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ON THE TORSION OF SPACES WITH CONNECTION

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Generally speaking, this paper is intended to show that some properties of a space with connection (of the first order) are naturally expressed in terms of the prolongation of its connection. (Some further results in this direction are outlined in [9].) EHRESMANN [6] has introduced higher order connections on a Lie groupoid and in the same situation he has defined the prolongation of a connection. This approach is really the most appropriate one from the conceptual view point, but it seems to be more convenient to replace Lie groupoids by principal fibre bundles in our investigations. An equivalent definition of higher order connections on a principal fibre bundle P was given by CENKL [1] who has used the higher order contact elements on P. Nevertheless, in the first part of this paper we present another equivalent definition of higher order connections on P, dealing with the elements of the corresponding prolongation of P. According to our opinion, this approach is convenient especially for the study of the development of a jet by means of an element of connection as well as for the study of the prolongation of a connection; it can be illustrated by Propositions 7, 8, 9.

In the second part of the paper, we show that the torsion form of a space with connection vanishes at a point if and only if the development of this space by means of the prolonged connection is holonomic. Further we treat a manifold with connection (see § 9), we define its reduced torsion form and we prove that this form vanishes at a point if and only if the contact element determined by the development of this manifold by means of the prolonged connection is holonomic. In these investigations, our main tool is the difference tensor of an arbitrary semi-holonomic 2-jet. We introduce this tensor in a formal way which is why we are often obliged to use direct computations in local coordinates even though the results are always invariant. We deduce that the curvature tensor of a connection of the first order coincides with the difference tensor of its prolongation. This assertion is very close to a theorem by Ehresmann [6]. To compare both points of view, we remark that our approach is formal; on the other hand, our consideration is direct, i.e. we do not use any auxiliary integrable connection. Then we show that the torsion tensor of a space with connection coincides in the main with the difference tensor of the development of this space by means of the prolonged connection and both above-mentioned results are simple consequences of this fact. Finally, we present a minor concrete example in § 10.

0. Our considerations are in the category C^∞. A fibered manifold E with base B and projection p is denoted by (E, p, B); E_x is the fibre over $x \in B$, $E_x = p^{-1}(x)$. $\tilde{J}^r(E, p, B)$ or $\bar{J}^r(E, p, B)$ or $J^r(E, p, B)$ means the r-th non-holonomic or semi-holonomic or holonomic prolongation of E respectively; we write also $\tilde{J}^r E$ or $\bar{J}^r E$ or $J^r E$ if there is no danger of confusion. The standard notation of the theory of jets is used throughout the paper, see [3]. In addition, j^r_s, $r < s$ denotes the canonical projection of s-jets into r-jets, so that $j^r_s X$ is the r-th part (or the underlying r-jet) of an s-jet X; naturally, $j^0_s X = \beta X$. Since the elements of $\tilde{J}^s E$ can be identified with some special elements of $\tilde{J}^s(B, E)$, the canonical projection of $\tilde{J}^s E$ into $\tilde{J}^r E$, $r < s$ will be denoted by the same symbol j^r_s. If $X \in \tilde{J}^s E$, $s > 1$ then we use already $j^0_s X$ for the canonical image of X in $J^0 E = E$ to avoid a possible obscurity whether βX means $j^{s-1}_s X$ or $j^0_s X$.

I

1. Let Φ be a Lie groupoid over B with projections a, b and let 1_x denote the unit of Φ over $x \in B$, see [10]. The partial composition law in Φ will be denoted by a dot so that $\theta' . \theta$ is defined for every $\theta', \theta \in \Phi$ satisfying $a(\theta') = b(\theta)$; then $b(\theta' . \theta) = b(\theta')$, $a(\theta' . \theta) = a(\theta)$. Let $i : \theta \mapsto \theta^{-1}$ be the inversion of Φ and let X be a non-holonomic r-jet of a manifold V into Φ, then we define $X^{-1} = iX$. We put $\Phi_x = a^{-1}(x)$ for every $x \in B$ and Φ_x will be considered as fibered manifold (Φ_x, b, B) unless otherwise stated.

