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MIXED PROBLEM FOR SEMILINEAR HYPERBOLIC EQUATION OF SECOND ORDER WITH DIRICHLET BOUNDARY CONDITION

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1. INTRODUCTION

Mixed problems for hyperbolic equations have been investigated by many authors. Various linear and some non-linear problems for hyperbolic equations or systems of equations of an arbitrary order were discussed for two independent variables (e.g. [1], [2], [3], [13]). In this case more detailed results can be reached by means of the method of characteristics. If we consider the case of more independent variables, then a similar situation when we can consider equations of arbitrary order (including systems of the first order), occurs only in the case of the Cauchy problem (e.g. [10] – in this work we can find some results for semilinear equations). In the case of the linear mixed problem some results for systems of the first order ([8]) or for equations of higher order ([11], [14], [15], [16]) were published, but a more detailed study was done in the case of one equation of second order with the Dirichlet or Neumann boundary conditions (e.g. [4], [5], [6], [7], [9], [11]).

The mixed problem of the Dirichlet type for one equation of the second order is considered also in the present work.

Let Ω be a bounded domain in $\mathbf{R}^n$, $0 < T < +\infty$, and let L be a linear differential operator on $Q = \Omega \times (0, T)$ of the following form:

$$L = \frac{\partial^2}{\partial t^2} + a_1(x, t; D)\frac{\partial}{\partial t} + a_2(x, t; D) \tag{1.1}$$

where $x \in \Omega$, $t \in (0, T)$,

$$a_1(x, t; D) = \sum_{i=1}^{n} h_i(x, t)\frac{\partial}{\partial x_i} + c_1(x, t)\,, \tag{1.2}$$

$$a_2(x, t; D) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x, t)\frac{\partial}{\partial x_i} + c_2(x, t)\,. \tag{1.3}$$

We shall suppose that the coefficients h_i ($i = 1, \ldots, n$) are real valued functions, and $a_2(x, t; D)$ is an elliptic operator satisfying

$$(1.4) \qquad a_{ij}(x, t) = \bar{a}_{ji}(x, t), \quad i, j = 1, \ldots, n, \quad (x, t) \in \bar{Q}$$

$$(1.5) \qquad \exists \delta > 0 \; \forall z \in \mathbf{C}^n \; \forall (x, t) \in Q : \sum_{i,j=1}^{n} a_{ij}(x, t)\, z_i \bar{z}_j \geqq \delta |z|^2 .$$

For this operator we shall consider the following problem: to find a function u satisfying (in a generalized sense which will be described later) the equation

$$(1.6) \qquad Lu = f\left(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right),$$

the initial conditions

$$(1.7) \qquad u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x), \quad x \in \Omega$$

and the boundary condition

$$(1.8) \qquad u(x, t) = g(x, t), \quad (x, t) \in \partial\Omega \times (0, T)$$

($\partial\Omega$ denotes the boundary of the domain Ω), where f, u_0, u_1, g are given functions.

In Theorem 5.1 we shall establish the existence of a unique local solution of this problem, and then in Theorem 5.4, under the assumption of an apriori estimate, the existence of a unique global solution on the whole domain Q. The regularity of these solutions is also included in our results and in §6 some sufficient conditions for the existence of apriori estimates are given.

This paper was inspired by S. Mizohata's work [10] and generalizes his results in the semilinear case from the Cauchy to the mixed problem. The results obtained can be also considered as a generalization from the linear to the semilinear case of some results of S. Mizohata [11] and M. Ikawa [4] concerning the mixed problem. In the examples 1, 2 from §6 we shall show that the results of J. Sather [17], [18] are included as a particular case.

2. NOTATIONS

Euclidean n-dimensional space is denoted by $\mathbf{R}^n$, $\mathbf{C}$ denotes the open Gauss complex plane.

Lebesgue spaces $L_p(V)$ on a domain $V \subset \mathbf{R}^n$ are defined in the usual way; for $1 \leqq p < \infty$ and an integer $k \geqq 0$ we denote by $W_p^{(k)}(V)$ the Sobolev space of all functions defined on V which have generalized derivatives up to the order k belonging to $L_p(V)$. $W_p^{(k)}$ is a Banach space with the norm

$$\|u\|_{k,p} = \left(\sum_{|i| \leqq k} \int_V |D^i u(x)|^p \, dx \right)^{1/p}.$$

For $p = 2$ it is a Hilbert space with the scalar product

$$(u, v)_k = \sum_{|i| \leqq k} \int_V D^i u(x) \, \overline{D^i v(x)} \, dx$$

and we shall denote the norm of $u \in W_2^{(k)}(V)$ briefly by $\|u\|_k$.

Spaces $C^{(k)}(V)$, $C^{(k),\lambda}(\overline{V})$, $C_0^\infty(V)$ are introduced as usual, the space $^\circ W_p^{(k)}(V)$ is defined as the closure of the set $C_0^\infty(V)$ in $W_p^{(k)}(V)$.

We shall use the definition of smoothness of a domain $V \subset \mathbf{R}^n$ (notation $V \in C^{(k),\lambda}$) from [12]. (Roughly speaking it means that there exists a finite system of functions of the class $C^{(k),\lambda}$ describing the boundary ∂V of the domain V.)

We shall often use the following well known

Gronwall lemma. *Let f, g be two non-negative functions defined on the interval $\langle 0, a \rangle$, $a > 0$, $f \in L_1(0, a)$, g non-decreasing, and let*

$$f(t) \leqq C \int_0^t f(s) \, ds + g(t) \quad \forall t \in \langle 0, a \rangle .$$

Then

$$f(t) \leqq e^{Ct} g(t) \qquad \forall t \in \langle 0, a \rangle .$$

For absolutely continuous functions this lemma can be written in the following form:

Lemma. *Let f be an absolutely continuous function on $\langle 0, a \rangle$, $a > 0$ and let $g \in L_1(0, a)$ be a non-negative function. Let*

$$f'(t) \leqq C f(t) + g(t) \quad \text{a.e. in} \quad \langle 0, a \rangle .$$

Then

$$f(t) \leqq e^{Ct} (f(0) + \int_0^t g(s) \, ds \quad \forall t \in \langle 0, a \rangle .$$

For $Q = V \times (0, T)$ ($V \subset \mathbf{R}^n$ a bounded domain) let $C^{(k,m)}(\overline{Q})$ be the space of all functions $f \in C(\overline{Q})$ such that f has all derivatives of the type $(\partial^p/\partial t^p) \, D_x^i f$, $0 \leqq p \leqq m$, $|i| \leqq k$, belonging to $C(\overline{Q})$.

Let B be a Banach space with the norm $\|\cdot\|_B$, $0 < T < \infty$, $k \geqq 0$ an integer. The space $C^{(k)}(0, T; B)$ is defined as the space of all functions defined on the interval

$\langle 0, T\rangle$ with their values in the space B which are k-times continuously differentiable in the norm $\|\cdot\|_B$. $C^{(k)}(0, T; B)$ is the Banach space with the norm

$$\max_{i=0,\dots,k} \sup_{t\in\langle 0,T\rangle} \|u^{(i)}(t)\|_B .$$

For the later use we shall introduce following formal notation:

let $V \subset \mathbf{R}^n$ be a domain, $k \geqq 0$ an integer. We shall say that a function u belongs to the space $C(0, T; H^k(V))$, if

$$u \in \bigcap_{i=0}^{k} C^{(i)}(0, T; W_2^{(k-i)}(V)) .$$

For $u \in C(0, T; H^k(V))$, $t \in \langle 0, T\rangle$ we define

$$|||u(t)|||_k = \|u(t)\|_k + \|u'(t)\|_{k-1} + \dots + \|u^{(k)}(t)\|_0 .$$

For $k \geqq 1$ we denote

$$C(0, T; {}^\circ H^k(V)) = \{u \in C(0, T; H^k(V));\, u(t) \in {}^\circ W_2^{(1)}(V),\ t \in \langle 0, T\rangle\} .$$

It is easily seen that this definition of $C(0, T; {}^\circ H^k(V))$ is equivalent with the conditions

$$u \in C(0, T; H^k(V))\ ;\quad u^{(i)}(t) \in {}^\circ W_2^{(1)}(V)\ ,\quad t \in \langle 0, T\rangle\ ,\quad i = 0, 1, \dots, k-1 .$$

The spaces $C(0, T; H^k(V))$ and $C(0, T; {}^\circ H^k(V))$ are evidently Banach spaces with the norm $\sup_{t\in\langle 0,T\rangle} |||u(t)|||_k$.

Finally, let us formally denote

$$C^{(1)}(0, T; H^k(V)) = \{u \in C(0, T; H^k(V));\ u' \in C(0, T; H^k(V))\} .$$

3. LINEAR PROBLEM

We shall solve the semilinear problem (briefly formulated in §1) by means of sucesive approximations. The solvabity of the mixed problem in the linear case plays here a basic role.

In this case we must find a solution of the linear equation

$$Lu(x, t) = f(x, t) \tag{3.1}$$

on $Q = \Omega \times (0, T)$, $\Omega \subset \mathbf{R}^n$ is a bounded domain, $0 < T < \infty$ (L is the linear differential operator introduced in §1 by relations (1.1)–(1.5)) satisfying the initial conditions

$$u(x, 0) = u_0(x)\ ,\quad u'(x, 0) = u_1(x) \quad \text{for} \quad x \in \Omega \tag{3.2}$$

and the Dirichlet boundary condition

$$(3.3) \qquad u(x, t) = g(x, t) \quad \text{for} \quad (x, t) \in \partial\Omega x(0, T).$$

The classical solution of this problem is a function $u \in C^{(2)}(\bar{Q})$ satisfying the equations (3.1)–(3.3) in the usual sense.

