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DEFORMATION AND EQUIVALENCE G-STRUCTURES. PART I. $\{e\}$-STRUCTURES

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This paper is a partial solution of the problem which was suggested to me by Professor ŠVEC. It is a contribution to the difficult and important problem of the equivalence of G-structures which has been already treated by many mathematicians.

Many important results in special cases were obtained already by E. CARTAN; the latest ones are due to GUILLEMIN, STERNBERG, SINGER and others.

The contents of the paper is the following. In the first chapter some necessary concepts from the theory of G-structures and the notion of contact of high order of submanifolds are introduced. The second chapter begins with the definition of deformation of G-structures and with some of its basic properties.

In the next part, deformations of $\{e\}$-structures and the relation between the deformation and the equivalence of $\{e\}$-structures is dealt with.

The deformation problem for $\{e\}$-structures is practically solved here but the study of general G-structures remains still open.

1. G-STRUCTURES

Throughout the paper differentiability means always differentiability of class C^∞; instead of differentiable manifolds we speak only manifolds, we use the usual summation convention and we take over much of the notation from the book by Sternberg [1].

We shall mention here only some definitions and propositions from the G-structures theory.

Let us denote by $F(M)$ the principal fibre bundle of all frames of a manifold M. The general linear group operates on $F(M)$ from the right. For $p \in F(M)$ and $a \in GI(n, \mathbf{R})$ [1]) we denote this operation by $p . a$, $\pi : F(M) \to M$ denotes the projection.

[1]) $GI(n, \mathbf{R})$ is the general linear group.

Then $F_x(M) = \pi^{-1}(x)$, i.e. the fibre at x is a submanifold of $F(M)$ diffeomorphic with $Gl(n, \mathbf{R})$. Any diffeomorphism $f : M_1 \to M_2$ induces isomorphisms $f_{*x} : T_x M_1 \to T_{f(x)} M_2$ of tangent spaces of the manifolds, thus defining a diffeomorphism $\tilde{f} : F(M_1) \to F(M_2)$ of the principal fibre bundles of frames.

Definition 1. Let G be a Lie subgroup of $Gl(n, \mathbf{R})$. *G-structure* $B \to M$ on M is a reduction of the principal fibre bundle $F(M)$ to the group G.

This is to say a G-structure $B \to M$ is a principal fibre bundle *over* M with a principal fibre bundle morphism $B \to F(M)$ which is an imbedding and induces the identity on M.

Proposition 1. *Let G be a Lie subgroup of $Gl(n, \mathbf{R})$. A submanifold $B \subset F(M)$ is a G-structure on M if and only if:*

(1) *The projection $\pi : F(M) \to M$ maps B onto M.*

(2) *If $p \in B$, $q \in F(M)$ such that $q = p \, . \, a$ then $q \in B$ if and only if $a \in G$.*

(3) *To any $x \in M$ there exists its neighborhood U and a cross-section $\sigma : U \to F(M)$ such that $\sigma(U) \subset B$.*

If $B \to M_1$ is a G-structure and $f : M_1 \to M_2$ is a diffeomorphism then the image $\tilde{f}(B)$ is a G-structure on M_2.

Definition 2. Let $B_1 \to M_1$ and $B_2 \to M_2$ be two G-structures. If there exists a diffeomorphism $f : M_1 \to M_2$ such that $\tilde{f}(B_1) = B_2$ then the G-structures are said *to be equivalent with the equivalence f.*

We say that G-structures $B_1 \to M_1$ and $B_2 \to M_2$ are *locally equivalent at a point* $(x, y) \in M_1 \times M_2$ if there exist neighborhoods U_1 of x and U_2 of y such that the G-structures B_1/U_1 and B_2/U_2 are equivalent with an equivalence f satisfying $f(x) = y$.

Remark. If $B \to M$ is a G-structure on M and $U \subset M$ an open subset, then $\pi^{-1}(U) \cap B$ is a G-structure on U which will be denoted by B/U.

For any Lie subgroup $G \subset Gl(n, \mathbf{R})$ we can define on $\mathbf{R}^n$ the standard flat G-structure $B_G^0 \to R^n$. Using the standard chart $(x^1, \ldots, x^n)$ on $\mathbf{R}^n$ we can define a global cross-section of $F(\mathbf{R}^n)$ by setting $\sigma(x) = (\partial/\partial x_1(x), \ldots, \partial/\partial x_n(x))$. A subset $B_G^0 \subset F(\mathbf{R}^n)$ consists of all elements of the type $\sigma(x) \, . \, a$, $x \in \mathbf{R}^n$, $a \in G$.

