Jaroslav Smítal

On approximation of Baire functions by Darboux functions

Czechoslovak Mathematical Journal, Vol. 21 (1971), No. 3, 418–423

Persistent URL: http://dml.cz/dmlcz/101042

Czechoslovak Mathematical Journal, 21 (96) 1971, Praha

ON APPROXIMATION OF BAIRE FUNCTIONS BY DARBOUX FUNCTIONS

JAROSLAV SMÍTAL, Bratislava

(Received April 15, 1970)

1. TERMINOLOGY AND NOTATION

Throughout this paper, unless otherwise specified, all functions will supposed to be real valued defined on a (possibly infinite) interval I.

We use the usual Borel classification of sets (see Kuratowski [4], page 250).

The class of Borel α functions is denoted as $\mathscr{B}_\alpha$ while the class of Baire α functions as Φ_α. As is well-known, for α finite we have $\Phi_\alpha = \mathscr{B}_\alpha$, but $\Phi_\alpha = \mathscr{B}_{\alpha+1}$ for α infinite. (For facts concerning Borel and Baire functions see KURATOWSKI [4], page 280, resp. 306.) $\mathscr{D}$ stands for the class of Darboux functions. For two classes $\mathscr{A}$ and $\mathscr{B}$ of functions let $\mathscr{A}\mathscr{B}$ denote the class $\mathscr{A} \cap \mathscr{B}$, e.g. $\mathscr{D}\mathscr{B}_\alpha$.

All limits of sequences of functions are pointwise limits. If $\mathscr{A}$ is a class of functions then $\mathscr{A}\uparrow$ (resp. $\mathscr{A}\downarrow$) denotes the set of all functions which are limits of increasing (resp. decreasing) sequences of functions in $\mathscr{A}$. Finally we write $\mathscr{A}\uparrow\downarrow$ for $(\mathscr{A}\uparrow)\downarrow$ and similarly with $\mathscr{A}\downarrow\uparrow$.

2. INTRODUCTION

It is known that each $f \in \Phi_\alpha$ with $\alpha \geqq 1$ is the limit of a sequence $\{f_n\}_{n=1}^\infty$ of functions such that each f_n is in $\mathscr{D}\Phi_{\alpha-1}$ if α is a non-limit ordinal, and $f_n \in \bigcup_{\beta<\alpha} \mathscr{D}\Phi_\beta$ otherwise (see [3], [5], [1], [6], and [7]). In the present paper a somewhat sharper result is given: For each ordinal $\alpha \geqq 1$, there is a lattice Ω_α of Darboux functions in Baire classes preceeding α such that Φ_α is the pointwise closure of Ω_α (see Theorem 2 below; the case $\alpha = 2$ is a simple consequence of Preiss' result [7]).

The following theorems have been stated in [2] by Ceder and Weiss:

Theorem A. $\mathscr{D}\uparrow\downarrow = \mathscr{D}\downarrow\uparrow$ *is the class of all functions.*

Theorem B. *Let* $\alpha \geqq 1$. *Then*

$$\Phi_\alpha = (\Phi_{\alpha-1}\uparrow\downarrow) \cap (\Phi_{\alpha-1}\downarrow\uparrow)$$

for α *a non-limit ordinal and*

$$\Phi_\alpha \supset (\bigcup_{\beta<\alpha} (\Phi_\beta\uparrow))\downarrow \cap (\bigcup_{\beta<\alpha} (\Phi_\beta\downarrow))\uparrow$$

when α *is a limit ordinal.*

The autors claimed that $\Phi_\alpha = (\bigcup_{\beta<\alpha} (\Phi_\beta\uparrow))\downarrow \cap (\bigcup_{\beta<\alpha} (\Phi_\beta\downarrow))\uparrow$ when α is a limit ordinal. Their proof is invalid and the question of equality remains open.

In connection with these results the following problem is posed in [2] by CEDER and WEISS: What is the class $\mathscr{D}\mathscr{B}_1\downarrow\uparrow \cap \mathscr{D}\mathscr{B}_1\downarrow\uparrow$? From a result [7] of D. PREISS it follows that $\mathscr{D}\mathscr{B}_1\uparrow\downarrow \cap \mathscr{D}\mathscr{B}_1\downarrow\uparrow = \mathscr{B}_2$. In the present paper it is shown that, in harmony with the above cited Theorem B a similar result holds for each class Φ_α where α is a non-limit ordinal > 0 (see Theorem 3 below).

