorem 3,8. *The semigroup B_0 does not contain a nilpotent ideal different from ω.*

To prove this it is sufficient to show that B_0 contains a unique non-zero minimal two sided ideal (kernel of B_0) which is not nilpotent. This result has been proved by Zareckij [7]. For the sake of completness we give here a modified proof of this statement.

Denote in the following by τ the relation with

$$M(\tau) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix},$$

i.e. $\tau = (a_n \times \Omega) \cup (\Omega \times a_1)$.

Proposition 3,9. B_0 *contains a unique minimal two-sided non-zero ideal, namely the set* $K = B_0 \tau B_0$.

Proof. Let $J \neq \omega$ be any two-sided ideal of B_0 and $\varrho \neq \omega$ any element $\in J$. We prove that $\tau \in B_0 \varrho B_0$, whence $B_0 \tau B_0 \subset B_0 \varrho B_0 \subset B_0 J B_0 \subset J$. This will prove that K is the kernel of B_0. Since τ is an idempotent, K is not nilpotent.

Since $\varrho \neq \omega$, there is a couple $(a_i, a_j) \notin \varrho$, i.e. $a_j \notin a_i \varrho$. Construct the following two relations $\sigma_1 = (a_n \times \Omega) \cup (\Omega \times a_i)$, $\sigma_2 = (a_j \times \Omega) \cup (\Omega \times a_1)$, both contained in B_0. We then have: $\sigma_1 \varrho = (a_n \times \Omega) \varrho \cup (\Omega \times a_i) \varrho = (a_n \times \Omega) \cup (\Omega \times \times a_i \varrho)$, $\sigma_1 \varrho \sigma_2 = [(a_n \times \Omega) \cup (\Omega \times a_i \varrho)] . [(a_j \times \Omega) \cup (\Omega \times a_1)] = [(a_n \times \Omega) \cup \cup (\Omega \times a_1)] = \tau$.

[We have used $(\Omega \times a_i \varrho)(a_j \times \Omega) = z$.] Now $\tau = \sigma_1 \varrho \sigma_2$ implies $\tau \in B_0 \varrho B_0$, whence (as above) $B_0 \tau B_0 \subset J$. This proves Proposition 3,9.

4

To get some further information concerning Hall relations we may proceed as follows. It is well known that to any $\varrho \in B_0$ there is $\pi \in \Pi$ such that

$$M(\pi \varrho \pi^{-1}) = \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix}. \tag{6}$$

Here A_{ii} is either an indecomposable $0-1$ square matrix or a zero matrix of order 1.

The following is again immediately to see:

Proposition 4,1. $\varrho \in B_0$ *is a Hall relation iff*

i) *there is no zero matrix of order* 1 *in the main diagonal of* (6),

ii) *each of the A_{ii} is a matrix representation of a Hall relation (of the corresponding order).*

It is therefore of some interest to study indecomposable Hall relations, which are not primitive.

Recall first some known facts concerning indecomposable relations. If $\varrho \in B_0$ is indecomposable and not primitive, then there is a least number $d = d(\varrho) > 1$ (called the index of imprimitivity of ϱ) such that ϱ^d is completely decomposable into d relations each of which is primitive (of the corresponding order). I.e., there is a $\pi \in \Pi$ such that

$$M(\pi^{-1}\varrho^d\pi) = \operatorname{diag}\left[C_{11}, C_{22}, \ldots, C_{dd}\right],$$

where each C_{ii} is $0-1$ square matrix representing a primitive relation. By this condition the integer $d > 1$ is uniquely determined.

Suppose that ϱ is indecomposable with index of imprimitivity $d > 1$. Then it is known that Ω can be written as a union of pairwise disjoint subsets $\Omega = A_1 \cup \cup A_2 \cup \ldots \cup A_d$, $A_i \cap A_j = \emptyset$ such that $\varrho \subset (A_1 \times A_2) \cup (A_2 \times A_3) \cup \ldots \cup (A_d \times \times A_1)$. Otherwise formulated there is a $\pi \in \Pi$ such that

$$M(\pi^{-1}\varrho\pi) = \begin{pmatrix} 0, & A_{12}, & 0, & \ldots, & 0 \\ 0, & 0, & A_{23}, & \ldots, & 0 \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots\dots} \\ 0, & 0, & 0, & \ldots, & A_{d-1,d} \\ A_{d1}, & 0, & 0, & \ldots, & 0 \end{pmatrix}, \tag{7}$$

where the zeros in the main diagonal denote square matrices with all elements equal to 0 and A_{ij} are rectangular $0-1$ matrices.