Definition 3. A G-structure $B \to M$ is said to be *flat* if it is equivalent with the standard flat G-structure on $\mathbf{R}^n$. It is called *locally flat* if it is locally equivalent with the standard flat G-structure at the point $(x, 0) \in M \times \mathbf{R}^n$ for any $x \in M$ $(0 = (0, \ldots, 0)$ is the origin in $\mathbf{R}^n)$.

G-structure $B \to M$ is locally flat if and only if to any $x \in M$ there exists a chart $(x^1, \ldots, x^n)$ around it such that the field of frames $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ belongs to B.

Examples of G-structures. (1) If $G = \{e\}$ is the trivial subgroup of $Gl(n, \mathbf{R})$ then an $\{e\}$-structure is a full parallelism on M.

(2) If $G = O(n)$ is the orthogonal group, then $O(n)$-structures are in $1 - 1$ correspondence with Riemannian metrics on M.

(3) If $G = C\,O(n)$ is the conformal group, $C\,O(n)$-structures are in $1 - 1$ correspondence with conformal structures on M.

Further examples of G-structures are almost complex structures, symplectic structures etc.

From this brief survey we can see that the majority of classical geometric structures belong to G-structures.

If G is a closed subgroup of $Gl(n, \mathbf{R})$ we can define G-structure also in another equivalent way. We know that the set of equivalent classes $Gl(n, \mathbf{R})/G$ is a homogeneous space and $Gl(n, \mathbf{R})$ operates from the left on it. We can define the fibre bundle with the standard fibre $Gl(n, \mathbf{R})/G$ associated with $F(M)$ and by Proposition 5,5 from [4] to identify it with the quotient space $F(M)/G$. The proof of the following proposition can be found in [4] Prop. 5.6.

Proposition 2. *Let G be a closed subgroup of $Gl(n, \mathbf{R})$. G-structures on M are in $1 - 1$ correspondence with sections of the fibre bundle $F(M)/G$.*

Thus for a closed subgroup $G \subset Gl(n, \mathbf{R})$ we get a commutative diagram

$$\begin{array}{ccc} F(M) & \xrightarrow{\tau_G} & F(M)/G \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_G} & \\ & M & \end{array}$$

of fibre bundles and their morphisms.

By Proposition 2 we can associate with any G-structure a unique cross-section $\sigma : M \to F(M)/G$. This cross-section will be called the representation of the G-structure. Speaking about representations of G-structures we shall always suppose that G is a closed subgroup of $Gl(n, \mathbf{R})$.

Further results in this direction can be found in [5]. If $B \to M_1$ is a G-structure on M_1 with the representation $\sigma : M_1 \to F(M_1)/G$ and $f : M_1 \to M_2$ is a diffeomorphism then f induces a mapping $\tilde{f}$ which is a diffeomorphism $F(M_1)/G$ with $F(M_2)/G$ and which maps the cross-section σ to the section $\tilde{f}(\sigma) : M_2 \to F(M_2)/G$. This cross-section is the representation of the G-structure $\tilde{f}(B)$ on M_2.

Thus f is an equivalence of G-structures if and only if $\tilde{f}$ maps a representation into a representation.

Definition 4. Let M be a manifold and let N_1 and N_2 be two m-dimensional submanifolds of M intersecting at the point P. We say that N_1, and N_2 have the *k-th* *order contact at P* if we can find a coordinate system for M, $(y_1, \ldots, y_n)$, defined in a neighborhood of P, such that the equations of N_1 are given by

$$y_{m+1} = 0, \ldots, y_n = 0$$

and the equations of N_2 by

$$\begin{aligned} y_{m+1} &= f_{m+1}(y_1, \ldots, y_m)\,, \\ &\vdots \\ y_n &= f_n(y_1, \ldots, y_m)\,, \end{aligned}$$

where $f_{m+1}, \ldots, f_n$ have vanishing derivatives of all orders less then or equal to k at the point P.

2. DEFORMATION AND EQUIVALENCE OF G-STRUCTURES

1. General definitions

Definition 1. Let G be a Lie subgroup of $Gl(n, \mathbf{R})$, M_1 and M_2 two manifolds. A diffeomorphism $f : M_1 \to M_2$ is said to be a *deformation of order k* at a point $u \in M$ of G-structures $B_1 \to M_1$ and $B_2 \to M_2$ if there exists a local diffeomorphism ψ of M_1 into M_1 with $\psi(u) = u$ and $p \in B_2$ with $\pi(p) = f(u)$ such that B_2 and $\tilde{f}\,\tilde{\psi}(B_1)$ have as submanifolds of $F(M_2)$ the contact of order k at the point p.