In the above cited paper [2] Ceder and Weiss give a characterization of the classes $\mathscr{D}\uparrow$ and $\mathscr{D}\downarrow$. In the present paper a similar characterization of the classes $(\mathscr{D}\Phi_\alpha)\uparrow$ and $(\mathscr{D}\Phi_\alpha)\downarrow$ with $\alpha > 1$ is given (see Theorem 1 below).

3. APPROXIMATION THEOREMS

We begin with two lemmas.

Lemma 1. *Let* $\{(I_n, A_n)\}_{n=1}^{\infty}$ *be a sequence of ordered pairs such that each* I_n *is an open interval and* A_n *a Borel set, and let for each* n, *the set* $I_n \cap A_n$ *be uncountable. Then there is a non-empty nowhere dense perfect set* $P \subset I_1 \cap A_1$ *such that for each* n, *the set* $I_n \cap A_n - P$ *is uncountable.*

Proof. Since $I_1 \cap A_1$ is an uncountable Borel set, it contains a non-empty nowhere dense perfect subset B (see Kuratowski [4], page 387). Define for each $n > 1$, the set C_n in this way: If the set $B \cap I_n \cap A_n$ is at most countable, let $C_n = \emptyset$. Otherwise $B \cap I_n \cap A_n$ as uncountable Borel set contains a non-empty perfect subset and hence a non-empty perfect subset which is nowhere dense in B. Denote this non-empty perfect set by C_n. Since $\bigcup_{n=1}^{\infty} C_n$ is of the first category in B and B is closed, the set $B - \bigcup_{n=1}^{\infty} C_n$ is non-empty. Moreover, it is an uncountable (B has no isolated points) Borel set. Hence there exists a non-empty nowhere dense perfect subset, say P, contained in $B - \bigcup_{n=1}^{\infty} C_n$. It is easy to verify that for each n, the set $I_n \cap A_n - P$ is uncountable, q.e.d.

Lemma 2. *Let $\{(I_n, A_n)\}_{n=1}^{\infty}$ be a sequence of ordered pairs such that I_n is an open interval and A_n a Borel set and assume for each n, that the set $I_n \cap A_n$ is uncountable. Then there are disjoint non-empty nowhere dense perfect sets $\{P_n\}_{n=1}^{\infty}$ such that $P_n \subset I_n \cap A_n$, for each n.*

Proof. By Lemma 1, there exists a non-empty nowhere dense perfect set P_1 such that $P_1 \subset I_1 \cap A_1$ and each set $I_n \cap A_n - P_1 = I_n \cap (A_n - P_1)$ is uncountable. In general, by induction let $P_1, P_2, \ldots, P_k$ be disjoint non-empty nowhere dense perfect sets such that $P_i \subset I_i \cap A_i$, $i = 1, 2, \ldots, k$, and for each n, $I_n \cap (A_n - \bigcup_{i=1}^{k} P_i)$ is uncountable. By applying the Lemma 1 to the sets $\{(I_n, A_n - \bigcup_{i=1}^{k} P_i)\}_{n=k+1}^{\infty}$ obtain a non-empty nowhere dense perfect set P_{k+1}. It is easy to verify that $P_{k+1} \subset I_{k+1} \cap \cap A_{k+1}$, that for each n, $I_n \cap (A_n - \bigcup_{i=1}^{k+1} P_i)$ is uncountable and that $P_1, P_2, \ldots, P_{k+1}$ are disjoint.

Now we are able to prove the following

Theorem 1. *For each ordinal $\alpha > 1$,*

$$(\mathcal{D}\Phi_\alpha)\uparrow = (\mathcal{D}\uparrow) \cap (\Phi_\alpha\uparrow) \quad \text{and} \quad (\mathcal{D}\Phi_\alpha)\downarrow = (\mathcal{D}\downarrow) \cap (\Phi_\alpha\downarrow).$$

