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RANDOMLY EULERIAN DIGRAPHS[1])

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1. INTRODUCTION

A *graph G is eulerian* if it has a circuit containing all vertices and edges of G. It is well known that a graph G is eulerian if and only if G is connected and every vertex of G has even degree. There is an analogous concept among directed graphs (digraphs); namely, a *digraph D is eulerian* if it has a (directed) circuit containing all vertices and arcs of D. Eulerian digraphs are characterized by the properties of being connected and having each of its vertices with equal indegree and outdegree.

In [4] ORE introduced an interesting class of eulerian graphs. An eulerian graph G is said to be *randomly eulerian from a vertex v* of G if the following procedure always results in an eulerian circuit of G: Begin a trail at v by choosing any edge incident with v. Next (and at each step thereafter), the trail is continued by selecting any edge not already chosen which is adjacent with the edge most recently selected. The process terminates when no such edge is available. Equivalently, a graph G is randomly eulerian from v if every trail of G beginning at v can be extended to an eulerian circuit of G.

The object of this article is to define digraphs which are randomly eulerian from a vertex and to present some of their properties.

2. FUNDAMENTAL TERMINOLOGY

In order to make this paper self-contained, we give here those fundamental definitions which are most pertinent to our discussion. For basic graph and digraph terminology, we follow [2], [3], respectively. For vertices u and v of a digraph D, a *$u - v$ arc sequence* is an alternating sequence

$$u = u_1, a_1, u_2, a_2, u_3, \ldots, u_{n-1}, a_{n-1}, u_n = v \tag{1}$$

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of vertices and arcs such that for $i = 1, 2, \ldots, n - 1$, a_i is the arc $u_i u_{i+1}$ (i.e., a_i is directed from u_i to u_{i+1}) or the arc $u_{i+1} u_i$. The digraph D is *connected* if for every two vertices u and v, there exists a $u - v$ arc sequence; otherwise D is disconnected. In the latter case, a (connected) *component* is a maximal connected sub-digraph of D.

A *$u - v$ trail* in a digraph D is a $u - v$ arc sequence (1) such that $a_i = u_i u_{i+1}$ for each i and such that no arc in (1) is repeated. In this case, it is customary and more convenient to express a $u - v$ trail as

$$u = u_1, u_2, u_3, \ldots, u_{n-1}, \quad u_n = v$$

since the arcs of the trail are evident.

A $u - v$ trail is called *open* or *closed* depending on whether $u \neq v$ or $u = v$. A closed trail (with at least two arcs) is also referred to as a *circuit.* A circuit in which no vertex is repeated (except that the last vertex is the same as the first) is a *cycle.* Also, an open $u - v$ trail in which no vertex is repeated is a *$u - v$ path.*

An open $u - v$ trail containing all vertices and arcs of a digraph D is called an *eulerian trail* of D, while a circuit containing all vertices and arcs of D is an *eulerian circuit* of D.

Let v be a vertex of a digraph D. The *indegree*, id v, of v is the number of arcs of D for which v is the terminal vertex and the *outdegree*, od v, of v is the number of arcs of D for which v is the initial vertex. If id v = od v, then we speak simply of the *degree* of v and write deg v = id v (= od v).

3. RANDOMLY EULERIAN DIGRAPHS

Following the terminology for graphs, we define a digraph D to be *randomly eulerian from a vertex v* if every trail with initial vertex v can be extended to an eulerian circuit of D. As an illustration, we consider the five eulerian digraphs shown in Figure 1. For $i = 0, 1, 2, 4, 6$, the digraph D_i is randomly eulerian from exactly

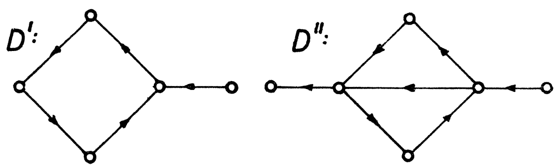

Fig. 1.

the i labeled vertices. Thus D_0 is randomly eulerian from no vertices. This follows since at each vertex v of D_0, a trail with initial vertex v and with four arcs exists which cannot be extended to an eulerian circuit of D_0.