If f is a deformation of order k at every point of M_1, we call it *a deformation of order k*. We say that $B_1 \to M_1$ and $B_2 \to M_2$ are *in deformation of order k at a point* $(u, v) \in M_1 \times M_2$ if there exists a local diffeomorphism from M_1 to M_2 with $\psi(u) = v$ such that ψ is a deformation of order k at u.

The immediate of the definition is the following:

Lemma 1. *Let $B_i \to M_i$, $i = 1, 2, 3$ be G-structures, $f : M_1 \to M_2$ and $g : M_2 \to M_3$ diffeomorphisms. Then*

(1) *If f is an equivalence of G-structures $B_1 \to M_1$ and $B_2 \to M_2$, and g a deformation of order k of G-structures $B_2 \to M_2$ and $B_3 \to M_3$ at a point $u \in M_2$ then $g \circ f$ is a deformation of order k of the G-structures $B_1 \to M_1$ and $B_3 \to M_3$ at the point $f^{-1}(u) \in M_1$.*

(2) *If g is an equivalence of G-structures $B_2 \to M_2$ and $B_3 \to M_3$ and f a deformation of order k at a point $u \in M_1$ of G-structures $B_1 \to M_1$ and $B_2 \to M_2$, then $g \circ f$ is a deformation of order k of the G-structures $B_1 \to M_1$ and $B_3 \to M_3$ at the point $u \in M_1$.*

Lemma 1 implies easily:

Lemma 2. *A diffeomorphism $f : M_1 \to M_2$ is a deformation of order k of G-structures $B_1 \to M_1$ and $B_2 \to M_2$ at a point $u \in M_1$ if and only if $Id : M_1 \to M_1$ is a deformation of order k of G-structures $B_1 \to M_1$ and $\tilde{f}^{-1}(B_2) \to M_1$ at the point $u \in M_1$.*

If G is a closed subgroup of $Gl(n, \mathbf{R})$ then any G-structure on M has a representation $\sigma : M \to F(M)/G$ (see 1) and the terms of it we can paraphrase the notion of deformation in the following way.

Proposition 1. *A diffeomorphism $f : M_1 \to M_2$ is a deformation of order k of a G-structure $B_1 \to M_1$ represented by a cross-section $\sigma_1 : M_1 \to F(M_1)/G$ with a G-structure $B_2 \to M_2$ represented by a cross-section $\sigma_2 : M_2 \to F(M_2)/G$ at a point $u \in M_1$ if and only if there exists a local diffeomorphism ψ of M_1 with $\psi(u) = u$ such that*

(1) $$j^k_{f(u)}(\tilde{f}\tilde{\psi}\sigma_1) = j^k_{f(u)}(\sigma_2)\,.$$

Remark. (1) can be written in an equivalent form

(2) $$j^k_u(\tilde{\psi}\sigma_1) = j^k_u(\tilde{f}^{-1}\sigma_2)$$

which will be frequently used in the sequel.

Proof. By Lemma 2 we can suppose $M_1 = M_2 = M$ and $f = Id$. Thus we have two G-structures $B_1 \to M$ and $B_2 \to M$ and a point $u \in M$. Further, for the sake of simplicity we can suppose that there exists $p \in B_1 \cap B_2$ with $\pi(p) = u$ which implies immediately $\pi^{-1}(u) \cap B_1 = \pi^{-1}(u) \cap B_2$ and therefore $\sigma_1(u) = \sigma_2(u)$. For our purpose it is sufficient to introduce on $F(M)$ special fibre coordinates related with the fiberings $F(M) \overset{\tau G}{\to} F(M)/G \overset{\pi G}{\to} M$.

Remark. We can see that the deformation of order k does not depend on the mapping ψ but only on its $(k + 1)$-jet at u. Thus we can use only $(k + 1)$-jet with the property (1) instead of ψ.

Now we can formulate naturally arising problems:

(1) Under which conditions is a given mapping a deformation or order k?
(2) Under which conditions are two G-structures in deformation of order k?
(3) Is it possible for a G-structure $B \to M$ to find a non-negative integer k such that $Id : M \to M$ is a deformation of order k the G-structure $B \to M$ with a G-structure $B_1 \to M$ then $B = B_1$?

In other words: Is it possible to find such k that a deformation of order k is already an equivalence?

In the sequel we shall study conditions equivalent to the notion of deformation and relations between a deformation and an equivalence especially in the case of $\{e\}$-structures.