Proof. To prove the theorem it suffices to show that $(\mathcal{D}\uparrow) \cap (\Phi_\alpha\uparrow) \subset (\mathcal{D}\Phi_\alpha)\uparrow$. (The proof for $(\mathcal{D}\downarrow) \cap (\Phi_\alpha\downarrow) \subset (\mathcal{D}\Phi_\alpha\downarrow)$ is similar.) We can without loss of generality assume that all functions in the sequel are defined on an open interval I. Let $f \in \in (\mathcal{D}\uparrow) \cap (\Phi_\alpha\uparrow)$. Let $\{(I_n, J_n)\}_{n=1}^{\infty}$ be an enumeration of all pairs (I_n, J_n) of intervals I_n, J_n with rational end-points, where I_n are open intervals which are contained in I, and J_n are intervals of the form $(r, r'\rangle$ $(= \{x; r < x \leqq r'\})$, and such that $I_n \cap \cap f^{-1}(J_n)$ is uncountable. Apply the Lemma 2 to obtain a sequence $\{P_n\}_{n=1}^{\infty}$ of disjoint non-empty nowhere dense perfect sets such that for each n, $P_n \subset I_n \cap f^{-1}(J_n)$.

If r_n is the left-side end-point of the interval J_n, let g_n be a continuous function defined on P_n which maps P_n onto the closed interval $\langle \min(-n, r_n - n), r_n \rangle$. Since f is in $\Phi_\alpha\uparrow$, there exists an increasing sequence $\{f'_n\}_{n=1}^{\infty}$ of Baire α functions such that $f = \lim_{n\to\infty} f'_n$. Define functions $\{f''_n\}_{n=1}^{\infty}$ as follows:

$$f''_n(x) = \begin{cases} g_m(x) & \text{if} \quad x \in P_m \quad \text{and} \quad m \geqq n\,, \\ f'_n(x) & \text{if} \quad x \notin \bigcup_{m=n}^{\infty} P_m\,. \end{cases}$$

Finally, for each n, let

$$f_n(x) = \max\left(f''_1(x), f''_2(x), \ldots, f''_n(x)\right).$$

It is easy to see that $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence of functions such that $\lim_{n\to\infty} f_n = f$.

As is well-known, the set Φ_α is the set of all Borel α functions if α is finite, and Φ_α is the set of all Borel $\alpha + 1$ functions if α is infinite (see [4], page 299). Hence to show that $f_n \in \Phi_\alpha$ it suffices to show that for each real λ, $[f_n'' < \lambda]$ and $[f_n'' > \lambda]$ are of the additive Borel class β where $\beta = \alpha$ if α is finite and $\beta = \alpha + 1$ otherwise. The fact that $[f_n'' < \lambda]$ is of the additive Borel class β follows from the equality

$$[f_n'' < \lambda] = ([f_n' < \lambda] \cap (I - \bigcup_{i=n}^{\infty} P_i)) \cup (\bigcup_{i=n}^{\infty} [g_i < \lambda]),$$

and from the fact that the first set on the right-hand side of this equality is of the additive Borel class β while the second set is of the type F_σ. The argument is similar for $[f_n'' > \lambda]$.

Finally, each f_n is in $\mathscr{D}$. To see it assume that $x < y$ and (say) $f_n(x) < \xi < f_n(y)$, where ξ is a real number (in the case $f_n(x) > f_n(y)$ the proof is similar). Since $\xi < f_n(y) \leqq f(y)$ we have $y \in [f > \xi]$. Since f is in $\mathscr{D}\uparrow$ the set $[f > \xi]$ is bilaterally c-dense in itself (see [2], Corollary 2 of Th. 3). Hence

$$\text{card}([f > \xi] \cap (x, y)) = c \tag{1}$$

(here c denotes the cardinality of the continuum). Let l be a natural number such that $-l < \xi$. From (1) it follows that there is a member (P_q, J_q) in the sequence $\{(I_k, J_k)\}_{k=l+n}^{\infty}$ such that $I_q \subset (x, y)$ and $J_q \subset (\xi, +\infty)$. Now from the definition of g_q we have $g_q(z) = \xi$ for some $z \in P_q \subset I_q$ and hence for some $z \in (x, y)$. Since $q > n$ it follows from the definition of the function f_n that $f_n(z) = g_q(z) = \xi$. Thus we have shown that f is the limit of an increasing sequence of functions in $\mathscr{D}\Phi_\alpha$, q.e.d.

The next Theorem 2 is an extension of results found in [3], [5], [1], [6], and [7].

Theorem 2. *For each ordinal $\alpha > 0$ there is a lattice Ω_α of functions defined on an interval I such that $\Omega_\alpha \subset \bigcup_{\beta<\alpha} \mathscr{D}\Phi_\beta$, and Φ_α is the pointwise closure of Ω_α.*

Proof. The case $\alpha = 1$ is trivial. D. Preiss [7] has shown that each function in Φ_2 is the pointwise limit of a sequence of approximately continuous functions. But the set of approximately continuous functions is a lattice and every approximately continuous function is a Darboux function. Hence from Preiss' result [7] follows the case $\alpha = 2$.