We now present a characterization of eulerian digraphs which are randomly eulerian from some vertex.

Theorem 1. *An eulerian digraph D is randomly eulerian from a vertex v of D if and only if every cycle of D contains v.*

Proof. Let D be an eulerian digraph having a vertex v which lies on every cycle of D, and assume that D is not randomly eulerian from v. Hence there exists some trail C with initial vertex v which cannot be extended to an eulerian circuit of D. Without loss of generality, we may assume that C can be extended no further; hence we may assume that C is a $v - v$ circuit which contains all arcs of D incident with v but not all arcs of D. Let D' be the digraph obtained from D by deleting all arcs of C. Every vertex of D' has equal indegree and outdegree, and, furthermore, v is isolated in D'. Since D' contains arcs, there exists a nontrivial component D_1 in D'. The component D_1 is itself an eulerian digraph; therefore D_1 has a circuit containing all vertices and arcs of D_1. This implies that D_1 has a cycle C_1, but C_1 does not contain v. This is a contradiction so that D is randomly eulerian from v.

Conversely, let D be a digraph which is randomly eulerian from a vertex v, and suppose there exists a cycle C not containing v. Denote by D' the digraph obtained from D by removing the arcs of C. Every vertex of D' again has equal indegree and outdegree. The component of D' containing v is itself eulerian and therefore has a $v - v$ circuit C' containing all vertices and arcs of D'. Necessarily, then, C' contains all arcs of D incident with v, and, therefore C' cannot be extended to include additional arcs of D. This, however, contradicts the fact that D is randomly eulerian from v.

In view of the preceding theorem, we arrive at the following consequence.

Corollary 1a. *An eulerian digraph D is randomly eulerian from all its vertices if and only if D is itself a cycle.*

Since every vertex of an eulerian digraph D has equal indegree and outdegree, every vertex of D has a degree defined. Denote by $\Delta(D)$ the maximum among the degrees of the vertices of D. We can now establish another corollary of Theorem 1.

Corollary 1b. *If an eulerian digraph D is randomly eulerian from the vertex v, then* $\Delta(D) = \deg v$.

Proof. Let $C : v = v_0, v_1, v_2, \ldots, v_{n-1}, v_n = v$ be an eulerian $v - v$ circuit of G. For any vertex $u \neq v$, every two occurrences of u must have an occurrence of v between them; otherwise, a cycle of D exists not containing v, which contradicts Theorem 1. Hence the number of occurrences of u cannot exceed the number of occurrences of v. Since every occurrence of a vertex in C indicates an indegree of one and an outdegree of one for that vertex, except for the first and last occurrences of v (which represents an outdegree of one for v and an indegree of one for v, respectively), the indegree (outdegree) of u cannot exceed the indegree (outdegree) of v. The result now follows.

Another corollary yet can be given.

Corollary 1c. *Let D be an eulerian digraph which is randomly eulerian from a vertex v. If u is any vertex of D such that* $\deg u = \deg v$, *then D is also randomly eulerian from u.*

Proof. Assume that the digraph D is not randomly eulerian from u. Thus, by Theorem 1, there is a cycle C which contains v but not u. Considering C as a $v - v$ trail, we extend C to an eulerian $v - v$ circuit C'. In that part of C' not including C, every two occurrences of u must have an occurrence of v between since every cycle of D contains v. However, since u does not occur in C, this implies that the arcs of C' contribute at least one less to the degree of u than the degree of v. Because $\deg u = \deg v$, there are at least two arcs incident with u in D that do not belong to C', which contradicts the fact that C' is an eulerian circuit of D. Hence the digraph D is randomly eulerian from u.

Corollary 1c further implies that if an eulerian digraph D is randomly eulerian from some vertex, then it is randomly eulerian from any vertex having degree $\Delta(D)$. Corollary 1b then implies that D is randomly eulerian from exactly those vertices of D having degree $\Delta(D)$. We now proceed to determine the possible number of vertices from which an eulerian digraph may be randomly eulerian. We first verify the following theorem.