2. Deformation of $\{e\}$-structures

We know from the preceding chapter that an $\{e\}$-structure on a manifold M is a full parallelism on M, i.e. a global cross-section $\mathbf{v} : M \to F(M)$. It can be expressed in the form $\mathbf{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ where $\mathbf{v}_i$, $i = 1, \ldots, n$ are global vector fields on M and for any $x \in M$ the vectors $\mathbf{v}_1(x), \ldots, \mathbf{v}_n(x)$ form a basis of $T_x(M)$.

If $\mathbf{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an $\{e\}$-structure on M, we can, using the Lie brackets, define the global vector fields $\mathbf{v}_{i_1 i_2}, \mathbf{v}_{i_1 i_2 i_3}, \ldots$ on M as follows

$$\mathbf{v}_{i_1 i_2} = [\mathbf{v}_{i_2}, \mathbf{v}_{i_1}], \quad \mathbf{v}_{i_1 i_2 i_3} = [\mathbf{v}_{i_3}, [\mathbf{v}_{i_2}, \mathbf{v}_{i_1}]], \ \ldots \tag{3}$$

Remark. Obviously any two G-structures are in deformation of order 0 so that we shall start from the deformation of order one.

Theorem. 1. *A diffeomorphism $f : M_1 \to M_2$ is a deformation of order k of $\{e\}$-structures $\mathbf{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ on M_1 and $\mathbf{w} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ on M_2 at a point $u \in M_1$ if and only if there exists an isomorphism $L: T_u M_1 \to T_{f(u)} M_2$ such that*

$$L(\mathbf{v}_{i_1}(u)) = \mathbf{w}_{i_1}(f(u)), \ldots, L(\mathbf{v}_{i_1 \ldots i_{k+1}}(u)) = \mathbf{w}_{i_1 \ldots i_{k+1}}(f(u)) \tag{4}$$

for all $i_1, \ldots, i_{k+1} = 1, \ldots, n$.

Proof. By Lemma 2 we can suppose $M = M_1 = M_2$ and $f = Id$. We shall proceed by induction. For $k = 0$ the theorem holds by the preceding remark.

Let us suppose that it holds for $k \geqq 0$. If $\mathbf{v}$ and $\mathbf{w}$ are in deformation of order k and ψ is a mapping realizing this deformation (in fact, a $(k + 1)$-jet of ψ) then the $\{e\}$-structure $\mathbf{v}^* = \{\mathbf{v}^*_1, \ldots, \mathbf{v}^*_n\}$ defined by $\mathbf{v}^*_i = \psi_* \mathbf{v}_i$, $i = 1, \ldots, n$ is equivalent with $\mathbf{v}$ and in deformation of order k with $\mathbf{w}$ at the point u through the identity map ($(k + 1)$-jet of the identity map), and by induction hypothesis we have

$$\mathbf{v}^*_i(u) = \mathbf{w}_i(u), \ldots, \mathbf{v}^*_{i_1 \ldots i_{k+1}}(u) = \mathbf{w}_{i_1 \ldots i_{k+1}} \quad \text{for all} \quad i_1 \ldots i_{k+1} = 1, \ldots, n\,. \tag{5}$$

Now because $\mathbf{v}^*$ and $\mathbf{v}$ are equivalent it suffices to prove the assertion for $\mathbf{v}^*$ and $\mathbf{w}$ (see Lemma 1).

Let us take a chart $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^n)$ around it such that $x^i(u) = 0$, $\mathbf{v}^*_i = X^\alpha_i \,\partial/\partial x^\alpha$ and $w_i = Y^\alpha_i \,\partial/\partial x^\alpha$ with $X^\alpha_i(0) = Y^\alpha_i(0) = \delta^\alpha_i$.

From the conditions (5) and the fact that Id realizes a deformation of order k we get immediately the equalities

$$\frac{\partial^l Y^\alpha_i}{\partial x^{j_1} \ldots \partial x^{j_1}}(0) = \frac{\partial^l X^\alpha_i}{\partial x^{j_1} \ldots \partial x^{j_1}}(0) \tag{6}$$

for $1 \leqq l \leqq k \leqq n$ and all $\alpha, i, j_1, \ldots, j_k = 1, \ldots, n$.