Ehresmann [6] has introduced a (non-holonomic) element of connection of order r (shortly: an r-element of connection) on Φ at $x \in B$ as a jet $X \in \tilde{J}^r(B, \Phi)$ satisfying $\alpha X = x$, $\beta X = 1_x$, $bX = j^r_x (= j^r_x id_B)$, $aX = j^r_x \hat{x}$ where $\hat{x}$ means the constant mapping $\hat{x}(t) = x$ for every $t \in B$. It is easy to see (and we proved it in detail in [8]) that this definition is equivalent to the following one: *a* (non-holonomic) *r-element of connection on Φ at $x \in B$* is an element $X \in \tilde{J}^r_x(\Phi_x, b, B)$ such that $j^0_r X = 1_x$. An r-element of connection X at $x \in B$ is said to be semi-holonomic or holonomic if $X \in \bar{J}^r \Phi_x$ or $X \in J^r \Phi_x$ respectively. $\tilde{Q}^r(\Phi)$ or $\bar{Q}^r(\Phi)$ or $Q^r(\Phi)$ denotes the fibered manifold of all non-holonomic or semi-holonomic or holonomic r-elements of connection on Φ. A non-holonomic or semi-holonomic or holonomic connection of order r (shortly: an *r-connection*) on Φ is a global cross section $C : B \to \tilde{Q}(\Phi)$ or $C : B \to \bar{Q}^r(\Phi)$ or $C : B \to Q^r(\Phi)$ respectively.

Suppose Φ is a groupoid of operators on a fibered manifold (E, p, B), see [5]. The action of Φ on E will be denoted by a dot so that $\theta . z$ is defined for every $\theta \in \Phi$ and $z \in E$ satisfying $a(\theta) = p(z)$; then $\theta . z \in E$ and $p(\theta . z) = b(\theta)$. In particular, if

$\theta \in \Phi_x$, then $\theta^{-1} . z$ belongs to E_x for every $z \in E$ such that the product $\theta^{-1} . z$ is defined. Let $x \in B$, let V be a manifold, $Z \in \tilde{J}^r_v(V, E)$, $p(\beta Z) = x$, and let $X \in \tilde{J}^r_x(\Phi_x, b, B)$, then the prolongation of the partial composition law $(\theta, z) \mapsto \theta . z$ defines

(1) $$X^{-1}(Z) = (X^{-1}pZ) . Z \in \tilde{J}^r_v(V, E_x),$$

cf. [6]. If $X \in \tilde{Q}^r_x(\Phi)$, then (1) will be called the *development of Z into E_x by means of X*, cf. [9]. (Note that Ehresmann has used the term "the absolute differential of Z with respect to X" for (1).) In particular, if σ is a local cross section of (E, p, B), then we shall write only $X^{-1}(\sigma)$ instead of $X^{-1}(j^r_x\sigma)$ and $X^{-1}(\sigma)$ will be said to be the *development of σ into E_x by means of X*. – We shall rewrite (1) in a more detailed form. Let $Z = j^r_v\, \zeta(y)$ where $\zeta(y)$ is a local cross section of $(\tilde{J}^{r-1}(V, E), \alpha, V)$ and let $X = j^1_x\, \psi(t)$ where $\psi(t)$ is a local cross section of $\tilde{J}^{r-1}(\Phi_x, b, B)$, then

(2) $$X^{-1}(Z) = j^1_v[\psi^{-1}(p(\beta\, \zeta(y)))\, p\, \zeta(y) . \zeta(y)] \quad \text{for} \quad r > 1,$$

(3) $$X^{-1}(Z) = j^1_v[\psi^{-1}(p\, \zeta(y)) . \zeta(y)] \quad \text{for} \quad r = 1.$$

2. Let $N = \bigcup_{x \in B} \tilde{J}^r(\Phi_x, b, B)$, which is a fibered manifold (N, aj^0_r, B). Φ acts (on the right) on N in the following way. Let $\theta \in \Phi$, $b(\theta) = x$, $a(\theta) = t$, then θ determines a mapping of Φ_x onto Φ_t, $\xi \mapsto \xi . \theta$, $\xi \in \Phi_x$. Now, let $X \in \tilde{J}^r(\Phi_x, b, B)$, then $X . \theta$ will be the image of X by this mapping; $X . \theta \in \tilde{J}^r(\Phi_t, b, B)$.