We shall find a generalized solution, more precisely a function $u \in C(0, T; H^2(\Omega))$ such that (3.1) holds a.e. in Q, and (3.2) holds a.e. in Ω. The equation (3.3) is now taken in the sense of traces (which is correct – by the assumption, $u(t) \in W_2^{(1)}(\Omega)$) and because a smooth function defined on $\partial\Omega$ can be extended on the whole domain Ω (see e.g. [12]), we shall assume the function g to be defined apriori on the whole domain Q and to be of the same class as the function u. The boundary condition (3.3) can be then written in the form

$$(3.4) \qquad u - g \in C(0, T; {}^{\circ}H^2(\Omega)).$$

Agreement. For the sake of brevity of notation we shall omit letter Ω in the Sobolev spaces and write only $W_2^{(k)}$ instead of $W_2^{(k)}(\Omega)$, $C(0, T; H^k)$ instead of $C(0, T; H^k(\Omega))$; full notation will be used only for other spaces as e.g. $L_2(Q)$, $C^{(k)}(\bar{Q})$.

If $u \in C(0, T; H^k)$, $k \geqq 2$, is a generalized solution of our problem, we have $f \in C(0, T; H^{k-2})$, $u_0 \in W_2^{(k)}$, $u_1 \in W_2^{(k-1)}$ from the equations (3.1), (3.2), and by agreement $g \in C(0, T; H^k)$. From the definition of the generalized solution it further follows that the following necessary conditions for the existence of a generalized solution $u \in C(0, T; H^k)$ must hold:

$$(3.5) \qquad u^{(i)}(0) - g^{(i)}(0) \in {}^{\circ}W_2^{(1)}, \quad i = 0, 1, \ldots, k - 1.$$

Differentiating with respect to t the equation (3.1) we obtain

$$(3.6) \qquad u^{(i)}(0) = f^{(i-2)}(0) - \sum_{j=0}^{i-2} \binom{i-2}{j} \left(a_1^{(i-2-j)}(x, 0; D)\, u^{(j+1)}(0) + a_2^{(i-2-j)}(x, 0; D)\, u^{(j)}(0)\right), \quad i = 2, 3, \ldots, k - 1$$

where we use the notation

$$(3.7) \qquad a_1^{(j)}(x, t; D) = \sum_{p=1}^{n} h_p^{(j)}(x, t) \frac{\partial}{\partial x_p} + c_1^{(j)}(x, t),$$

$$a_2^{(j)}(x, t; D) = - \sum_{p,q=1}^{n} \frac{\partial}{\partial x_p} \left(a_{pq}^{(j)}(x, t) \frac{\partial}{\partial x_q}\right) + \sum_{p=1}^{n} b_p^{(j)}(x, t) \frac{\partial}{\partial x_p} + c_2^{(j)}(x, t).$$

Now we see that $u^{(i)}(0)$, $i = 2, 3, \ldots, k - 1$ can be successively expressed from the relations (3.6) by means of the given functions u_0, u_1, f and their derivatives, hence the condition (3.5) contains only known functions. The necessary conditions (3.5) for the existence of a generalized solution of the class $C(0, T; H^k)$ $(k \geqq 2)$ will be called the compatibility conditions of order k.

Similar mixed linear problems have been solved in various papers, e.g. [4]–[7], [11]. By the same method as in [4] (by means of the theory of analytic semigroups) we can prove the following existence theorem for the linear problem:

Theorem 3.1. *Let $k \geqq 2$ be an integer and let the following assumptions hold:*

$$\Omega \in C^{(k+1),1}\,, \quad a_{ij} \in C^{(1,1)}(\bar{Q}) \cap C^{(k-1)}(\bar{Q})\,, \quad h_i \in C^{(\max 1, k-2)}(\bar{Q})\,,$$

$$b_i, c_p \in C^{(0,1)}(\bar{Q}) \cap C^{(k-2)}(\bar{Q})\,, \quad i, j = 1, \ldots, n\,, \quad p = 1, 2\,.$$

Let

$$f \in C(0, T; H^{k-2}) \cap C^{(k-1)}(0, T; L_2)\,,$$

$$u_0 \in W_2^{(k)}\,, \quad u_1 \in W_2^{(k-1)}\,, \quad g \in C^{(1)}(0\,, T; H^k)\,,$$

be such functions that the compatibility conditions (3.5) *of order k hold.*

Then there exists a generalized solution $u \in C(0, T; H^k)$ of the problem (3.1), (3.2), (3.4) *satisfying the energy inequality*

$$\|\|u(t)\|\|_k \leqq C(T)\Big(\|u_0\|_k + \|u_1\|_{k-1} + \|\|f(t)\|\|_{k-2} + \\ + \|\|f(0)\|\|_{k-2} + \int_0^t \|\|g'(s)\|\|_k \,\mathrm{d}s + \int_0^t \|f^{(k-1)}(s)\|_0 \,\mathrm{d}s\Big)\,. \tag{3.8}$$

Moreover, the solution of this problem is unique.

4. COMPOSITE FUNCTIONS AND EXTENSIONS OF INITIAL VALUES

Since we want to solve the semilinear equation (1.6), we shall state here first of all some propositions about composite functions of the required type. These propositions are of a form similar to that in [10], Chap. V, but the proofs are a bit different and we shall give them at least in a brief form.

We admit also complex-valued solutions of the equation (1.6), therefore we must take account of functions of the type $f(x, t, v)$ defined for $x \in \bar{\Omega} \subset \mathbf{R}^n$, $t \in \langle 0, T\rangle$, $v \in \mathbf{C}^m$. Such a function f of $n + 1 + m$ variables can be interpreted in the usual way as a function $\tilde{f}$ of $n + 1 + 2m$ real variables $(f(x, t, v) = \tilde{f}(x, t, \operatorname{Re} v, \operatorname{Im} v))$. We shall say that a function f of the mentioned type is of the class $\bar{C}^{(k),\lambda}(\bar{Q} \times \mathbf{C}^m)$ if $\tilde{f} \in C^{(k)}(\bar{Q} \times \mathbf{R}^{2m})$ and $D^\alpha \tilde{f}(x, t, .)$ is the λ-Hölder locally continuous function of $2m$ real variables $v_1, \ldots, v_{2m}$ for each $(x, t) \in \bar{Q}$, $|\alpha| = k$.

For partial derivatives of f with respect to the complex variable we shall use the evident notation $\partial f/\partial \operatorname{Re} v$, $\partial f/\partial \operatorname{Im} v$, $\partial f/\partial v = (\partial f/\partial \operatorname{Re} v; \partial f/\partial \operatorname{Im} v)$. Then if $f \in \bar{C}^{(1)}(\mathbf{C})$, $u \in C^{(1)}(\mathbf{R}^1)$, the function $F(x) = f(u(x))$ is of the class $C^{(1)}(\mathbf{R}^1)$ and

$$F'(x) = \frac{\mathrm{d}f}{\mathrm{d}u}(u(x)) \,.\, u'(x) \; (u'(x) = (\operatorname{Re} u'(x) \,; \;\; \operatorname{Im} u'(x)) \,.$$

Let us formulate one lemma which will be used later:

Lemma 4.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain of the class $C^{(0),1}$ and $u_i \in C^{(k)}(0, T; W_2^{([n/2]+N-k_i)})$, $i = 1, \ldots, q$, where $N \geqq 1$, $k \geqq 0$, $0 \leqq k_i \leqq [n/2] + N - 1$ are integers. Suppose that the n-indices v_i $(i = 1, \ldots, q)$ are such that $|v_i| \leqq [n/2] + N - k_i - 1$ and $\sum_{i=1}^{q} |v_i| \leqq [n/2] + N - \sum k_i$. Then*

$$D^{v_1} u_1 \,.\, D^{v_2} u_2 \,.\, \ldots \,.\, D^{v_q} u_q \in C^{(k)}(0, T; L_2)$$

and there exists a constant $C > 0$ independent of u_i, t such that

$$\left\| D^{v_1} u_1(t) \,.\, D^{v_2} u_2(t) \ldots D^{v_q} u_q(t) \right\|_0 \leqq C \prod_{i=1}^{q} \left\| u_i(t) \right\|_{[n/2]+N-k_i} .$$

This lemma can be proved by means of imbedding theorems in the Sobolev spaces.

For function $f \in \bar{C}^{(k)}(\bar{Q} \times \mathbf{C}^m)$ and $b \geqq 0$ let us denote

$$M(b) = \max_{|\alpha| \leqq k} \; \sup_{(x,t) \in Q} \; \sup_{|v_i| \leqq b} \left| D^\alpha f(x, t, v_1, \ldots, v^m) \right| . \tag{4.1}$$

Theorem 4.2. *Let $f \in \bar{C}^{(k)}(\bar{Q} \times \mathbf{C}^m)$, where $Q = \Omega \times (0, T)$, $\Omega \subset \mathbf{R}^n$ is a bounded domain of the class $C^{(0),1}$, $k \geqq [n/2] + 1$, and let functions $u_i \in C(0, T; H^k)$, $i = 1, \ldots, m$, satisfy the estimate*

$$|u_i(x, t)| \leqq b \,, \quad (x, t) \in \bar{Q} \,, \quad i = 1, 2, \ldots, m$$

for a constant $b \geqq 0$. If we set

$$F(x, t) = f(x, t, u_1(x, t), \ldots, u_m(x, t))$$

then:

(i) $F \in C(0, T; H^k)$;

(ii) *the generalized derivatives of the function F may be evaluated formally as derivatives of a composite function;*

(iii) *there exists a constant $C > 0$ independent of $f, x, t, u_1, \ldots, u_m$ such that for each $t \in \langle 0, T \rangle$ the estimate*

$$|||F(t)|||_k \leqq C \,.\, M(b) \, (1 + \sum_{i=1}^{m} |||u_i(t)|||_k^k) \tag{4.2}$$

holds.

Proof. Let us denote $u(x, t) = (u_1(x, t), \ldots, u_m(x, t))$.

1. Let $t \in \langle 0, T \rangle$ and let us prove $F(., t) \in W_2^{(k)}$: We have $u_i(., t) \in L_2$, $\partial u_i/\partial x_q \in L_2$, therefore $u_i(., t)$ are absolutely continuous on almost each parallel with the axis x_q and (if $[\partial/\partial x_q]$ denotes the derivative in the classical sense) $\partial u_i/\partial x_q = [\partial u_i/\partial x_q]$ a.e. in Ω (see e.g. [12]).

According to our assumptions, for each $(x, t) \in \bar{Q}$ we have $|F(x, t)| \leqq M(b)$ and therefore $F(., t) \in L_2$.