If there exists φ such that $J_0^{k+1}\tilde{\varphi}v_i^* = J_0^{k+1}w_i$ then necessarily $J_0^k\varphi = J_0^k(Id)$ and we must study the equality of the $(k+1)$ derivatives of the mappings:

$$X_i^\alpha \frac{\partial \varphi^\beta}{\partial x^\alpha} \quad \text{and} \quad Y_i^\beta \circ \varphi \quad \text{at} \quad 0 = (0, \ldots, 0)\,.$$

Taking $k+1$ derivatives at 0 we have the only nonzero terms

$$\frac{\partial^{k+1} X_i^\beta}{\partial x^{j_1} \ldots \partial x^{j_{k+1}}}(0) + \frac{\partial^{k+2} \varphi^\beta}{\partial x^i\, \partial x^{j_1} \ldots \partial x^{j_{k+1}}}(0) = \frac{\partial^{k+1} Y_i^\beta}{\partial x^{j_1} \ldots \partial x^{j_{k+1}}}(0) \tag{7}$$

i.e. we get the equality

$$\frac{\partial^{k+2} \varphi^\beta}{\partial x^i\, \partial x^{j_1} \ldots \partial x^{j_{k+1}}}(0) = \frac{\partial^{k+1} Y_i^\beta}{\partial x^{j_1} \ldots \partial x^{j_{k+1}}}(0) - \frac{\partial^{k+1} X_i^\beta}{\partial x^{j_1} \ldots \partial x^{j_{k+1}}}(0) \tag{8}$$

and the condition:

$$\frac{\partial^{k+1} Y_i^\beta}{\partial x^{j_1} \ldots \partial x^{j_{k+1}}}(0) - \frac{\partial^{k+1} X_i^\beta}{\partial x^{j_1} \ldots \partial x^{j_{k+1}}}(0) = \tag{9}$$

$$= \frac{\partial^{k+1} Y_{j_1}^\beta}{\partial x^i\, \partial x^{j_2} \ldots \partial x^{j_{k+1}}}(0) - \frac{\partial^{k+1} X_{j_1}^\beta}{\partial x^i\, \partial x^{j_2} \ldots \partial x^{j_{k+1}}}(0)\,.$$

Let us study the equality of two brackets:

$$[\mathbf{v}_{j_{k+1}}[\mathbf{v}_{j_k} \cdots [\mathbf{v}_{j_2}, \mathbf{v}_{j_1}] \ldots](0) = [\mathbf{w}_{j_{k+1}}[\mathbf{w}_{j_k} \cdots [\mathbf{w}_{j_2}, \mathbf{w}_{j_1}] \ldots](0)\,. \tag{10}$$

In the coordinate expression the brackets on the left and the right hand sides contain derivatives up to and including the order $(k+1)$ of the functions X_i^α and Y_i^α respectively at the point 0. The derivatives of functions X_i^α and Y_i^α up to and including the order k are equal to each other, therefore the only interesting terms are those including derivatives of order $k+1$ which are on the left-hand side:

$$\frac{\partial^{k+1} X_{i_1}^\alpha}{\partial x^{i_2} \ldots \partial x^{i_{k+2}}}(0) - \frac{\partial^{k+1} X_{i_2}^\alpha}{\partial x^{i_1}\, \partial x^{i_3} \ldots \partial x^{i_{k+2}}}(0)$$

and on the right-hand side:

$$\frac{\partial^{k+1} Y_{i_1}^\alpha}{\partial x^{i_2} \ldots \partial x^{i_{k+2}}}(0) - \frac{\partial^{k+1} Y_{i_2}^\alpha}{\partial x^{i_1}\, \partial x^{i_3} \ldots \partial x^{i_{k+2}}}(0)\,.$$

However, the equality of these two terms is exactly the equality (9). That is, to a $(k+1)$-jet a $(k+2)$-jet can be constructed if and only if the $(k+1)'$ brackets are equal.

With an $\{e\}$-structure $v = \{v_1, \ldots, v_n\}$ on M we can associate a system of differentiable functions $c^{\alpha}_{i_1 \ldots i_p} : M \to R(i_1 \ldots i_p, \alpha = 1, \ldots, n, p$ arbitrary) defined by

(11)
$$[\mathbf{v}_{i_2}, \mathbf{v}_{i_1}] = c^{\alpha}_{i_1 i_2} \mathbf{v}_{\alpha},$$
$$\vdots$$
$$[\mathbf{v}_{i_p}, [\mathbf{v}_{i_{p-1}}, \ldots [\mathbf{v}_{i_2}, \mathbf{v}_{i_1}] \ldots] = c^{\alpha}_{i_1 \ldots i_p} \mathbf{v}_{\alpha}.$$

In an equivalent way we can associate with an $\{e\}$-structure $\mathbf{v}$ a system of vector-valued functions on M, ${}^1\mathbf{c}_v, {}^2\mathbf{c}_v, \ldots$, so that

(12)
$${}^l\mathbf{c}_v : M \to HOM(\underbrace{\mathbf{R}^n \otimes \ldots \otimes \mathbf{R}^n}_{l+1}, \mathbf{R}^n)$$

is defined by

(13)
$${}^l\mathbf{c}_v(\mathbf{x}_0)(\mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_{l+1}}) = \mathbf{c}^{\alpha}_{i_1, \ldots i_{l+1}} \mathbf{e}_{\alpha}$$

$\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ being the standart bases on $\mathbf{R}^n$.