Theorem 2. *Let D be an eulerian digraph which is randomly eulerian from each vertex in a set S of vertices. Then the vertices of S appear in the same order in every cycle of D.*

Proof. If S contains one or two vertices, then the result is obvious; thus we assume that $|S| = n \geqq 3$. Suppose that the result is not true so that there exist cycles C_1 and C_2 which do not contain the vertices of S in the same order. By a suitable labeling, we may assume that the order of the vertices of S on C_1 is $u_1, u_2, \ldots, u_n$ while on C_2 the vertex following u_1 is u_j, $j \neq 2$. We construct a $u_2 - u_2$ circuit C as follows: Follow the circuit C_1 from u_2 to u_j, and then proceed along C_2 from u_j to u_2. The circuit C so constructed does not contain u_1, which contradicts the fact that D is randomly eulerian from u_1.

The following observation will prove useful.

Corollary 2a. *Let D be an eulerian digraph which is randomly eulerian from each of the vertices $v_1, v_2, \ldots, v_n$, $n \geqq 3$, and assume that these vertices appear in this order in every cycle of D. Then for any $j (1 \leqq j \leqq n)$, every $v_j - v_i$ path $(i \neq j + 1$, where $v_{n+1} = v_1)$ contains v_{j+1}.*

Proof. Suppose P is a $v_j - v_i$ path (as in the statement of the corollary) which does not contain v_{j+1}. Since D is randomly eulerian from v_j, P can be extended to a $v_j - v_j$ cycle C; otherwise, a cycle would exist not containing v_j. However, since v_{j+1} lies on C, the vertices v_i $(1 \leqq i \leqq n)$ do not belong to C in the appropriate order, which contradicts Theorem 2. The desired result now follows.

This brings us to the result with which we are mainly concerned.

Theorem 3. *If an eulerian digraph D of order $p(\geqq 2)$ is randomly eulerian from exactly n vertices, then either* $0 \leqq n \leqq [p/2]$ *or* $p = n$.

Proof. Suppose that D is randomly eulerian from the vertices $v_1, v_2, \ldots, v_n$ and that this is the order in which these n vertices appear on every cycle of D.

Let d be the maximum number of arcs in any $v_1 - v_2$ path in D. We distinguish two cases here.

Case 1. $d = 1$. In this case v_2 is adjacent from v_1 but from no other vertices. Hence $\operatorname{od} v_1 = \operatorname{id} v_2 = 1$ which implies that $\deg v_1 = \deg v_2 = 1$ as well as $\deg v_i = 1$ for $1 \leqq i \leqq n$. Since $\Delta(D) = 1$, every vertex of D has degree one. Therefore D is a cycle of order p so that $p = n$.

Case 2. $d > 1$. Here there exists a vertex $v \neq v_i$ $(1 \leqq i \leqq n)$ which lies on some $v_1 - v_2$ path. If $\operatorname{od} v_1 = 1$, then $\Delta(D) = 1$ implying that D is a cycle which contradicts the fact that D is not randomly eulerian from v. Therefore, $\operatorname{od} v_1 > 1$, so that $\deg v_i > > 1$ for $i = 1, 2, \ldots, n$. Also, for each such i, no $v_i - v_{i+1}$ path (where $v_{n+1} = v_1$) contains a vertex v_j $(j \neq i, i + 1)$ by Corollary 2a. Hence for each i, some $v_i - v_{i+1}$ path contains a vertex $u_i \neq v_j$ $(1 \leqq j \leqq n)$. The vertices u_i are distinct; for otherwise we have a contradiction to Corollary 2a. From this, we conclude that $p \geqq 2n$ or, equivalently, $n \leqq [p/2]$.

4. RANDOMLY TRAVERSABLE DIGRAPHS

In this, the concluding section, we discuss a concept very closely related to that of randomly eulerian digraphs.

A *graph* has been called *traversable* if it contains an eulerian trail. A necessary and sufficient condition for a graph G to be traversable is that G be connected and contain exactly two vertices of odd degree. In [1] a traversable graph G was defined to be *randomly traversable from a vertex* v (having odd degree) if every trail of G with initial vertex v can be extended to an eulerian trail of G. Analogues of these concepts for digraphs may now be given.