From the above mentioned facts and the assumptions of the theorem we immediately get: $F(., t)$ is absolutely continuous on almost each parallel with the axis x_q and and $[\partial F/\partial x_q](., t) \in L_2$, therefore $\partial F/\partial x_q = [\partial F/\partial x_q]$ and consequently $F(., t) \in W_2^{(1)}$ (and the item (ii) from our theorem holds).

This process can be repeated up to order k, except that we must use Lemma 4.1 to prove that $[D_x^\alpha F](., t) \in L_2$, $|\alpha| > 1$.

Finally, we obtain $F(., t) \in W_2^{(k)}$ and from the expression for the derivatives of F we obtain (again using Lemma 4.1) the estimate

$$\|F(t)\|_k \leqq C \, . \, M(b) \, (1 + \sum_{i=1}^{m} \|u_i(t)\|_k^k) \, .$$

2. F has the generalized derivative with respect to t: The proof is analogous as in the item 1, because if $u \in C^{(1)}(0, T; L_2)$, then u has the generalized derivative $\partial u/\partial t \in L_2(Q)$.

3. $$\frac{\partial F}{\partial t}(., t) \in W_2^{(k-1)} \quad \text{for} \quad t \in \langle 0, T \rangle :$$

We have

$$\left[\frac{\partial F}{\partial t}\right](x, t) = \frac{\partial f}{\partial t}(x, t, u(x, t)) + \sum \frac{\partial f}{\partial u_i}(x, t, u(x, t)) \, u_i'(x, t) \quad \text{for a.e.} \quad x \in \Omega \, .$$

Applying to this expression the same method as in the item 1 we obtain the required proposition.

4. In this way we obtain successively: there exists $(\partial^i F/\partial t^i)(., t) \in W_2^{(k-i)}$, $i = 0, 1, \ldots, k$ and for each $t \in \langle 0, T \rangle$ the inequality (4.2) holds.

5. Continuity and differentiability of F in the required norms can be proved by writting the derivatives of F as derivatives of a composite function with the help of Lemma 4.1.

Theorem 4.3. *Let Ω, Q, k satisfy the assumptions of Theorem* 4.2, *let $f \in \bar{C}^{(k),\lambda}(\bar{Q} \times \mathbf{C}^m)$, $\lambda \in (0, 1\rangle$ and let functions $u_{ij} \in C(0, T; H^k)$, $i = 1, \ldots, m$, $j = 1, 2$ satisfy the inequality*

$$\max_{t,i,j} |||u_{ij}(t)|||_k \leqq b$$

for some constant $b \geqq 0$. *If we set*

$$F_j(x, t) = f(x, t, u_{1j}(x, t), \ldots, u_{mj}(x, t)) , \quad j = 1, 2 ,$$

then $F_j \in C(0, T; H^k)$ *and there exists a constant* $K(b) \geqq 0$ *independent of* u_{ij}, t *such that for each* $t \in \langle 0, T \rangle$ *we have*

$$(4.3) \qquad |||F_1(t) - F_2(t)|||_k \leqq K(b) \, . \max_{\mu=\lambda,1} \sum_{i=1}^{m} |||u_{i1}(t) - u_{i2}(t)|||_k^{\mu} \, .$$

Proof. Since $k \geqq [n/2] + 1$, the Sobolev imbedding theorem implies that there exists a constant $S > 0$ such that

$$\max_{t,j} |u_{ij}(x, t)| \leqq S \, . \max_{t,j} |||u_{ij}(t)|||_k \leqq Sb \, , \quad (x, t) \in \bar{Q}$$

and hence Theorem 4.2 yields $F_j \in C(0, T; H^k)$. According to this theorem we also obtain an explicit expression for the derivatives of F_j and the estimate (4.3) can be derived by direct computation with the help of Lemma 4.1.

From Theorems 4.2, 4.3 it immediately follows

Theorem 4.4. *Let* Ω, Q, k *satisfy the assumptions of Theorem* 4.2, *let* $f \in \bar{C}^{(k)}(\bar{Q} \times \mathbf{C}^{n+2})$ *and let functions* $u_i \in C(0, T; H^{k+1})$, $i = 1, 2$ *satisfy the inequality*

$$\max_{t,i} |||u_i(t)|||_{k+1} \leqq b$$

for a constant $b \geqq 0$. *If we set*

$$F_i(x, t) = f\left(x, t, u_i(x, t) , \quad u_i'(x, t) , \quad \frac{\partial u_i}{\partial x_1}(x, t), \ldots, \frac{\partial u_i}{\partial x_n}(x, t)\right), \quad i = 1, 2 ,$$

then

(i) $F_j \in C(0, T; H^k)$, $i = 1, 2$;

(ii) *the generalized derivatives of the functions* F_i *may be evaluated formally as derivatives of a composite function*;

(iii) *there exists a constant* $C > 0$ *independent of* f, x, t, u_i *such that for each* $t \in \langle 0, T \rangle$ *the estimates*

$$(4.4) \qquad |||F_i(t)|||_k \leqq C \, . \, M(Sb) \, (1 + |||u_i(t)|||_{k+1}^{k}) \, , \quad i = 1, 2$$

hold;

(iv) *moreover, if* $f \in \bar{C}^{(k),\lambda}(\bar{Q} \times \mathbf{C}^{n+2})$ *for* $\lambda \in (0, 1\rangle$, *then there exists a constant* $K(b) > 0$ *independent of* t, u_i *such that for each* $t \in \langle 0, T \rangle$ *we have*

$$(4.5) \qquad |||F_1(t) - F_2(t)|||_k \leqq K(b) \, . \max_{\mu=\lambda,1} (|||u_1(t) - u_2(t)|||_{k+1}^{\mu}) \, .$$

We shall solve the semilinear problem by means of successive approximations. Taking into account the compatibility conditions, we shall have to make a suitable choice of the initial approximation. To this purpose we shall use the following

Theorem 4.5. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain of the class $C^{(0),1}$, let $k \geqq 1$ be an integer, $T > 0$ and let $u_i \in W_2^{(k-i)}$, $i = 0, 1, \ldots, k-1$.*

Then there exists a function $U \in C(0, T; H^k)$ such that

$$U^{(i)}(0) = u_i \quad \textit{for} \quad i = 0, 1, \ldots, k-1$$

and

$$|||U(t)|||_k \leqq C_T \sum_{i=0}^{k-1} \|u_i\|_{k-i}$$

where $C_T \leqq 0$ is a constant independent of u_i.

This theorem can be proved analogously as in [19], Chap. III and therefore we omit the proof.

5. SEMILINEAR PROBLEM

In the present paragraph we shall solve the semilinear problem indicated in §1.

Let L be a linear hyperbolic differential operator given by relations (1.1)–(1.5) on a domain $Q = \Omega \times (0, T)$, where Ω is a bounded domain in $\mathbf{R}^n$, $0 < T < \infty$. Given functions $u_0(x)$, $u_1(x)$, $g(x, t)$, $h(x, t)$ and a sufficiently smooth function $f(x, t, v_1, \ldots, v_{2+n})$ (defined on $\bar{Q} \times \mathbf{C}^{2+n}$) we want to find a function $u \in C(0, T; H^k)$, $k \geqq 2$ such that

$$u(0) = u_0, \quad u'(0) = u_1$$

$$u - g \in C(0, T; {}^{\circ}H^k) \tag{5.1}$$

$$Lu(x, t) = f(x, t, u(x, t), u'(x, t), \frac{\partial u}{\partial x_1}(x, t), \ldots, \frac{\partial u}{\partial x_n}(x, t)) + h(x, t).$$

Naturally, the compatibility conditions must hold also for the semilinear problem (5.1), i.e., the relations

$$u^{(i)}(0) - g^{(i)}(0) \in {}^{\circ}W_2^{(1)}, \quad i = 0, 1, \ldots, k-1 \tag{5.2}$$

must be satisfied. If we denote

$$F(x, t) = f\left(x, t, u(x, t), \quad u'(x, t), \quad \frac{\partial u}{\partial x_1}(x, t), \ldots\right),$$

we obtain from the equation (5.1):

$$u^{(i)}(0) = F^{(i-2)}(0) + h^{(i-2)}(0) - \sum_{j=0}^{i-2} \binom{i-2}{j} \cdot \tag{5.3}$$
$$\cdot \left(a_1^{(i-2-j)}(x, 0; D)\, u^{(j+1)}(0) + a_2^{(i-2-j)}(x, 0; D)\, u^{(j)}(0)\right)$$

for $i = 2, 3, \ldots, k - 1$. But (if f is sufficiently smooth) according to Theorem 4.4 we have the expression

$$F^{(i-2)}(x, 0) = f^{(i-2)}\left(x, 0, u_0(x), u_1(x), \frac{\partial u_0}{\partial x_1}(x), \ldots, \frac{\partial u_0}{\partial x_n}(x)\right) + \tag{5.4}$$
$$+ \sum_\alpha C_\alpha D^\alpha_{t,v} f(x, 0, u_0(x), \ldots) \,.\, S_\alpha(x)$$

for the derivatives of F. Here $S_\alpha(x)$ are products containing derivatives of the functions u, u', $\partial u/\partial x_i$ with respect to t of order at most $i - 2$, i.e. derivatives of the functions $u(0), u'(0), \ldots, u^{(i-1)}(0)$ with respect to x_i. Therefore (as well as in the linear case) we can successively express $u^{(i)}(0)$ by means of u_0, u_1, f, h and, consequently, the compatibility conditions (5.2) contain only known functions.

We shall suppose that for the coefficients of the operator L and for the domain Ω the assumptions of Theorem 3.1 hold for an integer $k \geqq [n/2] + 2$. Further, let

$$u_0 \in W_2^{(k)}, \quad u_1 \in W_2^{(k-1)}, \quad f \in \bar{C}^{(k-1),\lambda}(\bar{Q} \times \mathbf{C}^{n+2}), \quad \lambda \in (0, 1\rangle,$$
$$h \in C(0, T; H^{k-2}) \cap C^{(k-1)}(0, T; L_2), \quad g \in C^{(1)}(0, T; H^k)$$

be such functions that the compatibility conditions (5.2) hold

As we said, the solution of the problem (5.1) will be found by means of successive approximations. First of all, according to Theorem 4.5 there exists a function $v_0 \in C(0, T; H^k)$ such that

$$v_0^{(i)}(0) = u^{(i)}(0), \quad i = 0, 1, \ldots, k - 1 \tag{5.5}$$

(where $u^{(i)}(0)$ are the known functions from (5.3)).