Proposition 2. *Diffeomorphism $f: M_1 \to M_2$ is a deformation of order k of $\{e\}$-structures $\mathbf{v}$ and $\mathbf{w}$ on M_1 and M_2 respectively at a point $u \in M$ if and only if the functions c_v and ${}^i c_w$ just defined satisfy*

(14)
$$c^i_v(u) = c^i_w(f(u)) \quad \text{for} \quad i = 1, \ldots, k.$$

Proof. By the theorem it suffices to study the existence of an isomorphism L with the properties (4). Defining L on the bases of the tangent spaces by

(15)
$$L(\mathbf{v}_{\alpha}(u)) = \mathbf{w}_{\alpha}(f(u))$$

then

(16)
$$L([\mathbf{v}_i, \mathbf{v}_j])(u) = L(c^{\alpha}_{vij}(u)\, \mathbf{v}_{\alpha}(u)) = c^{\alpha}_{vij}(u)\, \mathbf{w}_{\alpha}(f(u))$$

and therefore

(17)
$$L([\mathbf{v}_i, \mathbf{v}_j](u)) = [\mathbf{w}_i, \mathbf{w}_j](f(u)) = c^{\alpha}_{wij}(f(u))\, \mathbf{w}_{\alpha}(f(u))$$

if and only if

(18)
$$c^{\alpha}_{vij}(u) = c^{\alpha}_{wij}(f(u)).$$

Similarly we find that $L(\mathbf{w}_{i_1 \ldots i_l}(u)) = \mathbf{w}_{i_1 \ldots i_l}(f(u))$ if and only if

$${}^l c_v(u) = {}^l c_w(f(u)).$$

Remark. The functions $c^{\alpha}_{vi_1 \ldots i_p}$ and ${}^p c_v$ will be called structural functions of order p of a structure $\mathbf{v}$.

Corollary. *A diffeomorphism $f : M_1 \to M_2$ is a deformation of order k of $\{e\}$-structures* $\mathbf{v}$ *and* $\mathbf{w}$ *on M_1 and M_2 respectively if and only if*

$$ {}^i c_v \circ f = {}^i c_w \,, \quad i = 1, \ldots, k \,. \tag{19} $$

The next paragraph will be devoted to the relation of deformation and equivalence of $\{e\}$-structures.

3. Deformations and equivalence of $\{e\}$-structures

If $\mathbf{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an $\{e\}$-structure on M and $f : M \to R$ a differentiable function we define i^{th} covariant derivative $f_{;i}$ with respect to v by

$$ f_{;i} = \mathbf{v}_i f \,. \tag{20} $$

If F is a system of differentiable functions on M then the rank $r_p(F)$ of the system F at a point p is defined to be the dimension of the *suppose* of T_p^*M spanned by the differentials $\{\mathrm{d}f_p, f \in F\}$. System F is called regular at p if $r_q(F) = r_p(F)$ for all q from a neighborhood of p. We get easily:

Lemma 3. *If F is regular at p and $r_p(F) = k$, then there exists a chart $(x^1, \ldots, x^n)$ around p such that $x^1, \ldots, x^k \in F$ and every functions $f \in F$ can be expressed in the form $f = f(x^1, \ldots, x^k)$. Any such chart is said to be associated with F at p.*

Our definition of covariant derivative is equivalent to that from the book by Sternberg [1, K7]. We use also freely some definitions and lemmas from that book.

Our structural functions $c^{\alpha}_{i_1 \ldots i_{k+1}}$ defined in § 2 satisfy the equalities

$$ c^{\gamma}_{i_1 i_2 i_3} = \mathbf{v}_{i_3} c^{\gamma}_{i_1 i_2} + c^{\alpha}_{i_1 i_2} c^{\gamma}_{i_3 \alpha} = c^{\gamma}_{i_1 i_2 ; i_3} + c^{\alpha}_{i_1 i_2} c^{\gamma}_{i_3 \alpha} \,. \tag{21} $$

Further, for any positive integer k we have

$$ c^{\gamma}_{i_1 \ldots i_k} = c^{\gamma}_{i_1 \ldots i_{k-1} ; i_k} + c^{\alpha}_{i_1 \ldots i_{k-1}} c^{\gamma}_{i_k \alpha} \,. \tag{22} $$

We denote by F_0 the system functions $\{c^{\gamma}_{i_1 i_2}\}$ on M and similarly

$$ F_s = \{c^{\gamma}_{i_1 i_2}, \ldots, c^{\gamma}_{i_1 \ldots i_{s+2}}\} \,. $$

We define $k_s(p) = r_p(F_s)$.