We define a *digraph* D to be *traversable* if D contains an eulerian trail. It is easy to establish the following characterization of traversable digraphs: A digraph D is traversable if and only if D is connected and each vertex of D has equal indegree and outdegree with the exception of two vertices, one of which has its indegree exceeding its outdegree by one and the other has its outdegree exceeding its indegree by one.

Let D be a traversable digraph with vertices u and v such that $\operatorname{id} u = \operatorname{od} u + 1$ and $\operatorname{od} v = \operatorname{id} v + 1$. Then D is defined to be *randomly traversable from* v (or simply *randomly traversable*) if every trail of D with initial vertex v can be extended to an eulerian trail of D. For example, the traversable digraph D' of Figure 2 is randomly traversable while the traversable digraph D'' is not randomly traversable.

A characterization of randomly traversable digraphs exists very much like that for digraphs which are randomly eulerian from some vertex given in Theorem 1.

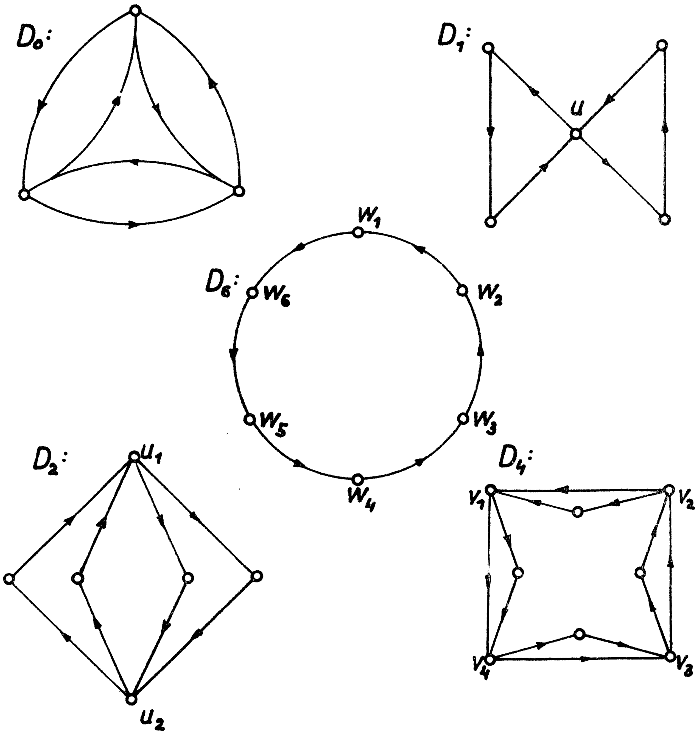

Fig. 2.

Theorem 4. *Let D be a traversable digraph with vertices u and v such that* $\text{id } u = \text{od } u + 1$ *and* $\text{od } v = \text{id } v + 1$. *Then D is randomly traversable (from v) if and only if u is contained in every cycle of D.*

Proof. Let D be a traversable digraph which is randomly traversable from v, and assume D has a cycle C not containing u. Denote by D' the digraph obtained by deleting the arcs of C from D. In D' every vertex has equal indegree and outdegree except for u and v, in which case $\text{id } u = \text{od } u + 1$ and $\text{od } v = \text{id } v + 1$. Necessarily, u and v belong to the same component D_1 of D'. The digraph D_1 is therefore traversable and has a $v - u$ trail T containing all vertices and arcs of D_1. The trail T contains all arcs of D incident with u; hence T cannot be extended to an eulerian trail of D. This contradicts the fact that D is randomly traversable from v.

For the converse, suppose every cycle of a traversable digraph D contains u but that D is not randomly traversable from v. Hence there is a trail of D with initial vertex v which cannot be extended to an eulerian trail of D. This implies that there is a $v - u$ trail T which is not an eulerian trail and which cannot be extended, i.e., T contains all arcs of D incident with u. Delete from D all arcs of T obtaining a digraph D' with arcs. Every vertex of D' has equal indegree and outdegree and, moreover, u is isolated in D'. Let D_1 be a nontrivial component of D'. The digraph D_1 is eulerian and therefore has a circuit containing all vertices and arcs of D_1. Furthermore, D_1 has cycles, none of which contains u. This contradicts the hypothesis and completes the proof.

References

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