Suppose now that ϱ is indecomposable with index of imprimitivity $d > 1$ and that ϱ is a Hall relation. The condition $|N| \leqq |N\varrho|$ (for any subset $N \subset \Omega$) implies $|A_1| \leqq |A_2|$, $|A_2| \leqq |A_3|, \ldots, |A_{d-1}| \leqq |A_d|$, $|A_d| \leqq |A_1|$. Hence $|A_1| = |A_2| = \ldots \ldots = |A_d|$ and therefore $d \mid n$. (It should be explicitly remarked that, in general, for an indecomposable relation $d \mid n$ need not hold.) All matrices in (7) are square matrices of order n/d. In this case it follows from (7) that there exist $\pi_1, \pi_2 \in \Pi$ such that

$$M(\pi_1\varrho\pi_2) = \operatorname{diag}\left[A_{12}, A_{23}, \ldots, A_{d1}\right]. \tag{8}$$

Since ϱ is a Hall relation, each of the square matrices $A_{12}, \ldots, A_{d1}$ represents a Hall relation on a subset of Ω of cardinality n/d.

We have:

Theorem 4,2. *If ϱ is an indecomposable Hall relation with $d > 1$, then $M(\varrho)$ is partly decomposable into d (non-zero) $0-1$ matrices along the main diagonal each of which has the same order.*

Example 4,3. It would be nice if it were possible to state that the A_{ij} in (8) are all indecomposable (or even fully indecomposable). But this need not be true. Consider the relation ϱ with

$$M(\varrho) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

This relation is an indecomposable Hall relation with $d(\varrho) = 2$. With suitably chosen π_1, π_2 we have

$$M(\pi_1 \varrho \pi_2) = \operatorname{diag}[A_{12}, A_{21}],$$

where

$$A_{12} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Here A_{12} is decomposable.

In general we may state:

Theorem 4,4. *An indecomposable relation ϱ with $d(\varrho) > 1$ is a Hall relation iff the following holds:*

i) $d \mid n$;

ii) *Each A_{ij} in (7) is a square matrix of order n/d;*

iii) *Each A_{ij} represents a Hall relation.*

Example 4,5. None of these conditions can be omited. E.g., the relation ϱ with

$$M(\varrho) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is indecomposable, $d = 2$ divides $n = 6$, but ϱ is not a Hall relation.

5

We conclude this paper by an open problem of a combinatorial character.

It is known that to any $\varrho \in P$ there is a least integer $k = k(\varrho) \leqq (n-1)^2 + 1$ such that $\varrho^k = \omega$.

Hence to any $\varrho \in P$ there is a least integer $l_1 = l_1(\varrho) \geqq 1$ such that $\varrho^{l_1} \in H$ and a least integer $l_2 = l_2(\varrho) \geqq l_1$ such that $\varrho^{l_2} \in F$.

The problem to find exact upper bounds for l_1 and l_2 (in terms of n) seems to be (at this writing) rather difficult. There are some reasons for the following

Conjecture. *For any $\varrho \in P$, we have $l_2 = l_2(\varrho) \leqq n$.*

The number n to the right cannot be certainly replaced by a smaller one. For, if $\varrho^{l_2} \in F$, then with respect to Theorem 1,2, $\varrho^{l_2(n-1)} = \omega$, hence $l_2(n-1) \geqq k(\varrho)$, $l_2 \geqq k(\varrho)/(n-1)$. Now it is know that for the relation ϱ_1 considered in Example 2,6 we have $k(\varrho_1) = (n-1)^2 + 1$, hence $l_2(\varrho_1) \geqq n - 1 + 1/(n-1)$. As a matter of fact $l_2(\varrho_1) = n$, since $\varrho_1^{(n-1)^2} \neq \omega$, while $\varrho_1^{n(n-1)} = \omega$.

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