Let $C : B \to \tilde{Q}^r(\Phi)$ be an r-connection on Φ, then $C_1 = j^1_r C$ is a 1-connection on Φ. Consider a point $x \in B$, then $C_1(x) = j^1_x\, \varrho(t)$ where ϱ is a local cross section of (Φ_x, b, B), $\varrho(x) = 1_x$. We define

(4) $$C'(x) = j^1_x[C(t) . \varrho(t)].$$

Since $C(t) . \varrho(t)$ is a local cross section of $\tilde{J}^r(\Phi_x, b, B)$, it is $C'(x) \in \tilde{J}^{r+1}_x(\Phi_x, b, B)$; further we have $j^0_{r+1}\, C'(x) = 1_x$ so that $C'(x) \in \tilde{Q}^{r+1}_x(\Phi)$. Thus, $C' : B \to \tilde{Q}^{r+1}(\Phi)$ is an $(r + 1)$-connection on Φ, which is called the *prolongation of C*, cf. [6]. The k-th prolongation $C^{(k)}$ of C is defined by iteration $C^{(k)} = C^{(k-1)\prime}$

Having a fibered manifold (E, p, B) and a manifold V, we define

(5) $$\tilde{F}^r(V, (E, p, B)) = \bigcup_{x \in B} \tilde{J}^r(V, E_x);$$

we shall also write $\tilde{F}^r(V, E)$ if there is no danger of confusion. In the semi-holonomic or holonomic case, we analogously define $\bar{F}^r(V, (E, p, B))$ or $F^r(V, (E, p, B))$ respectively. Let Φ be a groupoid of operators on E and let $\theta \in \Phi$, $a(\theta) = x$, $b(\theta) = t$. Let $S \in \tilde{J}^r(V, E_x)$ and let $\theta . S$ be the image of S by the mapping $\theta : E_x \to E_t$, then Φ becomes a groupoid of operators (on the left) on $(\tilde{F}^r(V, E), p\beta, B)$. Further, let $Z \in \tilde{J}^{r+1}_v(V, E)$, $p(\beta Z) = x$, then the development of Z by means of an element of the prolonged connection $C'^{-1}(x)\,(Z) \in \tilde{J}^{r+1}_v(V, E_x)$ can be also described in the following manner. It is $Z = j^1_v\, \zeta(y)$ where $\zeta(y)$ is a local cross section of V into $\tilde{J}^r(V, E)$. Developing each jet $\zeta(y)$ by means of the corresponding element of C, we get a local cross section $\zeta^{(r)}$ of V into $\bar{F}^r(V, E)$, $\zeta^{(r)}(y) = C^{-1}(p(\beta\,\zeta(y)))\,(\zeta(y))$.

Proposition 1. *It holds*

$$C_1^{-1}(x)\,(j_v^1\,\zeta^{(r)}) = C'^{-1}(x)\,(Z)\,. \tag{6}$$

Proof. It is $C_1(x) = j_x^1\,\varrho(t)$, $C'^{-1}(x) = j_x^1(C(t)\,.\,\varrho(t))$, so that $C'^{-1}(x)\,(Z) = j_v^1[((\varrho^{-1}(p(\beta\,\zeta(y)))\,.\,C^{-1}(p(\beta\,\zeta(y)))\,p(\zeta(y)))\,.\,\zeta(y)] = j_v^1[\varrho^{-1}(p(\beta\,\zeta(y)))\,.\,C^{-1}(p\beta\,\zeta(y)))\,p(\zeta(y))\,.\,\zeta(y)] = j_v^1[\varrho^{-1}(p(\beta\,\zeta^{(r)}(y)))\,.\,\zeta^{(r)}(y)] = C_1^{-1}(x)\,(j_v^1\zeta^{(r)})$, QED.

Proposition 1 gives a very instructive description of the development by means of a prolonged connection when one develops a local cross section σ of (E, p, B), i.e. $Z = j_x^{r+1}\,\sigma(t)$. Then $\sigma^{(r)}(t) = C^{-1}(t)\,(\sigma)$ is a local cross section of $(\bar{F}^r(B, E), p\beta, B)$ and the development of σ into E_x by means of $C'^{-1}(x)$ coincides with the development of $\sigma^{(r)}$ by means of the element of the underlying 1-connection $C_1(x)$. In particular, let C be a 1-connection on Φ and consider the sequence $C, C', \ldots, C^{(r)}, \ldots$ of its prolongations; C will be called the *fundamental connection of this sequence.* If σ is a local cross section of (E, p, B), then the local cross section $\sigma^{(r)}(t) = C^{(r-1)-1}(t)\,(\sigma)$ of $\bar{F}^r(B, (E, p, B))$ can be called the *r-th development of* σ. We have

