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COMMENTS CONCERNING FLOWS NEAR COMPACT INVARIANT SETS

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I. Introduction. For (X, π) a (global, bilateral, continuous) flow and $x \in X$ we denote the positive limit set of x by L_x^+, the negative limit set of x by L_x^-, and the trajectory through x by C_x. ([1] or [2]).

The following theorem or its generalization [3, Theorem 9.1] is known as the Kimura-Ura Theorem [4, Proposition 14]:

Let (X, π) be a flow on the locally compact Hausdorff space X and let $M \subset X$ be compact and invariant. Then one of the following holds:

(i) *M is positively asymptotically stable.*

(ii) *M is negatively asymptotically stable.*

(iii) *$\exists\, x_1, x_2 \notin M$ such that $L_{x_1}^+ \neq \emptyset$, $L_{x_2}^- \neq \emptyset$, $L_{x_1}^+ \subset M$, and $L_{x_2}^- \subset M$.*

(iv) *For every neighborhood $\mathcal{U}$ of M in X $\exists\, x \in \mathcal{U}$ such that $C_x \subset \mathcal{U} \setminus M$.*

For the flow on the plane, E^2, arising from the system:

$$\frac{dx}{dt} = x^2 - y^2 ,$$

$$\frac{dy}{dt} = 2xy ,$$

(Fig. 1) if $M = \{(0, 0)\}$, then (iii) and (iv) both occur. We show that, in general, the absence of (iii) does not insure a stronger version of (iv).

II. Theorem. *Let (X, π) be a flow with X homeomorphic to E^2 or S^2. Let $M \subset X$ be compact and invariant. If* (i), (ii), *and* (iii) *of the Kimura-Ura Theorem do not occur, then*

(iv)′ *For each neighborhood $\mathcal{U}$ of M in X $\exists\, x \in \mathcal{U}$ such that $\bar{C}_x \subset \mathcal{U} \setminus M$.*

Proof. Let a neighborhood $\mathscr{U}$ of M in X be given. Then $\exists$ neighborhoods W, V of M in X such that $W \subset \overline{W} \subset V \subset \overline{V} \subset \mathscr{U}$, $\overline{V}$ is compact, and for each $x \in V \setminus M$ either $\emptyset \neq L_x^+ \not\subset M$ or $\emptyset \neq L_x^- \not\subset M$. Since (iv) of the Kimura-Ura Theorem must hold, $\exists\, \omega \in W$ such that $C_\omega \subset W \setminus M$ and $\bar{C}_\omega = L_\omega^- \cup C_\omega \cup L_\omega^+$ is compact. Each of the sets L_ω^+ and L_ω^- is nonempty, compact, invariant, and contained in $\mathscr{U}$. If either one does not meet M, (iv)′ holds and we are done.

Otherwise, WLOG assume that $L_\omega^+ \cap M \neq \emptyset$ and $L_\omega^+ \cap (\mathscr{U} \setminus M) \neq \emptyset$. Let $z \in L_\omega^+ \cap (\mathscr{U} \setminus M)$. Now L_ω^+ cannot be a cycle since $L_\omega^+ \cap M$ is invariant. Hence, either z is a rest point or L_z^+ and L_z^- are nonempty continua of rest points [5, Proposition 1.11]. If z is a rest point or either L_z^+ or L_z^- is not contained in M, we are done. Otherwise, $L_z^+ \subset M$, $L_z^- \subset M$ and this contradicts the choice of V. The proof is complete.

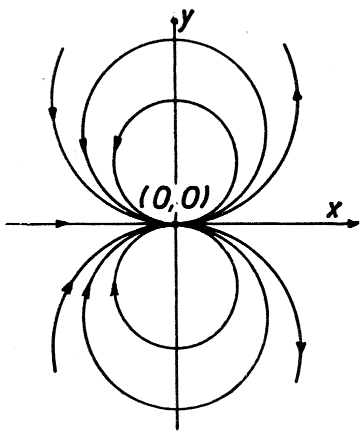

Fig. 1.