It remains to prove the theorem for $\alpha > 2$. Let $\{I_n\}_{n=1}^{\infty}$ be an enumeration of all open subintervals of I with rational end-points. In Lemma 2 put $J_n = (-\infty, +\infty)$ for each n, to obtain a sequence $\{P_n\}_{n=1}^{\infty}$ of disjoint non-empty nowhere dense perfect sets such that $P_n \subset I_n$. Let g_n be a continuous function defined on P_n which maps this set onto the closed interval $\langle -n, n\rangle$. For each n, let V_n be an operation on the set $\bigcup_{\beta<\alpha} \Phi_\beta$ of functions in Baire classes preceding α such that for each $f \in \bigcup_{\beta<\alpha} \Phi_\beta$, $V_n(f)$ is a function defined as follows:

$$V_n(f)(x) = \begin{cases} g_m(x) & \text{if } x \text{ is in } P_m \text{ with } m \geqq n, \\ f(x) & \text{otherwise}. \end{cases}$$

Each $V_n(f)$ is in $\bigcup_{\beta<\alpha} \Phi_\beta$. To see it assume that f is in some $\mathscr{B}_\beta$ with $\beta < \alpha$ if α is a finite ordinal and $\beta \leqq \alpha$ otherwise. We assert that $V_n(f)$ is in $\mathscr{B}_{\max(\beta,2)} \subset \bigcup_{\gamma<\alpha} \Phi_\gamma$. Indeed, let λ be a real number. Consider the set

$$[V_n(f) > \lambda] = \bigcup_{i=n}^{\infty} [g_i > \lambda] \cup ([f > \lambda] - \bigcup_{i=n}^{\infty} P_i).$$

The set $\bigcup_{i=n}^{\infty} [g_i > \lambda]$ is clearly of the type F_σ. The set $([f > \lambda] - \bigcup_{i=n}^{\infty} P_i)$ is adifference of two sets, the first of the additive Borel class β, and the second of the type F_σ. It is easily checked that the difference is in the additive max $(\beta, 2)$. Hence $[V_n(f) > \lambda]$ as the union of two sets, the first of the type F_σ and the second of the additive Borel class max $(\beta, 2)$ is itself of the additive Borel class max $(\beta, 2)$. For $[V_n(f) < \lambda]$ the argument is similar and hence we conclude that $V_n(f) \in \mathscr{B}_{\max(\beta,2)} \subset \bigcup_{\gamma<\alpha} \Phi_\gamma$.

To prove that each $V_n(f)$ is also in $\mathscr{D}$ it suffices to show that $V_n(f)$ takes on each real value on each non-empty open interval J. Let p be a positive integer. J contains some rational open interval I_r with $r > n + p$ hence J contains the set P_r; from the definition of V_n it follows that $V_n(f)(x) = g_r(x)$ for $x \in P_r \subset J$, hence $V_n(f)$ takes on each value $y \in \langle -r, r\rangle \supset \langle -p, p\rangle$ on the interval J.

Since the set $\bigcup_{\beta<\alpha} \Phi_\beta$ is a lattice of functions it follows that for each n, the set $V_n(\bigcup_{\beta<\alpha} \Phi_\beta)$ is a lattice of Darboux functions in $\bigcup_{\beta<\alpha} \Phi_\beta$: Clearly $V_n(\bigcup_{\beta<\alpha} \Phi_\beta) \subset V_{n+1}(\bigcup_{\beta<\alpha} \Phi_\beta)$, for each n. Hence $\Omega_\alpha = \bigcup_{n=1}^{\infty} V_n(\bigcup_{\beta<\alpha} \Phi_\beta)$ is a lattice of Darboux function sand $\Omega_\alpha \subset \bigcup_{\beta<\alpha} \Phi_\beta$.

Finally, let $h \in \Phi_\alpha$. There exists a sequence $\{h_n\}_{n=1}^{\infty}$ of functions in $\bigcup_{\beta<\alpha} \Phi_\alpha$ such that $\lim_{n\to\infty} h_n = h$. It is easy to see that $\lim_{n\to\infty} V_n(h_n) = h$, q.e.d.