Corollary 1. *The* $(r + 1)$*-th development* $C^{(r)-1}(t)\,(\sigma)$ *of a local cross section* σ *of* (E, p, B) *coincides with the development of the r-th development* $C^{(r-1)-1}(t)\,(\sigma)$ *by means of the fundamental connection C.*

3. We define a (non-holonomic) *distribution* δ *of order* r (briefly: an r-distribution) *on a fibered manifold* (E, p, B) as a global cross section of fibered manifold $(\tilde{J}^rE, j_r^0, E)$, so that δ assigns to every $z \in E$ an element $\delta(z) \in \tilde{J}^rE$ such that $j_r^0\,\delta(z) = z$. If $\delta(E) \subset \bar{J}^rE$ or $\delta(E) \subset J^rE$, then δ is said to be semi-holonomic or holonomic respectively. The *prolongation* δ' *of an r-distribution* δ can be introduced as follows. It is $j_r^1\,\delta(z) \in J^1E$, so that $j_r^1\,\delta(z) = j_x^1\,\gamma(t)$ where $\gamma(t)$ is a local cross section of (E, p, B); then we define

$$\delta'(z) = j_x^1\delta(\gamma(t))\,. \tag{7}$$

As $\delta(\gamma(t))$ is a local cross section of $(\tilde{J}^rE, \alpha, B)$, it is $\delta'(z) \in \tilde{J}^{r+1}E$. Thus, δ' is an $(r + 1)$-distribution on E. The s-th prolongation $\delta^{(s)}$ of δ is defined by iteration $\delta^{(s)} = \delta^{(s-1)\prime}$.

Proposition 2. *Let* δ *be a semi-holonomic r-distribution on E. If* δ' *is also semi-holonomic, then* $\delta^{(s)}$ *is semi-holonomic for every s.*

Proof. Consider an element $z \in E$, $p(z) = x$. Let $j_r^1\,\delta(z) = j_x^1\,\gamma(t)$, then $\delta'(z) = j_x^1\,\delta(\gamma(t))$ and $\delta''(z) = j_x^1\,\delta'(\gamma(t))$. If δ' is semi-holonomic, then $\delta'(\gamma(t))$ is a local cross section of $(\bar{J}^{r+1}E, \alpha, B)$ and it holds $j_x^1 j_{r+1}^r\,\delta'(\gamma(t)) = j_x^1\,\delta(\gamma(t)) = \delta'(z)$ which proves that $\delta''(z) \in \bar{J}^{r+2}E$. Thus, δ'' is semi-holonomic and by iterated application of this result we deduce that $\delta^{(s)}$ is semi-holonomic for every s.

Proposition 3. *Let δ be a distribution of the first order on (E, p, B), then $\delta^{(s)}$ is semi-holonomic for every s.*

Proof. We have only to prove that δ' is semi-holonomic. If $\delta(z) = j_x^1 \gamma(t)$, then $\delta'(z) = j_x^1 \delta(\gamma(t))$ and $j_x^1 \beta\, \delta(\gamma(t)) = j_x^1 \gamma(t) = \delta(z)$, which implies $\delta'(z) \in \bar{J}^2E$.

4. Let $P(B, G)$ be a principal fibre bundle with base B and structure group G and let $\pi : P \to B$ be its projection. Suppose a left action of G on a manifold F is given, $(g, a) \mapsto ga$, $g \in G$, $a \in F$. Consider the associated fibre bundle $E(B, F, G, P)$ with standard fibre F and denote its projection by p, $p : E \to B$. Each element $z \in E$ is an equivalence class $\{(u, a)\}$ with respect to the equivalence relation $(ug^{-1}, ga) \sim (u, a)$, $g \in G$. Every $u \in P$ can be considered as a diffeomorphism $u : F \to E_x$ defined by $u(a) = \{(u, a)\}$, $x = \pi(u)$. But u can also be interpreted as the inverse diffeomorphism $u^{-1} : E_x \to F$. Under this interpretation, we shall write P^{-1} instead of P and $u^{-1} . z$ instead of $u^{-1}(z)$. Further, let Φ be the groupoid associated with P, $\Phi = PP^{-1}$, see [6], [10], then Φ is a groupoid of operators on E as follows. If $\theta = u'u^{-1}$ and $z \in E$, then $\theta . z = u'(u^{-1} . z)$. In particular, Φ is a groupoid of operators on P by $\theta . u = (u'u^{-1}) . u = u'$.