An obvious consequence:

Corollary. *Under the hypothesis of the theorem, if M is connected and there is a simply-connected neighborhood $\mathscr{U}$ of M in X such that $\overline{\mathscr{U}}$ is compact and $\mathscr{U} \setminus M$ is free of rest points, then every neighborhood V of M in X contains a cycle σ such that $M \subset B_\sigma \equiv$ the (a, for S^2) bounded component of $X \setminus \sigma$.*

Proof. Let V be any neighborhood of M in X. $\exists\, \omega \in V \cap \mathscr{U} = W$ such that $\bar{C}_\omega \subset W \setminus M$. Then $\bar{C}_\omega$ contains a nonempty compact minimal set, hence, a cycle σ since $\mathscr{U} \setminus M$ is free of rest points. [1, Theorem 12.8]. Since σ lies in $\mathscr{U}$, B_σ lies in $\mathscr{U}$. But $\sigma \cup B_\sigma$ contains a rest point [6], say p. Then $p \in M$ and $M \cap B_\sigma \neq \emptyset$. Since M is connected, $M \subset B_\sigma$. This completes the proof.

III. An example. For each natural number N let $r_N = \frac{1}{3}(1/N - 1/(N+1)) = 1/3N(N+1)$ and let T_N be the torus given by

$$T_N = \{(X_N, Y_N, Z_N) \in E^3 \mid X_N = \left(\frac{1}{N} + r_N \cos 2\pi\theta\right) \cos 2\pi\phi - \left(\frac{1}{N} + r_N\right),$$

$$Y_N = \left(\frac{1}{N} + r_N \cos 2\pi\theta\right) \sin 2\pi\phi\,, \quad Z_N = \frac{1}{N} + r_N \sin 2\pi\theta\,,$$

$$0 \leqq \phi < 1,\ 0 \leqq \theta < 1\}\,.$$

Let π_N be a sub-flow on T_N with just two trajectories: a single rest point, P_N, at

$(0, 0, 1/N)$ and the trajectory C_N of a motion everywhere dense on T_N and stable in the sense of Poisson in both directions [2, Example 4.06]. Let π_0 be the flow on $\{(0, 0, 0)\}$ such that $P_0 = (0, 0, 0)$ is a rest point. Put $\Pi = \bigcup_{i=0}^{\infty} \pi_i$ (Fig. 2). Then Π is a flow [1, Theorem 12.1]. Let $M = \{P_j \mid j = 0, 1, 2, \ldots\}$. Then M is compact and invariant. Further, (i), (ii), (iii) of the Kimura-Ura Theorem do not hold but (iv)′ of section II does not hold.

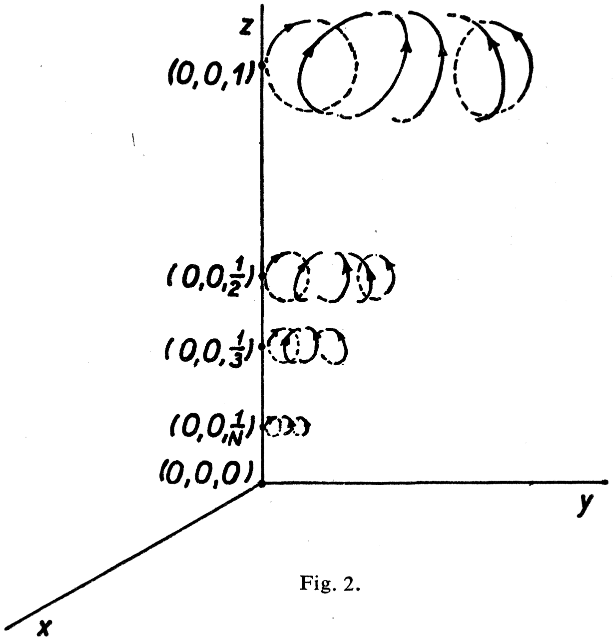

Fig. 2.

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