Next theorem gives a somewhat sharper result than the above cited Theorems A and B.

Theorem 3. *For each non-limit ordinal* $\alpha > 0$

$$\Phi_\alpha = (\mathscr{D}\Phi_{\alpha-1})\uparrow\downarrow \cap (\mathscr{D}\Phi_{\alpha-1})\downarrow\uparrow .$$

Proof. From the above cited Theorem B it follows that $\Phi_\alpha \supset (\mathscr{D}\Phi_{\alpha-1})\uparrow\downarrow \cap \cap (\mathscr{D}\Phi_{\alpha-1})\downarrow\uparrow$. Thus suppose f to be in Φ_α. By Th. 2 there is a lattice Ω_α of Darboux Baire $\alpha - 1$ functions such that Φ_α is the pointwise closure of Ω_α. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in Ω_α converging to f. Put

$$g_{n,k} = \max(f_n, f_{n+1}, \ldots, f_{n+k}), \quad h_{n,k} = \min(f_n, f_{n+1}, \ldots, f_{n+k}),$$

and

$$g_n = \sup(f_n, f_{n+1}, \ldots), \quad h_n = \inf(f_n, f_{n+1}, \ldots).$$

It is easy to see that the functions $g_{n,k}$ and $h_{n,k}$ are in $\mathscr{D}\Phi_{\alpha-1}$, that each g_n is in $(\mathscr{D}\Phi_{\alpha-1})\uparrow$ and each h_n is in $(\mathscr{D}\Phi_{\alpha-1})\downarrow$, and that functions g_n decrease pointwise to f and functions h_n increase pointwise to f, q.e.d.

Remark. If $\alpha > 2$, then the Theorems 2 and 3 can be stated for functions with a more general domain, e.g. for functions defined on a complete separable metric space which is dense in itself. In the proof of Theorem 2 it suffices to replace the rational open intervals $\{I_n\}_{n=1}^{\infty}$ by an open basis $\{G_n\}_{n=1}^{\infty}$, and similarly as in the proof of Lemmas 1 and 2 apply the Alexandroff-Hausdorff theorem (see [4], p. 355) which states that each uncountable Borel set contains a set P which is topologically equivalent to the Cantor set C. Thus the following theorems can be proved:

Theorem 4. *Let X be a complete separable metric space which is dense in itself and let $\alpha > 2$ be an ordinal; there is a lattice Ω_α of real-valued functions defined on X such that $\Omega_\alpha \subset \bigcup_{\beta<\alpha} \Phi_\beta$, each $f \in \Omega_\alpha$ takes on each real value on each non-empty open subset of X, and Φ_α is the pointwise closure of Ω_α.*

Theorem 5. *Let X be a complete separable metric space which is dense in itself and let $\alpha > 2$ be a non-limit ordinal; if $\tilde{\mathscr{D}}\Phi_\beta$ denote the set of all real-valued functions defined on X, in Baire class β, which take on each real value on each non-empty open subset of X then*

$$\Phi_\alpha = (\bigcup_{\beta<\alpha} \tilde{\mathscr{D}}\Phi_\beta)\uparrow\downarrow \bigcup_{\beta<\alpha} \tilde{\mathscr{D}}\Phi_\beta)\downarrow\uparrow .$$

References

[1] *A. M. Bruckner, J. G. Ceder* and *R. Keston:* Representations and Approximations by Darboux Functions in the First Class of Baire. Rev. Math. Pures Appl. *XIII* (1968), 1247—1254.

[2] *J. Ceder* and *M. Weiss:* Some Approximation Theorems for Baire Functions and Darboux Functions, Rev. Math. Pures Appl. XII (1967), 527—535.

[3] *А. Б. Гуревич:* О Д-непрерывных компонентах серпинского, ДАН БССР *10* (1966), 539—541.

[4] *C. Kuratowski:* Topologie, Vol. 1, Warszava 1958.

[5] *L. Mišík:* Zu zwei Sätzen von W. Sierpiński, Rev. Math. Pures Appl. *XII* (1967), 849—860.

[6] *D. Preiss:* Limits of Derivatives and Darboux Baire Functions, Rev. Math. Pures Appl. *14* (1969), 1201—1206.

[7] *D. Preiss:* Limits of Approximately Continuous Functions, Czechoslovak Math. Journal *21* (*96*), (1971), 371—372.

Author's address: Bratislava, Šmeralova 2b, ČSSR (Univerzita Komenského).