The formulas (21) and (22) imply immediately that the rank of the system F is equal to the rank of the system

$$ \tilde{F}_s = \{c^{\gamma}_{i_1, i_2}, \ldots, c^{\gamma}_{i_1 i_2 ; i_3 ; i \ldots ; i_{s+1}}\} \,. \tag{23} $$

From the inclusion $F_s \subset F_{s+1}$ we can see that

$$ 0 \leqq k_0(p) \leqq k_1(p) \leqq \ldots \leqq k_s(p) \leqq n \,. \tag{24} $$

Lemma 4 ([1]). *Let F_s be regular at p. If $k_s(p) = k_{s+1}(p)$ then $k_t(p) = k_s(p)$ for $t \geqq s$.*

Proof. See Sternberg [1].

Definition 2. A point $p \in M$ is called *a regular point* of an $\{e\}$-structure $\mathbf{v}$ if there exists s such that F_s is regular at p and $r_s(p) = r_{s+1}(p)$. The smallest s having this property will be called *the order at the point* p of the $\{e\}$-structure $\mathbf{v}$ and will be denoted by $r(p)$. The number $k_p = k_s(p)$ will be called *the rank at the point* p of the $\{e\}$-structure $\mathbf{v}$. A chart around p associated with F_s will be called associated at the point p with the $\{e\}$-structure $\mathbf{v}$.

Definition 3. A point $p \in M$ is called an *s-general point* of an $\{e\}$-structure $\mathbf{v}$ if $r_s(p) = n$. The smallest s having this property is called *the order of generality* of the point p and will be denoted by $\deg_{\mathbf{v}}(p)$.

A point p is s-general if we can find a functions belonging to F_s and linearly independent at p. Of course then there exists a neighborhood of p such that for any q from this neighborhood there is $r_s(q) = n$ and the point p is therefore a regular point.

0-general point will be called simply a general point. At such a point differentials of the functions c^{γ}_{ij} generate the contangent space.

Proposition 3. *Let $f : M_1 \to M_2$ be a deformation of order two of an $\{e\}$-structure $\mathbf{v}$ on M_1 with $\mathbf{w}$ on M_2, $u \in M_1$ a general point of $\mathbf{v}$. Then $f(u)$ is a general point of $\mathbf{w}$ and f is a local equivalence of $\mathbf{v}$ with $\mathbf{w}$ at the point $(u, f(u)) \in M_1 \times M_2$.*

Proof. By Lemma 2 § 2 we can again suppose $M_1 = M_2 = M$ and $f = Id$. Thus we have two $\{e\}$-structures $\mathbf{v}$ and $\mathbf{w}$ on M and a general point u of the $\{e\}$-structure $\mathbf{v}$. Proposition 2 implies immediately the equality of structures functions

$$c^{\gamma}_{\mathbf{v}ij}(x) = c^{\gamma}_{\mathbf{w}ij}(x),\ c^{\gamma}_{\mathbf{v}ijk}(x) = c^{\gamma}_{\mathbf{w}ijk}(x) \quad \text{for all} \quad i, j, k, \gamma = 1, \ldots, n$$

on M. From these two equalities and (21) we get e.g.

$$dc^{\gamma}_{ij}(x)\,(\mathbf{v}_k(x) - \mathbf{w}_k(x)) = 0\,, \quad x \in M\,.$$

So we have $\mathbf{v}_k(x) = \mathbf{w}_k(x)$ at all points x at which the functions $\{c^{\gamma}_{ij}\}$ have the rank n. Because of u_0 being a general points of v this condition is satisfied on a neighborhood of u_0. The first part of the assertion is evident.

Proposition 3 can be generalized to:

Proposition 4. *Let $f : M_1 \to M_2$ be a deformation of order $(s + 2)$ of $\{e\}$-structures $\mathbf{v}$ and $\mathbf{w}$ on M_1 and M_2 respectively If $u_0 \in M_1$ is an s-general point of $\mathbf{v}$*

then $f(u_0)$ is an s-general of **w** *and f is a local equivalence of* **v** *with* **w** *at the point $(u_0, f(u_0)) \in M_1 \times M_2$.*

Proof. We proceed as in the proof of Proposition 3. f being a deformation of order $(s + 2)$ we get on M the equations (for all $i_1, \ldots, i_{s+2}, k, \gamma = 1, \ldots, n$).