Let $Y \in \bar{J}^r(P, \pi, B)$, $\alpha Y = x$, $j_r^0 Y = u$. Denote by Yu^{-1} the image of Y by the mapping $u^{-1} : P \to \Phi_x$, $u' \mapsto u'u^{-1}$, then Yu^{-1} is an element of $\bar{J}_x^r(\Phi_x, b, B)$ such that $j_r^0 Yu^{-1} = 1_x$, i.e. $Yu^{-1} \in \tilde{Q}_x^r(\Phi)$. Further, it holds $(Yg)(ug)^{-1} = Yu^{-1}$, $g \in G$ where Yg means the image of jet Y by the mapping $g : P \to P$. Conversely, let $X \in \tilde{Q}_x^r(\Phi)$ and let $X . u$ denote the image of X by the mapping $u : \Phi_x \to P$, $\theta \mapsto \theta . u$, then $X . u \in \bar{J}_x^r(P, \pi, B)$, $j_r^0 X . u = u$. We shall say that $X . u$ is the *representant of $X \in \tilde{Q}_x^r(\Phi)$ at $u \in P$*. Obviously, it holds $X . (ug) = (X . u)\, g$.

Let φ be a mapping of P_x into $\bar{J}_x^r(P, \pi, B)$ such that $j_r^0\, \varphi(u) = u$ for every $u \in P_x$ and such that $\varphi(ug) = \varphi(u)\, g$ for every $u \in P$, $g \in G$. Then $\varphi(P_x)$ will be called an *invariant system of elements of $\bar{J}^rP$ along P_x*. We have deduced.

Proposition 4. *An element of connection $X \in \tilde{Q}_x^r(\Phi)$ is equivalent to an invariant system of elements of $\bar{J}^rP$ along P_x.*

Further, let $Y \in \bar{J}^rP$, $\alpha Y = x$. If we consider P as P^{-1}, then we write Y^{-1} instead of Y. Let $Z \in \bar{J}_v^r(V, E)$, $p(\beta Z) = x$, then the prolongation of the partial composition law $(u, z) \mapsto u^{-1} . z$ determines the product

$$Y^{-1}(Z) = (Y^{-1}pZ) . Z \in \bar{J}_v^r(V, F) \tag{8}$$

which will be called the *development of Z into F by means of Y*. In particular, if σ is a local cross section of (E, p, B), then we shall write only $Y^{-1}(\sigma)$ instead of $Y^{-1}(j_x^r\sigma)$

and $Y^{-1}(\sigma)$ will be said to be the *development of σ into F by means of Y.* Now, we can state two obvious propositions.

Proposition 5. *Let Y_1, Y_2 be two representants of the same element of connection, $Y_1 = Y_2 g$, then it is $g(Y_1^{-1}(Z)) = Y_2^{-1}(Z)$ where $g(Y_1^{-1}(Z))$ means the image of jet $Y_1^{-1}(Z)$ by the mapping $g : F \to F$.*

Proposition 6. *Let $X \in \tilde{Q}_x^r(\Phi)$, $Z \in \tilde{J}^r(V, E)$, $u \in P_x$ and let $Y = X \,.\, u$, then it holds $u(Y^{-1}(Z)) = X^{-1}(Z)$.*

We underline that $X^{-1}(Z)$ is an r-jet of V into E_x, while $Y^{-1}(Z)$ is an r-jet of V into the standard fibre F.

An r-distribution δ on principal fibre bundle P is said to be *invariant*, if its restriction to every fibre is an invariant system. Let $C : B \to \tilde{Q}^r(\Phi)$ be an r-connection on Φ, then the set of all representants of the elements of C is an invariant r-distribution on P which will be called the *representant of connection C*. Conversely, every invariant r-distribution on P represents a connection on Φ which is why we may also say that an invariant r-distribution $\Gamma : P \to \tilde{J}^r P$ is an r-connection on P.