Put

$$F_0(x, t) = f\left(x, t, v_0(x, t), v_0'(x, t), \frac{\partial v_0}{\partial x_1}(x, t), \ldots\right).$$

From Theorem 4.4 it follows that $F_0 \in C(0, T; H^{k-1})$, and if we express the derivatives of F_0 analogously as in (5.4) we see (taking into account (5.5)) that for u_0, u_1, g, $F_0 + h$ the "linear" compatibility conditions of order k hold. Consequently, Theorem 3.1 implies the existence of a function $v_1 \in C(0, T; H^k)$ such that

$$v_1 - g \in C(0, T; {}^\circ H^k), \quad v_1(0) = u_0, \quad v_1'(0) = u_1, \quad Lv_1(t) = F_0(t) + h(t).$$

Put $F_1(x, t) = f(x, t, v_1(x, t), v_1'(x, t), \ldots)$. It is easily seen that for $u_0, u_1, g, F_1 + h$ the "linear" compatibility conditions hold again and we can repeat the above reasoning.

Step by step we construct a sequence $\{v_q\}_{q=0}^{\infty} \subset C(0, T; H^k)$ such that

$$v_q(0) = u_0\,, \quad v_q'(0) = u_1\,, \quad v_q - g \in C(0, T; {}^\circ H^k)\,, \qquad (5.6)$$
$$Lv_q(t) = F_{q-1}(t) + h(t)\,, \quad q = 1, 2, 3, \ldots$$

Here we use the notation

$$F_q(x, t) = f\left(x, t, v_q(x, t), v_q'(x, t), \frac{\partial v_q}{\partial x_1}(x, t), \ldots\right). \qquad (5.7)$$

Now we shall prove that there exists a constant $B > 0$ and $\Delta \in (0, T\rangle$ such that

$$|||v_q(t)|||_k \leqq B\,, \quad t \in \langle 0, \Delta\rangle\,, \quad q = 0, 1, \ldots \qquad (5.8)$$

Let us denote

$$\beta = \sup_{\langle 0,T\rangle} |||v_0(t)|||_k\,, \qquad (5.9)$$

$$\gamma = \|u_0\|_k + \|u_1\|_{k-1} + |||g(0)|||_k + 2|||F_q(0)|||_{k-2} + \qquad (5.10)$$
$$+ \int_0^T |||g'(s)|||_k \,\mathrm{d}s + 2 \sup_{\langle 0,T\rangle} |||h(t)|||_{k-2} + \int_0^T \|h^{(k-1)}(s)\|_0 \,\mathrm{d}s\,.$$

Since $|||F_q(0)|||_{k-2}$ does not depend on $q = 0, 1, \ldots$ our definition of γ is correet.

Now for $F_q \in C(0, T; H^{k-1})$ we can write

$$|||F_q(t)|||_{k-2} \leqq |||F_q(0)|||_{k-2} + \int_0^t |||F_q'(s)|||_{k-2} \,\mathrm{d}s$$

therefore the energy inequality (3.8) implies

$$|||v_q(t)|||_k \leqq C(T)\left(\gamma + \int_0^t |||F_{q-1}(s)|||_{k-1} \,\mathrm{d}s\right). \qquad (5.11)$$

Put

$$B = C(T)(\beta + \gamma + 1)\,. \qquad (5.12)$$

Evidently $|||v_0(t)|||_k \leqq B$ for $t \in \langle 0, T\rangle$ and then we have from Theorem 4.4 for $s \in \langle 0, T\rangle$

$$|||F_0(s)|||_{k-1} \leqq C\,.\,M(SB)\,.\,(1 + |||v_0(s)|||_k^k) \leqq C\,.\,M(SB)\,.\,(1 + B^k)\,.$$

Hence we obtain from (5.11) for $t \in \langle 0, \Delta \rangle$ where $\Delta \in (0, T\rangle$,

$$|||v_1(t)|||_k \leqq C(T) \, . \left(\gamma + \int_0^\Delta |||F_0(s)|||_{k-1} \, ds \right) \leqq$$

$$C(T) \, . \, (\gamma + \Delta(C \, . \, M(SB) \, (1 + B^k))) \, .$$

If we put

$$\Delta = \min \left(T, \frac{1}{C \, . \, (1 + M(SB)) \, . \, (1 + B^k)} \right), \tag{5.13}$$

we have $|||v_1(t)|||_k \leqq C(T) \, . \, (\gamma + 1) \leqq B$. Now we can prove by induction that for our B, Δ (5.8) holds.

Further we shall show that $\{v_q\}$ is a Cauchy sequence in $C(0, \Delta; H^k)$:

From (5.6) we have

$$(v_{q+1} - v_q)(0) = 0 \, , \quad (v_{q+1} - v_q)'(0) = 0 \, , \quad v_{q+1} - v_q \in C(0, \Delta; {}^\circ H^k) \, ,$$
$$L(v_{q+1} - v_q)(t) = F_q(t) - F_{q-1}(t) \, , \quad q = 1, 2, \ldots$$

Consequently, it follows from the energy inequality (3.8)

$$|||v_{q+1}(t) - v_q(t)|||_k \leqq C(T) \int_0^t |||F_q(s) - F_{q-1}(s)|||_{k-1} \, ds \, , \quad q = 1, 2, \ldots \, .$$

Since (5.8) holds, we have from Theorem 4.4

$$|||F_q(s) - F_{q-1}(s)|||_{k-1} \leqq K(B) \max_{\mu = \lambda, 1} |||v_q(s) - v_{q-1}(s)|||_k^\mu$$

for $q = 1, 2, \ldots$, $s \in \langle 0, \Delta \rangle$ ($K(B)$ does not depend on q, s). Consequently

$$|||v_{q+1}(t) - v_q(t)|||_k \leqq C(T) \, K(B) \int_0^t \max_{\mu = \lambda, 1} |||v_q(s) - v_{q-1}(s)|||_k^\mu \, ds$$

for $q = 1, 2, \ldots$, $t \in \langle 0, \Delta \rangle$.

Now, since the constant C from Theorem 4.4 can be taken $\geqq 1$, we have $\Delta \leqq 1$ and if we denote

$$A = \max(1, C(T) \, K(B)) \, , \quad a = \max\left(\sup_{\mu = \lambda, 1 \; \langle 0, \Delta \rangle} |||v_1(s) - v_0(s)|||_k^\mu, 1 \right)$$

we obtain:

$$|||v_2(t) - v_1(t)|||_k \leqq aAt \, ,$$

$$|||v_3(t) - v_2(t)|||_k \leqq A_0 \int_0^t ((aAs)^\lambda + aAs) \, ds \leqq aA^2 \int_0^t (s^\lambda + s) \, ds \leqq$$

$$\leqq a \, . \, 2A^2 t^{\lambda+1} (\lambda + 1)^{-1} \leqq a \, . \, 2A^2 t^{\lambda+1} (\lambda + 1)^{-\lambda} \, .$$

Let us introduce λ_q, $q = 0, 1, 2, \ldots$ by relations

$$\lambda_0 \equiv 1\,; \quad \lambda_1 \equiv \lambda\,, \quad \lambda_2 \equiv \lambda^\lambda, \ldots, \lambda_{q+1} = \lambda_q^\lambda, \ldots$$

(evidently $\lambda = \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq 1$).

Now we prove by induction

$$(5.14) \qquad |||v_{q+1}(t) - v_q(t)|||_k \leqq a\,.\,2^{q-1} A^q t^{((q-1)\lambda)^{\lambda_{q-2}}+1} \prod_{p=1}^{q-1} (((p\lambda)^{\lambda_{p-1}} + 1)^{-\lambda})$$

for $q = 2, 3, 4, \ldots$ For $q = 2$ (5.14) holds, and if we suppose that (5.14) holds for some $q > 2$, we have

$$|||v_{q+2}(t) - v_{q+1}(t)|||_k \leqq A \int_0^t \sum_{\mu=\lambda,1} |||v_{q+1}(s) - v_q(s)|||_k^\mu \,\mathrm{d}s \leqq$$

$$\leqq a\,.\,2^q A^{q+1} t^{(((q-1)\lambda)^{\lambda_{q-2}}+1)^\lambda+1} [(((q-1)\,\lambda)^{\lambda_{q-2}} + 1)^\lambda + 1]^{-1}\,.$$

$$.\,[\prod_{p=1}^{q-1} ((p\lambda)^{\lambda_{p-1}} + 1)]^{-\lambda} \leqq$$

$$\leqq a\,.\,2^q A^{q+1} t^{(q\lambda)^{\lambda_{q-1}}+1} ((q\lambda)^{\lambda_{q-1}} + 1)^{-1} (\prod_{p=1}^{q-1} ((p\lambda)^{\lambda_{p-1}} + 1))^{-\lambda}$$

because

$$((q-1)\,\lambda)^{\lambda_{q-2}} + 1 \geqq ((q-1)\,\lambda)^{\lambda_{q-2}} + \lambda^{\lambda_{q-2}} \geqq (q\lambda)^{\lambda_{q-2}}\,.$$

Further

$$(q\lambda)^{\lambda_{q-1}} + 1 \geqq ((q\lambda)^{\lambda_{q-1}} + 1)^\lambda$$

and immediately we see that (5.14) holds for $q + 1$.

From (5.14) it follows ($t \leqq \Delta \leqq 1$):

$$|||v_{q+1}(t) - v_q(t)|||_k \leqq a\,.\,2^{q-1} A^q \prod_{p=1}^{q-1} ((p\lambda)^{\lambda_{p-1}} + 1)^{-\lambda} \equiv \alpha_q$$

for $q = 2, 3, \ldots$, and from this relation it is seen that $\{v_q\}$ is a Cauchy sequence in $C(0, \Delta; H^k)$, because $\sum_{q=2}^{\infty} \alpha_q$ is a convergent series (we have $\lim_{q\to\infty} \lambda_q = 1$).