$$dc^{\gamma}_{i_1 i_2}(\mathbf{v}_k(x) - \mathbf{w}_k(x)) = 0\,, \tag{25}$$
$$\vdots$$
$$dc^{\gamma}_{i_1 \ldots i_{s+2}}(\mathbf{v}_k(x) - \mathbf{w}_k(x)) = 0\,.$$

Further, since $dc^{\gamma}_{i_1 i_2}(x), \ldots, dc^{\gamma}_{i_1 \ldots i_{s+2}}(x)$ generate $T^*_x M$ for all x from a neighborhood of u_0

$$\mathbf{v}_k(x) = \mathbf{w}_k(x)\,, \quad k = 1, \ldots, n$$

holds on this neighborhood. Globally this proposition can be formulated as:

Proposition 5. *Let* **v** *be an $\{e\}$-structure on M_1 such that all points from M_1 are s-general points of* **v**, **w** *an $\{e\}$-structure on M_2, and $f : M_1 \to M_2$ a diffeomorphism. If f is a deformation of order $(s + 2)$ then f is an equivalence.*

Proof. Proposition 5 is an immediate consequence of Proposition 4.

If an $\{e\}$-structures on M has no s-general points for any s, i.e. its rank is always less than n, then it is not possible under the hypothesis that f is a deformation of arbitrarily high rank to prove that f is an equivalence. It is only possible to prove the existence of a local equivalence from the existence of a deformation of a certain order under further additional suppositions.

If for example **v** is an $\{e\}$-structure with all structural functions of order one vanishing (and then all structural functions vanish) then an arbitrary diffeomorphism of M to itself is a deformation of arbitrarily high rank but is not necessarily a local equivalence. But is such a case a local equivalence always exists.

Using Proposition 4 from Sternberg [1] it is possible to prove:

Proposition 6. *Let p be a regular point of an $\{e\}$-structure* **v** *on M_1 of order r and rank k, q a regular point of an $\{e\}$-structure* **w** *on M_2 of order r and rank k. If there exists a deformation $f : M_1 \to M_2$ of order $r + 2$ of the $\{e\}$-structure* **v** *with* **w** *such that $f(p) = q$ then there exists a local equivalence* **v** *with* **w** *at the point $(p, q) \in M_1 \times M_2$.*

Proof. We show that under our hypothesis the hypothesis of Proposition 4,1 from [1] is satisfied. The existence of a deformation of order $(r + 2)$ implies immediately the equality ${}^i\tilde{c} = {}^i c$ for structural functions up to and including the order $(r + 1)$. If $(x^1, \ldots, x^n)$ is a chart associated with **v** at p then it is also associated with **w**. From the equality of the structural functions of order $r + 1$ the condition (iii) from Sternberg [1] follows immediately.

4. Flatness of G-structures

In the end I would like to present some results from a paper of Guillemin [2]. Let us introduce the following notation and definitions (the rest of notions used here can be fould in [2]).

A G-structure $B \to M$ is called *uniformly k-flat* if it is in deformation of order k with the standard flat G-structure at any point $(x, 0) \in M \times R^n$.

If $B_1 \to M_1$ and $B_2 \to M_2$ are G-structures, we say that a diffeomorphism $f : M_1 \to \to M_2$ *preserves the structure up to the order k at* $u \in M_1$ if there exists $p \in \pi_2^{-1}(f(u))$ such that the G-structures $B_2 \to M_2$ and $\tilde{f}(B_1) \to M_1$ are in contact or oder k at p.

Again this is only a property of the $(k + 1)$-jet of f and not of f itself, and thus we shall speak about $(k + 1)$-jet preserving the structure up to the order k.

If $B \to M$ is a uniformly k-flat G-structure we can define a principal fibre bundle $\pi^k : E^k \to M$ of all $(k + 1)$-jets with a source $0 \in R^n$ and target in M preserving the structure up to the order k. On E^k, a canonical 1-form with values in $V + \mathfrak{Y} + \dots .. + \mathfrak{Y}^{(k-1)}$ and structural functions

$$c^k : E^k \to H^{k,2}(\mathfrak{Y})$$

can be defined.

$(H^{k,l}(\mathfrak{Y})$ is the spencer cohomology of $\mathfrak{Y})$. Then we have:

Proposition 7. *Let $B \to M$ be uniformly k-flat G-structure. $B \to M$ is in deformation of order $(k + 1)$ at $(x, 0) \in M \times R^n$ with the standard flat G-structure on R^n if and only if $c^k(p) = 0$ for some element $p \in (\pi^k)^{-1}(x)$. If $H^{k,2}(\mathfrak{Y}) = 0$, then the uniform k-flatness implies the uniform $(k + 1)$-flatness.*

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