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ON UNISOLVENT SYSTEMS*)

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The purpose of this paper is to give some miscellaneous results concerning unisolvent systems of equations.

Let $f_1, f_2, \ldots, f_n$ be real valued functions defined on a set S. Denote by $|f_i(x_j)|_{i,j=1}^n$ the n by n determinant

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \ldots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \ldots & f_2(x_n) \\ \vdots & & & \\ f_n(x_1) & f_n(x_2) & \ldots & f_n(x_n) \end{vmatrix}$$

where $x_1, x_2, \ldots, x_n$ is a subset of S consisting of n distinct points.

Definition. The system of n functions $f_1, f_2, \ldots, f_n$ is *unisolvent* on the set S if and only if $|f_i(x_j)|_{i,j=1}^n \neq 0$ for every selection of n distinct points in S [1, page 31].

Theorem 1. *Let $f_1, f_2, \ldots, f_n$ be an even number of continuous functions which are unisolvent on the closed interval $[a, b]$.*

Suppose that f_{n+1} is continuous on the open interval (a, b) with $\lim_{x \to a} f_{n+1}(x) = -\infty$ and $\lim_{x \to b} f(x) = +\infty$. Then the set $f_1, f_2, \ldots, f_{n+1}$ cannot be unisolvent on (a, b).

Proof. Let the numbers c and d be chosen such that $a < c < d < b$. Then f_{n+1} is bounded on $[c, d]$ and the f_i's, $i = 1, 2, \ldots, n$ are bounded on $[a, b]$. Let M be a number which is greater than the absolute value of all of these upper and lower bounds. Since the expansion of $|f_i(x_j)|_{i,j=1}^n$ contains $n!$ terms, with n factor in each term, it follows that an upper bound for the absolute value of $|f_i(x_j)|_{i,j=1}^n$ is $n!\, M^n$. Choose $n + 1$ points such that $c \leqq x_1 < x_2 < \ldots < x_n < x_{n+1} \leqq d$ and consider $|f_i(x_j)|_{i,j=1}^{n+1}$. Denote the cofactor of $f_i(x_j)$ by $F_i(x_j)$ and we have

$$|f_i(x_j)|_{i,j=1}^{n+1} = f_{n+1}(x_1)\, F_{n+1}(x_1) + \ldots + f_{n+1}(x_{n+1})\, F_{n+1}(x_{n+1})$$

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If we hold $x_2, x_3, \ldots, x_{n+1}$ fixed and let x_1 tend toward a, the sign of $|f_i(x_j)|_{i,j=1}^{n+1}$ will be dominated by the sign of $f_{n+1}(x_1)\, F_{n+1}(x_1)$. To see this, note that (1) $F_{n+1}(x_1)$ is a constant, since it does not contain x_1, (2) $F_{n+1}(x_1)$ is not zero, since $f_1, \ldots, f_n$ are unisolvent and (3) $F_{n+1}(x_i)$, $i = 2, \ldots, n$ is bounded in absolute value by $n!\, M^n$.

On the other hand, if we hold $x_1, x_2, \ldots, x_n$ fixed and let x_{n+1} tend toward b, the sign of $|f_i(x_j)|_{i,j=1}^{n+1}$ is dominated by $f_{n+1}(x_{n+1})\, F_{n+1}(x_{n+1})$. Now $F_{n+1}(x_1)$ has the same sign as $F_{n+1}(x_{n+1})$ since the $f_1, f_2, \ldots, f_n$ are continuous and unisolvent and the deminsions of $|f_1(x_j)|_{i,j=1}^{n+1}$ is odd. Since the determinant $|f_i(x_j)|_{i,j=1}^{n+1}$ takes on values continuously, and $f_{n+1}(x_1)$ and $f_{n+1}(x_n)$ have opposite signs in the limits above, it follows that $|f_i(x_j)|_{i,j=1}^{n+1}$ must be zero for some value of x_1 or x_{n+1}. Therefore $f_1, \ldots$ $\ldots, f_{n+1}$ cannot be unisolvent on (a, b).

Theorem 2. *Let $f_1, f_2, \ldots, f_n$ be an odd number of continuous functions which are unisolvent on the closed interval $[a, b]$, Suppose that f_{n+1} is continuous on the open interval (a, b) with*

$$\lim_{x\to a} f_{n+1}(x) = \lim_{x\to b} f_{n+1}(x) = \infty\,.$$

Then $f_1, f_2, \ldots, f_{n+1}$ cannot be unisolvent on (a, b).

The proof is similar to that of Theorem 1.

Theorem 3. *Let $f_1, \ldots, f_n$ be n functions defined on a set S. If any $n - k$, $0 \leqq k \leqq$ $\leqq n - 2$, of these functions have common values for $k + 2$ points in S, then $f_1, \ldots, f_n$ are not unisolvent.*

Proof. We may assume that $f_1, \ldots, f_{n-k}$ have common values at the points $x_1, x_2, \ldots, x_{k+2}$. Choose any other $n - (k + 2)$ points of S and expand $|f_i(x_j)|_{i,j=1}^{n}$ by minors with respect to the last row. After k expansions, we have for a first term

$$f_n(x_1)\, f_{n-1}(x_2) \ldots f_{n-k+1}(x_k) \begin{vmatrix} f_1(x_{k+1}) & \ldots f_1(x_n) \\ \vdots & \\ f_{n-k}(x_{k+1}) & \ldots f_{n-k}(x_n) \end{vmatrix}.$$

But the first two columns of this determinant are identical. Thus this term is 0. A similar argument hold for each term. Therefore $f_1, \ldots, f_n$ are not unisolvent.

References

[1] *Davis, Philip J.:* Interpolation and Approximation, Blaisdell Publishing Company, New York, 1963.

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