Therefore there exists $u \in C(0, \Delta; H^k)$ such that $u = \lim_{q\to\infty} v_q$ in $C(0, \Delta; H^k)$. From (5.6) it follows

$$u - g \in C(0, \Delta; {}^\circ H^k)\,, \quad u(0) = u_0\,, \quad u'(0) = u_1$$

and

$$(5.15) \qquad |||u(t)|||_k \leqq B \quad \text{for} \quad t \in \langle 0, \Delta \rangle\,.$$

Further $Lv_q \to Lu$ in $C(0, \Delta; W_2^{(k-2)})$ when $q \to \infty$, because L is a continuous operator from $C(0, \Delta; H^k)$ to $C(0, \Delta; W_2^{(k-2)})$.

From (5.8), (5.15) and Theorem 4.4 applied to the function

$$F(x, t) = f\left(x, t, u(x, t), \quad u'(x, t) \frac{\partial u}{\partial x_1}(x, t), \ldots, \frac{\partial u}{\partial x_n}(x, t)\right)$$

we have

$$\|F(t) - F_{q-1}(t)\|_{k-2} \leqq K(B) \max_{\mu=\lambda,1} |||u(t) - v_{q-1}(t)|||_{k-1}^{\mu}$$

and consequently $F(t) = \lim F_q(t)$ in $W_2^{(k-2)}$ and u is the required solution of our problem (5.1) on the interval $\langle 0, \Delta \rangle$. Hence we completed the proof of the following

Theorem 5.1 — local existence. *Let the coefficients of the operator L (defined by (1.1)–(1.5)) and the domain $\Omega \subset R^n$ satisfy the assumptions of Theorem* 3.1 *for an integer $k \geqq [n/2] + 2$. Let*

$$u_0 \in W_2^{(k)}, \quad u_1 \in W_2^{(k-1)}, \quad h \in C(0, T; H^{k-2}) \cap C^{(k-1)}(0, T; L_2),$$
$$g \in C^{(1)}(0, T; H^k), \quad f \in \bar{C}^{(k-1),\lambda}(\bar{Q} \times \mathbf{C}^{n+2}), \quad 0 < \lambda \leqq 1$$

be such functions that the compatibility conditions (5.2) *hold.*

Then there exists $\Delta \in (0, T\rangle$ such that the mixed semilinear problem (5.1) *has on $\langle 0, \Delta\rangle$ a unique solution $u \in C(0, \Delta; H^k)$.*

Uniqueness follows from the following more general

Theorem 5.2 — uniqueness. *Let the assumptions of Theorem* 5.1 *hold. Then the mixed semilinear problem* (5.1) *has at most one solution on an arbitrary interval $\langle 0, t\rangle \subset \langle 0, T\rangle$.*

Proof. If $u, v \in C(0, t; H^k)$ are two solutions of (5.1), we put $w = u - v$, $b = \sup_{\langle 0,t\rangle} (|||u(s)|||_k + |||v(s)|||_k)$. Then from the energy inequality and from Theorem 4.4 we have

$$|||w(s)|||_k \leqq C(T)\, K(b) \int_0^s \max_{\mu=\lambda,1} |||w(r)|||_k^{\mu}\, dr\,.$$

From this inequality we obtain analogously as in the proof of convergence of successive approximations $|||w(s)|||_k = 0$, i.e. $u = v$.

Now we shall consider the question of the existence of a global solution on the whole interval $\langle 0, T\rangle$. First we shall introduce an apriori estimate:

Definition 5.3. Let $L, \Omega, u_0, u_1, h, g, f, k$ satisfy the assumptions of Theorem 5.1. We shall say that the apriori estimate for the semilinear mixed problem (5.1) holds, if

$$\exists C_A \geqq 0\ \forall t \in (0, T\rangle : u \in C(0, t; H^k) \quad \text{is a solution of (5.1)} \Rightarrow$$
$$\Rightarrow |||u(s)|||_k \leqq C_A\ \forall s \in \langle 0, t\rangle\,.$$

Further we shall suppose that the apriori estimate holds.

A global solution will be found by continuing the known local solution on $\langle 0, \Delta\rangle$ by means of Theorem 5.1.

Using this theorem one can find a local solution on the interval $\langle 0, \Delta\rangle$, where Δ is given by the relation (5.13). For the constant γ from (5.10) we have the estimate

$$\gamma \leqq \sup_{\langle 0,\Delta\rangle} |||u(t)|||_k + |||g(0)|||_k + 2C \,.\, M(S \,.\, \sup_{\langle 0,\Delta\rangle} |||u(t)|||_k) \,.$$

$$. (1 + \sup_{\langle 0,\Delta\rangle} |||u(t)|||_k^k) +$$

$$+ \int_0^T (|||g'(s)|||_k + \|h^{(k-1)}(s)\|_0) \, ds + 2 \,.\, \sup |||h(t)|||_{k-2} \leqq$$

$$\leqq C_A + C \,.\, M(SC_A) \,.\, (1 + C_A^k) + |||g(0)|||_k + \ldots + \equiv \gamma_0$$

because the apriori estimate holds. The constant γ_0 does not depend on u, t.

For the initial approximation v_0 we have from Theorem 4.5

$$\beta = \sup_{\langle 0,T\rangle} |||v_0(t)|||_k \leqq C(T) \sum_{i=0}^{k-1} \|u^{(i)}(0)\|_{k-i} \leqq C(T)\, C_A \equiv \beta_0 \,.$$

Therefore we have an estimate for the constant B defined by (5.12):

$$B \leqq C(T)\,(\beta_0 + \gamma_0 + 1) \equiv B_0$$

where B_0 does not depend on u, t.

Then (5.13) implies

$$\Delta \geqq \frac{1}{C(1 + M(SB_0))\,(1 + B_0^k)} \equiv \Delta_0 \,.$$

Now if we have a solution u on some interval $\langle 0, t\rangle \subset \langle 0, T\rangle$, then according to Theorem 5.1 there exists $\Delta(t) > 0$ and $v \in C(t, t + \Delta(t); H^k)$ such that

$$v(t) = u(t)\,, \quad v'(t) = u'(t)\,, \quad v - g \in C(t, t + \Delta(t); {}^\circ H^k)\,,$$

$$Lv(s) = f(x, s, v(x, s)\,, v'(x, s), \ldots) + h(x, s) \quad \text{for} \quad s \in \langle t, t + \Delta(t)\rangle$$

(in fact, we can use Theorem 5.1 because the compatibility conditions at the point t are automatically satisfied as a necessary conditions for the existence of the solution of the problem (5.1) on the interval $\langle 0, t\rangle$).

But for $\Delta(t)$ we have the expression

$$\Delta(t) = \min\left(T - t;\ \frac{1}{C(1 + M(SB(t)))\,(1 + B(t)^k)}\right)$$

where $B(t)$ is given in the same way as B from (5.12).

As the function

$$w(s) = \begin{cases} u(s) & \text{for} \quad s \in \langle 0, t \rangle \\ v(s) & \text{for} \quad s \in \langle t, t + \Delta(t) \rangle \end{cases}$$

is apparently a solution of (5.1) on $\langle 0, t + \Delta(t) \rangle$, it follows again from the apriori estimate that $\Delta(t) \geqq \Delta_0$.

Thus we have established the existence of a solution $v(t)$ of (5.1) on $(\Delta_0, 2\Delta_0)$ and the function

$$U(t) = \begin{cases} u(t) & \text{for} \quad t \in \langle 0, \Delta_0 \rangle \\ v(t) & \text{for} \quad t \in \langle \Delta_0, 2\Delta_0 \rangle \end{cases}$$

is the solution of (5.1) on $\langle 0, 2\Delta_0 \rangle$. We can continue the described process, and because $\Delta(t) \geqq \Delta_0$ for each t, after a finite number of steps we obtain a solution on the whole interval $\langle 0, T \rangle$. So we have

Theorem 5.4 – global existence. *Let the assumptions of Theorem 5.1 be satisfied and let, moreover, the apriori estimate holds. Then there exists a unique solution $u \in C(0, T; H^k)$ of the semilinear mixed problem (5.1) on the whole interval $\langle 0, T \rangle$.*

Remark 5.5. In the special case when our non-linear term f has the simpler form $f = f(x, t, u)$, Theorems 5.1–5.4 hold also for $k = [n/2] + 1$.

6. APRIORI ESTIMATE

In the present paragraph we shall prove some sufficient conditions on the non-linear term f which establish that the apriori estimate holds (and consequently there exists a global solution).

The apriori estimate was introduced in Definition 5.3. In this definition we ought to have spoken more precisely about "the apriori estimate of order k" – according to the norm for which the estimate is required. Therefore we shall prove first that it is enough to obtain "the apriori estimate of order $[n/2] + 2$":

Theorem 6.1. *The apriori estimate for the problem (5.1) holds if and only if*

$$\exists C_A > 0 \ \forall t \in (0, T\rangle : u \in C(0, t; H^k) \ \text{ is a solution of (5.1)} \Rightarrow \tag{6.1}$$
$$\Rightarrow |||u(s)|||_{[n/2]+2} \leqq C_A \ \forall s \in \langle 0, t \rangle .$$

Remark 6.2. We can see, analogously as in Remark 5.5 that for the simpler equation $Lu(t) = f(x, t, u(x, t))$ it is enough to verify the apriori estimate for $k = [n/2] + 1$.

Proof of Theorem 6.1:

As $|||u(t)|||_{k'} \leqq |||u(t)|||_k$ for $k' \leqq k$, the first implication of Theorem is evident.

Let (6.1) hold and let $k > [n/2] + 2$ be an integer. We have from (3.8) for the solution u of (5.1) on $\langle 0, t\rangle$

$$|||u(s)|||_k \leqq C(T)\left(\gamma + |||F(s)|||_{k-2} + \int_0^s \|F^{(k-1)}(r)\|_0 \,\mathrm{d}r\right) \tag{6.2}$$

where γ is the constant from (5.10), $F(x, t) = f(x, t, u(x, t), \ldots)$. But from Theorem 4.4 it follows

$$|||F(s)|||_{k-2} \leqq C \,.\, M(S \,.\, \sup_{\langle 0,t\rangle} |||u(r)|||_{k-1})\,(1 + |||u(s)|||_{k-1}^{k-2}) \,. \tag{6.3}$$

If we express $F^{(k-1)}(r)$ as the derivative of a composite function (according to Theorem 4.4), with help of Lemma 4.1 we get the estimate

$$\|F^{(k-1)}(r)\|_0 \leqq C \,.\, M(S \,.\, \sup |||u(\tau)|||_{k-1}) \,.\, (1 + (\sup |||u(\tau)|||_{k-1}^{k-2})\,(1 + |||u(r)|||_k) \tag{6.4}$$

Substituting (6.4) and (6.3) into (6.2) and using the Gronwall Lemma we see that $|||u(s)|||_k$ can be estimated by means of $|||u(s)|||_{k-1}$. Now we prove our assertion by repeating this procedure.

Following two propositions are easy to see:

Theorem 6.3. *Let $n = 1$ and let us suppose that there exists a constant $C \geqq 0$ such that*

$$\left|\frac{\partial f}{\partial t}(x, t, z)\right| \leqq C(1 + \sum_{i=1}^{3} |z_i|) \,, \tag{6.5}$$

$$\left|\frac{\partial f_i}{\partial z}\right|(x, t, z) \leqq C \,, \quad i = 1, 2, 3 \tag{6.6}$$

for each $(x, t, z) \in Q \times \mathbf{C}^3$.

Then the apriori estimate for the problem (5.1) holds.

Proof. According to Theorem 6.1 it is enough to estimate $|||u(t)|||_2$. From (3.8) we have for the solution u:

$$|||u(t)|||_2 \leqq C(T)\left(\gamma + \int_0^t \|F'(s)\|_0 \,\mathrm{d}s\right) \tag{6.7}$$

(γ is the constant from (5.10)). But

$$F' = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial z_1} u' + \frac{\partial f}{\partial z_2} u'' + \frac{\partial f}{\partial z_3} \frac{\partial u'}{\partial x} \,,$$

therefore using (6.5), (6.6) we immediately see

$$\|F'(s)\|_0^2 \leqq C(1 + |||u(s)|||_2^2)$$

and our assertion follows from the Gronwall Lemma.

Theorem 6.4. *Let $\Omega, L, u_0, u_1, f, h$ satisfy the assumptions of Theorem* 5.1 *and let f be bounded on $\bar{Q} \times \mathbf{C}^{n+2}$ together with all derivatives up to order $[n/2] + 1$. Then the apriori estimate for the problem* (5.1) *holds.*

We shall omit the proof because it can be carried out by almost the same method as the proofs of Theorems 6.1, 6.3.

Theorem 6.5. *Let $g = 0$ and let Ω, $L, u_0, u_1, h, f(x, t, z)$ satisfy the assumptions of Theorem* 5.1. *Let for $u \in C(0, t; {}^\circ H^2)$ $(t \in (0, T\rangle)$ be*

$$(6.8) \qquad Lu(s) = f(x, s, u(x, s)) + h(s)\,, \quad u(0) = u_0\,, \quad u'(0) = u_1\,.$$

Let there exist a real-valued function $\Phi(x, t, z)$ defined on $\bar{Q} \times \mathbf{C}$ such that for each $(x, t, z) \in \bar{Q} \times \mathbf{C}$,

$$(6.9) \qquad \frac{\partial \Phi}{\partial(\operatorname{Re} z)}(x, t, z) = \operatorname{Re} f(x, t, z)\,, \quad \frac{\partial \Phi}{\partial(\operatorname{Im} z)}(x, t, z) = \operatorname{Im} f(x, t, z)\,,$$

$$(6.10) \qquad \Phi(x, t, z) \leqq C_\Phi\,, \quad C_\Phi > 0$$

and either

$$(6.11a) \qquad -\frac{\partial \Phi}{\partial t}(x, t, z) \leqq C'_\Phi(C_\Phi - \Phi(x, t, z))\,, \quad C'_\Phi > 0$$

or

$$(6.11b) \qquad \left|\frac{\partial \Phi}{\partial t}(x, t, z)\right| \leqq C'_\Phi(1 + |z|^2)\,.$$

Then there exists a constant $C_1 > 0$ such that

$$(6.12) \qquad |||u(s)|||_1 \leqq C_1 \quad s \in \langle 0, t\rangle$$

and consequently the apriori in the case $n = 1$ holds.

Proof. If we put

$$E(s) = (u'(s), u'(s))_0 + \sum_{i,j=1}^{n} \left(a_{ij}(s)\frac{\partial u}{\partial x_j}(s)\,, \frac{\partial u}{\partial x_i}(s)\right)_0$$

then from the elipticity of a_{ij} and from the equivalence of norms in ${}^\circ W_2^{(1)}$ (Friedrich's inequality) we have

$$(6.13) \qquad E(s) \geqq C|||u(s)|||_1^2\,.$$

Further

$$\frac{\mathrm{d}E}{\mathrm{d}s}(s) = 2\mathrm{Re}\,(u''(s), u'(s))_0 + \tag{6.14}$$

$$+ \sum_{i,j} \left[\left(a'_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i} \right)_0 + \left(a_{ij} \frac{\partial u'}{\partial x_j}, \frac{\partial u}{\partial x_i} \right)_0 + \left(a_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u'}{\partial x_i} \right)_0 \right].$$

The second term on the right hand side is

$$\leqq C|||u(s)|||_1^2 \leqq C \, . \, E(s) \, .$$

The last two terms are

$$= \left(a_{ji} \frac{\partial u'}{\partial x_i}, \frac{\partial u}{\partial x_j} \right)_0 + \left(a_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u'}{\partial x_i} \right)_0 = \left(\bar{a}_{ij} \frac{\partial u'}{\partial x_i}, \frac{\partial u}{\partial x_j} \right)_0 + \ldots =$$

$$= 2 \,\mathrm{Re} \left(a_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u'}{\partial x_i} \right)_0 .$$

As $u'(s) \in {}^\circ W_2^{(1)}$, the integration by parts yields

$$\left(a_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u'}{\partial x_i} \right)_0 = - \left(\frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right), u' \right)_0 .$$

Therefore

$$\frac{\mathrm{d}E}{\mathrm{d}s}(s) \leqq CE(s) + 2 \,\mathrm{Re} \left(u'' - \sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right), u' \right) =$$

$$= CE(s) + 2 \,\mathrm{Re} \left(- \sum_i \left(\left(h_i \frac{\partial u'}{\partial x_i}, u' \right)_0 + \left(b_i \frac{\partial u}{\partial x_i}, u' \right)_0 \right) - \right.$$

$$\left. - (c_1 u', u')_0 - (c_2 u, u')_0 + (h, u')_0 + (f(s, u(s)), u')_0 \right) \leqq$$

$$\leqq C(E(s) + \|h(s)\|_0^2) + 2 \,\mathrm{Re}\,(f(s, u(s)), u'(s))_0$$

because h_i are real-valued functions and $u' \in {}^\circ W_2^{(1)}$. Taking into account (6.9) we have

$$\mathrm{Re}\,(f, u')_0 = \int (\mathrm{Re}\, f \,\mathrm{Re}\, u' + \mathrm{Im}\, f \,\mathrm{Im}\, u') \,\mathrm{d}x =$$

$$= \frac{\mathrm{d}}{\mathrm{d}s} \int \Phi(x, s, u(x, s)) \,\mathrm{d}x - \int \frac{\partial \Phi}{\partial s}(x, s, u(x, s)) \,\mathrm{d}x \, .$$

Finally we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s} \left[E(s) + \int_\Omega 2(C_\Phi - \Phi(x, s, u(x, s))) \,\mathrm{d}x \right] \leqq C \left(C + E(s) - 2 \int_\Omega \frac{\partial \Phi}{\partial s} \,\mathrm{d}s \right) .$$

If (6.11a) holds, the right hand side is

$$\leqq C\left(C + E(s) + 2C'_\Phi \int_\Omega (C_\Phi - \Phi)\,\mathrm{d}x\right) \leqq C\left(1 + E(s) + \int_\Omega 2(C_\Phi - \Phi)\,\mathrm{d}x\right).$$

If (6.11b) holds, the right hand side is

$$\leqq C(C + E(s) + C'_\Phi(2 \operatorname{mes} \Omega + 2\|u(s)\|_0^2)) \leqq C(1 + E(s) + |||u(s)|||_1^2) \leqq$$
$$\leqq C(1 + E(s)) \leqq C\left(1 + E(s) + \int_\Omega 2(C_\Phi - \Phi)\,\mathrm{d}x\right)$$

because from (6.10) it follows $C_\Phi - \Phi \geqq 0$.

Using the Gronwall Lemma we obtain in both cases

$$E(s) + 2\int_\Omega (C_\Phi - \Phi)\,\mathrm{d}x \leqq C'_1\,, \quad s \in \langle 0, t\rangle$$

and because $C_\Phi - \Phi \geqq 0$, we have $E(s) \leqq C'_1$ for $s \in \langle 0, t\rangle$. But then (6.13) implies (6.12).

Remark 6.6. Theorem 6.5 can be modified also for some non-homogeneous boundary conditions $g \neq 0$. Then we must use a function $E_1(s) = E(s) + \|u(s)\|_0^2$, because now (6.13) need not hold. Further we must somehow eliminate integrals over the boundary $\partial\Omega$, generally appearing in the integration by parts. These integrals vanish, if e.g. $g = g(x)$ does not depend on time t, because then $u'(s) \in {}^\circ W_2^{(1)}$ for each $s \in \langle 0, t\rangle$.

Theorem 6.7. *Let $\Omega, L, u_0, u_1, h, g, f(x, t, z)$ satisfy the assumptions of Theorem 5.1 and let $u \in C(0, t; H^2)$ be such a solution of the problem (5.1) that (6.12) holds. Suppose that the function $f(x, t, z)$ satisfy further*

(6.15)
$$\left|\frac{\partial f}{\partial t}(x, t, z)\right| \leqq C_f(1 + |z|^{a+1})$$

(6.16)
$$\left|\frac{\partial f}{\partial z}(x, t, z)\right| \leqq C_f(1 + |z|^a)$$

for each $(x, t, z) \in \bar{Q} \times \mathbf{C}$, where

$$a = \frac{2}{n-2} \quad \textit{for} \quad n > 2\,, \quad 0 \leqq a < \infty \quad \textit{for} \quad n \leqq 2\,,$$

$C_f \geqq 0$ is a constant.

Then there exists a constant $C_2 > 0$ such that

(6.17) $$|||u(s)|||_2 \leqq C_2\,, \quad s \in \langle 0, t\rangle$$

and consequently the apriori estimate in the cases $n = 2$, $n = 3$ holds.

Proof. From the energy inequality we have again

(6.18) $$|||u(s)|||_2 \leqq C(T)\left(\gamma + \int_0^s \|F'(r)\|_0 \,dr\right).$$

Now $|F'(x, r)|^2 \leqq 2(|(\partial f/\partial r)(x, r, u(x, r))|^2 + |\partial f/\partial z|^2 |u'|^2)$, therefore

(6.19) $$\|F'(r)\|^2 \leqq 2\left(\int_\Omega \left|\frac{\partial f}{\partial r}\right|^2 dx + \int_\Omega \left|\frac{\partial f}{\partial z}\right|^2 |u'|^2 \,dx\right) = 2(J_1 + J_2)\,.$$

From (6.15) it follows

$$J_1 \leqq C_f(\text{mes}\,\Omega + \|u(r)\|^{2(a+1)}_{L_{2(a+1)}})$$

but because $u \in W_2^{(1)}$, the choice of a implies the imbedding $W_2^{(1)} \subset L_{2(a-1)}$, consequently

(6.20) $$J_1 \leqq C(\text{mes}\,\Omega + \|u(r)\|_1^{2(a+1)}) \leqq$$
$$\leqq C(\text{mes}\,\Omega + |||u(r)|||_1^{2(a+1)}) \leqq C(\text{mes}\,\Omega + C_1^{2(a+1)})$$

because (6.12) holds. Further, from (6.16) it follows

$$J_2 \leqq C_f\left(\int_\Omega |u'(x, r)|^2 \,dx + \int_\Omega |u(x, r)|^{2a} |u'(x, r)|^2 \,dx\right).$$

By means of Hölder inequality we obtain

$$J_2 \leqq C_f\left(\|u'(r)\|_0^2 + \left(\int_\Omega |u(x, r)|^{an} \,dx\right)^{2/n} \left(\int_\Omega |u'(x, r)|^{2n/(n-2)}\right)^{(n-2)/n}\right).$$

But $an = 2n/(n-2)$ and because $u(r), u'(r) \in W_2^{(1)}$, the imbedding $W_2^{(1)} \subset L_{2n/(n-2)}$ implies

(6.21) $$J_2 \leqq C(\|u'(r)\|_0^2 + \|u(r)\|_1^{4/(n-2)} \|u'(r)\|_1^2) \leqq$$
$$\leqq C(1 + C_1^{4/(n-2)})\,|||u(r)|||_2^2\,.$$

Now from (6.18)–(6.21) it follows

$$|||u(s)|||_2 \leqq C\left(1 + \int_0^s |||u(r)|||_2 \,dr\right)$$

and we obtain our assertion again by means of the Gronwall Lemma.

Example 1. We shall consider the following mixed semilinear problem of the special form:

$$(6.22) \qquad Lu(t) + \varphi(x, t)\, u(t)\, |u(t)|^a = h(x, t), \quad a > 0,$$

$$u(0) = u_0, \quad u'(0) = u_1, \quad u - g \in C(0, T; {}^\circ H^2)$$

on a domain $Q = \Omega \times (0, T)$, where $\Omega \subset \mathbf{R}^n$ is a bounded domain of the class $C^{(2),1}$, $n = 1, 2, 3$, $0 < T < \infty$.

Here L is an operator defined by (1.1)–(1.5), its coefficients are assumed to fulfil

$$a_{ij} \in C^{(1,1)}(\bar{Q}), \quad h_i \in C^{(1)}(\bar{Q}), \quad b_i, c_q \in C^{(0,1)}(\bar{Q}), \quad i, j = 1, \dots, \quad n, q = 1, 2.$$

Let further

$$u_0 \in W_2^{(2)}, \quad u_1 \in W_2^{(1)}, \quad h \in C^{(1)}(0, T; L_2), \quad g \in C^{(1)}(0, T; H^2).$$

If $\varphi \equiv 0$, then the problem (6.22) is linear and from Theorem 3.1 the existence of a unique solution $u \in C(0, T; H^2)$ follows, if only the compatibility conditions of order 2

$$(6.23) \qquad u_0 \in g(0) + W_2^{(2)} \cap {}^\circ W_2^{(1)}, \quad u_1 \in g'(0) + {}^\circ W_2^{(1)}$$

hold.

For $\varphi \equiv 1$, $g = 0$ and the particular case of the operator $L = \Box = \partial^2/\partial t^2 - \Delta$, the problem (6.22) was solved in [17] by J. SATHER. There the existence and uniqueness of a weak global solution was shown (even for higher dimensions).

Let us show that the existence of a global solution of more general problem (6.22) follows from Theorem 5.3.

Since we consider only the case $n \leqq 3$, the compatibility conditions of order $[n/2] + 1 \leqq 2$ are the same as in the linear case, i.e. (6.23). We must only show the required smoothness of the non-linear term and the apriori estimate. Therefore we shall consider the function

$$f(x, t, z) = -\varphi(x, t)\, z|z|^a$$

defined on $\bar{Q} \times C$ which can be taken as a function

$$f(x, t, v_1, v_2) = -\varphi(x, t)\,(v_1 + iv_2)\,(v_1^2 + v_2^2)^{a/2}$$

defined on $\bar{Q} \times \mathbf{R}^2$. For the existence of a local solution, Theorem 5.1 requires $f \in \bar{C}^{(1),\lambda}(\bar{Q} \times \mathbf{C})$ for some $\lambda \in (0, 1\rangle$. If we suppose

$$(6.24) \qquad \varphi \in C^{(1)}(\bar{Q})$$

then $f \in \bar{C}^{(1)}(\bar{Q} \times \mathbf{C})$ because

$$(6.25) \quad \frac{\partial f}{\partial \operatorname{Re} z} \equiv \frac{\partial f}{\partial v_1} = -\varphi(x, t)\,(v_1^2 + v_2^2)^{a/2} \left(1 + \frac{a v_1^2}{v_1^2 + v_2^2} + i \frac{a v_1 v_2}{v_1^2 + v_2^2}\right)$$

$$\frac{\partial f}{\partial \operatorname{Im} z} \equiv \frac{\partial f}{\partial v_2} = -\varphi(x, t)\,(v_1^2 + v_2^2)^{a/2} \left(\frac{a v_1 v_2}{v_1^2 + v_2^2} + i \left(1 + \frac{a v_2^2}{v_1^2 + v_2^2}\right)\right)$$

(and obviously $= 0$ for $(v_1, v_2) = (0, 0)$) are continuous functions in $\bar{Q} \times \mathbf{R}^2$ for each $a > 0$. If $a \geqq 2$, we can differentiate f at least once more with respect to v_1, v_2 and therefore then $f \in \bar{C}^{(1),1}(\bar{Q} \times \mathbf{C})$ (if a is big enough, we can obtain the local existence also for higher dimensions). But even for $a < 2$ the functions from (6.25) are locally $a/2$ – Hölder continuous (see Appendix) and so we have $f \in \bar{C}^{(1),a/2}(\bar{Q} \times \times \mathbf{C})$. Consequently, for each $a > 0$ the assumptions of Theorem 5.1 are satisfied and the problem (6.22) has a unique local solution.

If we define a function F for $(x, t, z) \in \bar{Q} \times \mathbf{C}$ by the relation

$$(6.26) \qquad F(x, t, z) = -\frac{\varphi(x, t)\,|z|^{a+1}}{2(a + 1)}$$

we see that

$$\frac{\partial F}{\partial \operatorname{Re} z} = \operatorname{Re} f\,, \quad \frac{\partial F}{\partial \operatorname{Im} z} = \operatorname{Im} f$$

and

$$\frac{\partial F}{\partial t} = -\frac{\partial \varphi}{\partial t}(x, t)\,\frac{|z|^{a+1}}{2(a + 1)}\,.$$

Thus, if the conditions (6.10) and either (6.11a) or (6.11b) held for F, then we should obtain from Theorem 6.5 the apriori estimate in the case $n = 1$, $g = 0$ (or $g = g(x)$ according to Remark 6.6).

Let

$$(6.27) \qquad \varphi(x, t) \geqq 0\,, \quad (x, t) \in \bar{Q}\,.$$

Then $F(x, t, z) \leqq 0$ and (6.10) holds.

Further, if for some constant $C > 0$

$$(6.28) \qquad \frac{\partial \varphi}{\partial t}(x, t) \leqq C\,.\,\varphi(x, t) \quad (x, t) \in \bar{Q}$$

holds (e.g. if $\varphi(x, .)$ is a non-increasing function), we have

$$-\frac{\partial F}{\partial t} \leqq C\,.\,\varphi\,\frac{|z|^{a+1}}{2(a + 1)} = -C\,.\,F$$

and consequently (6.11a) holds. If instead of (6.28) the condition

$$(6.29) \qquad \left|\frac{\partial \varphi}{\partial t}(x, t)\right| \leqq C, \quad (x, t) \in \bar{Q}, \quad a \leqq 1$$

holds, then evidently (6.11b) applies to F.

Summary. The problem (6.22) has a unique global solution in the case $n = 1$, $g = g(x)$ if (6.24), (6.27) and either (6.28) or (6.29) hold.

Let us consider the case $n = 2, 3$. Then $[n/2] + 1 = 2$ and so we must prove the apriori estimate "of order 2". For this purpose we shall use Theorem 6.7. First, the above mentioned conditions are supposed in order to establish an estimate in the norm $|||.|||_1$. Further we have

$$\left|\frac{\partial f}{\partial t}(x, t, z)\right| \leqq \left|\frac{\partial \varphi}{\partial t}(x, t)\right| |z|^{a+1},$$

therefore (6.15) holds if

$$(6.30) \qquad \left|\frac{\partial \varphi}{\partial t}(x, t)\right| \leqq C, \quad (x, t) \in \bar{Q}$$

and

$$(6.31) \qquad 0 < a < \infty \quad \text{for} \quad n = 2; \quad 0 < a \leqq 2 \quad \text{for} \quad n = 3.$$

Condition (6.16) follows from the condition

$$(6.32) \qquad |\varphi(x, t)| \leqq C, \quad (x, t) \in Q$$

and from (6.31), because $|\partial f/\partial z| \leqq |\partial f/\partial v_1| + |\partial f/\partial v_2| \leqq 2|\varphi| \cdot |z|^a (1 + 2a)$. If (6.15) and (6.16) hold, we can use Theorem 6.7 which implies the required apriori estimate.

Summary. The problem (6.22) has a unique global solution in the case $n = 2, 3$, $g = g(x)$ if the conditions (6.24), (6.31)

$$(6.33) \qquad 0 \leqq \varphi(x, t) \leqq C, \quad (x, t) \in \bar{Q}$$

and either (6.28), (6.30) or (6.29) hold.

Now we see that our results include the results from [17] (where the same condition (6.31) concerning a is required), because $\varphi \equiv 1$ evidently satisfies our assumptions. But we can also establish the regularity of the solution. We can prove analogously as in Appendix

$$\varphi \in C^{(k)}(\bar{Q}), \quad a > k - 1 \Rightarrow f \in \bar{C}^{(k),a/2}(\bar{Q} \times \mathbf{C}), \quad k = 1, 2, \ldots$$

Therefore if $a > [n/2] - 1$ and if $\varphi \in C^{([n/2])}(\bar{Q})$, we have $f \in \bar{C}^{([n/2]),a/2}(\bar{Q} \times \mathbf{C})$ and we can obtain a local solution in higher dimensions. On the other hand, in the case $n \leqq 3$, $a > k - 1$, $k = 1, 2, \ldots$, we obtain smoother global solution $u \in C(0, T; H^{k+2})$ – according to the smoothness of u_0, u_1, g, h and the coefficients of the operator L.

Example 2. In [18] J. Sather proved the existence of a global classical solution of the problem

$$(6.35) \qquad \Box u + u^3 = h\,, \quad u(0) = u_0\,, \quad u'(0) = u_1\,, \quad u = 0 \quad \text{on} \quad \partial\Omega \times (0, T)$$

in the class of real-valued functions ($\Omega \subset \mathbf{R}^n$, $n = 1, 2, 3$).

If we consider also only real-valued functions, we can put in Theorem 6.5 $\Phi(z) = -z^4/4$ (we have $f(z) = -z^3$). Then it is easily seen from Theorems 6.5, 6.7 that the apriori estimate for $n = 1, 2, 3$ holds. Consequently we have a unique global solution $u \in C(0, T; {}^\circ H^2)$ for

$$h \in C^{(1)}(0, T; L_2)\,, \quad u_0 \in W_2^{(2)} \cap {}^\circ W_2^{(1)}\,, \quad u_1 \in {}^\circ W_2^{(1)}\,, \quad \Omega \in C^{(3),1}$$

and for

$$h \in C(0, T; H^2) \cap C^{(3)}(0, T; L_2)\,, \quad u_0 \in W_2^{(4)} \cap {}^\circ W_2^{(1)}\,, \quad u_1 \in W_2^{(3)} \cap {}^\circ W_2^{(1)}$$

$\Omega \in C^{(5),1}$ satisfying the compatibility conditions of order 4 we obtain the global classical solution

$$u \in C(0, T; {}^\circ H^4) \subset C^{(2)}(\bar{Q})\,.$$

This fact again includes the results proved in [18].

APPENDIX

Proposition. *Let* $0 < a < 1$. *Then functions*

$$A(v_1, v_2) = (v_1^2 + v_2^2)^a\,, \quad B_q(v_1, v_2) = v_q^2(v_1^2 + v_2^2)^{a-1}\,, \quad q = 1, 2\,,$$
$$C(v_1, v_2) = v_1 v_2 (v_1^2 + v_2^2)^{a-1}$$

are locally a-Hölder continuous for $(v_1, v_2) \in \mathbf{R}^2$.

Proof. Our assertion is evident for A and we shall prove it only for B_1, because we can apply the same method to prove it for B_2, C.

So we must prove the implication

$$\forall b > 0 \ \exists C(b) : |v_p| \leqq b\,, \quad |w_p| \leqq b\,, \quad p = 1, 2 \Rightarrow$$
$$\Rightarrow |B_1(v_1, v_2) - B_1(w_1, w_2)| \leqq C(b) \sum_{p=1}^{2} |v_p - w_p|^a\,.$$

Taking into account the definition of B_1 we see that we can consider only $v_p, w_p \in \in \langle 0, b \rangle$. Now

$$B_1(v_1, v_2) - B_1(v_1, w_2) = 0 \quad \text{if} \quad v_1 = 0 \quad \text{or} \quad v_2 = w_2$$

and for $v_1 \neq 0$ we can write (if e.g. $v_2 < w_2$)

$$|B_1(v_1, v_2) - B_1(v_1, w_2)| = \left| \int_0^1 \frac{\partial B_1}{\partial v_2} (v_1, v_2 + s(w_2 - v_2)) (w_2 - v_2) \, ds \right| =$$

$$= \left| \int_0^1 2(a - 1) v_1^2 (v_1^2 + (v_2 + s(w_2 - v_2))^2)^{a-2} (v_2 + s(w_2 - v_2)) (w_2 - v_2) \, ds \right| \leqq$$

$$\leqq 2(1 - a) \int_0^1 (v_2 + s(w_2 - v_2))^{2a-1} (w_2 - v_2) \, ds = 2(1 - a) \int_{v_2}^{w_2} s^{2a-1} \, ds =$$

$$= (1 - a) a^{-1} (w_2^{2a} - v_2^{2a}) \leqq (2b)^a a^{-1} (1 - a) |w_2 - v_2|^a .$$

Analogously

$$|B_1(v_1, w_2) - B_1(w_1, w_2)| = \left| \int_0^1 \frac{\partial B_1}{\partial v_1} (v_1 + s(w_1 - v_1), w_2) (w_1 - v_1) \, ds \right| =$$

$$= \left| \int_0^1 2[(v_1 + s(w_1 - v_1)) + (a - 1) ((v_1 + s(w_1 - v_1))^2 + w_2^2)^{-1} \, . \right.$$

$$\left. . (v_1 + s(w_1 - v_1))^3] ((v_1 + s(w_1 - v_1))^2 + w_2^2)^{a-1} (w_1 - v_1) \right| ds \leqq$$

$$\leqq 2(w_1 - v_1) \left[\int_0^1 (v_1 + s(w_1 - v_1))^{2a-1} \, ds + (1 - a) \int_0^1 (v_1 + s(w_1 - v_1)^{2a} \, ds \right] \leqq$$

$$\leqq 2(1 + (1 - a) b) \int_0^1 (v_1 + s(w_1 - v_1))^{2a-1} (w_1 - v_1) \, ds \leqq C(b) (w_1 - v_1)^a .$$

Since moreover

$$|B_1(v_1, v_2) - B_1(w_1, w_2)| \leqq$$
$$\leqq |B_1(v_1, v_2) - B_1(v_1, w_2)| + |B_1(v_1, w_2) - B_1(w_1, w_2)|$$

we see that the required implication holds.

References

[1] *В. Э. Аболиня, А. Д. Мышкис:* О смешанной задаче для линейной гиперболической системы на плоскости. Уч. зап. Латвий. гос. унив. *XX* (1958), 87—104.

[2] *В. Э. Аболиня, А. Д. Мышкис:* Смешанная задача для почти линейной гиперболической системы на плоскости. Матем. Сборник *50* (1960), 423—442.

[3] *R. Courant:* Partial Differential Equations. (Russian) Moskva 1964.
[4] *M. Ikawa:* Mixed problem for hyperbolic equation of second order. J. Math. Soc. Japan *20* (1968), 580—608.
[5] *M. Ikawa:* A Mixed Problem for Hyperbolic Equation of Second Order with a First Order Derivative Boundary Condition. Publ. RIMS Kyoto Univ. *5* (1969), 119—147.
[6] *M. Ikawa:* A Mixed Problem for Hyperbolic Equation of Second Order with Non-homogeneous Neumann Type Boundary Condition. Osaka J. Math. *6* (1969), 339—374.
[7] *M. Ikawa:* On the Mixed Problem for Hyperbolic Equation of Second Order with the Neumann Boundary Condition. Osaka J. Math. *7* (1970), 203—223.
[8] *H.-O. Kreiss:* Initial Boundary Value Problems for Hyperbolic Systems. Comm. Pure Appl. Math. 9 *23* (1970), 277—298.
[9] *О. А. Ладыженская:* Смешанная задача для гиперболического уравнения. Москва 1953.
[10] *S. Mizohata:* Lectures on the Cauchy Problem. Tata Institute of Fundamental Research, Bombay 1965.
[11] *S. Mizohata:* Quelques problémes au bord, du type mixte, pour des équations hyperboliques. Séminaire sur les équations aux dérivées partielles, Collége de France (1966—67), 23—60.
[12] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia Praha 1967.
[13] *Б. Л. Рождественский, Н. Н. Яненко:* Системы квазилинейных уравнений. Москва 1968.
[14] *R. Sakamoto:* Mixed Problems for Hyperbolic Equations I. Energy Inequalities. J. Math. Kyoto Univ. *10* (1970), 349—373.
[15] *R. Sakamoto:* Mixed Problems for Hyperbolic Equations II. J. Math. Kyoto Univ. *10* (1970), 403—417.
[16] *R. Sakamoto:* Iterated Hyperbolic Mixed Problems. Publ. RIMS Kyoto Univ. *6* (1970), 1—42.
[17] *J. Sather:* The initial-boundary value problem for a non-linear hyperbolic equation in relativistic quantum mechanics. J. Math. Mech. vol. 16, 1966/1, 27—50.
[18] *J. Sather:* The existence of a Global Classical Solution of the Initial-Boundary Value Problem for $\Box u + u^3 = f$. Arch. Rat. Mech. Anal. *22* (1966), 292—307.
[19] *С. Л. Соболев:* Некоторые пнименения функционального анализа в математической физике. Новосибирск 1962.

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