Proposition 7. *If an r-distribution Γ on P represents an r-connection C on Φ, then the prolongation Γ' of Γ represents the prolongation C' of C.*

Proof. Let $u \in P_x$, $\Gamma(u) = C(x) \,.\, u$, then $j_r^1\, \Gamma(u) = (j_r^1\, C(x)) \,.\, u$. If $j_r^1\, C(x) = j_x^1\, \varrho(t)$, then $j_r^1\, \Gamma(u) = j_x^1(\varrho(t) \,.\, u)$ and it holds $C'(x) \,.\, u = j_x^1[C(t) \,.\, \varrho(t)] \,.\, u = j_x^1[C(t) \,.\, (\varrho(t) \,.\, u)] = j_x^1 \Gamma(\varrho(t) \,.\, u) = \Gamma'(u)$, QED.

From Propositions 2 and 7 we obtain directly

Proposition 8. *Let C be a semi-holonomic r-connection on Φ. If C' is also semi-holonomic, then $C^{(s)}$ is semi-holonomic for every s.*

From Propositions 3 and 7 one can deduce again the well-known result by Ehresmann [6] that all prolongations of a connection of the first order are semi-holonomic.

The following assertion is a direct analogy of Proposition 1.

Proposition 9. *Let $\Gamma : P \to \tilde{J}^r P$ be an r-connection on P and let $j_r^1\, \Gamma(u) = j_x^1\, \gamma(t)$ where $\gamma(t)$ is a local cross section of P. Further, let $Z \in \tilde{J}_v^{r+1}(V, E)$, $p(\beta Z) = x$, $Z = j_v^1\, \zeta(y)$ where $\zeta(y)$ is a local cross section of $\tilde{J}^r(V, E)$. If we develop every $\zeta(y)$ by means of the corresponding element of Γ along the local cross section γ, then we get a local cross section of $\tilde{J}^r(V, F)$ which determines $\Gamma'^{-1}(u)\,(Z)$, i.e.*

$$\Gamma'^{-1}(u)\,(Z) = j_v^1[\Gamma^{-1}(\gamma(p(\beta\, \zeta(y))))\,(\zeta(y))] \in \tilde{J}_v^{r+1}(V, F)\,. \tag{9}$$

Proof. Let Γ represent an r-connection C on Φ, then it is $u(\Gamma'^{-1}(u)\,(Z)) = C'^{-1}(x)\,(Z)$ by Propositions 6 and 7. It is evident that $j_r^1 \Gamma$ represents $j_r^1 C$, so that

$j^1_r\, C(x) = j^1_x\, \varrho(t)$ where $\varrho(t) = \gamma(t)\, u^{-1}$. Then $u = \varrho^{-1}(t) \,.\, \gamma(t)$ and we further get (the second or third equality is based on Proposition 6 or 1 respectively)

$$u j^1_v [\Gamma^{-1}(\gamma(p(\beta\, \zeta(y))))\, (\zeta(y))] =$$
$$= j^1_v \varrho^{-1}(p(\beta\, \zeta(y))) \,.\, \gamma(p(\beta\, \zeta(y)))\, [\Gamma^{-1}(\gamma(p(\beta\, \zeta(y))))\, (\zeta(y))] =$$
$$= j^1_v \varrho^{-1}(p(\beta\, \zeta(y))) \,.\, C^{-1}(p(\beta\, \zeta(y)))\, (\zeta(y)) = C'^{-1}(x)\, (Z)\,,$$

which is equivalent to our assertion.

In what follows we shall use the following special case of Proposition 9.

Corollary 2. *Let Γ be a connection of the first order on P and let $\gamma(t)$ be a local cross section of P such that $\Gamma(u) = j^1_x\, \gamma(t)$. Let σ be a local cross section of E, then $\Gamma^{-1}(\gamma(t))\, (\sigma)$ is a local cross section of $J^1(B, F)$ which determines $\Gamma'^{-1}(u)\, (\sigma)$, i.e.*

(10) $$\Gamma'^{-1}(u)\, (\sigma) = j^1_x[\Gamma^{-1}(\gamma(t))\, (\sigma)] \in \bar{J}^2_x(B, F)\,.$$

II

5. Consider the set $\bar{L}^2_{n,m}$ of all semi-holonomic 2-jets of $\mathbf{R}^m$ into $\mathbf{R}^n$ with source 0 and target 0. The usual coordinates in $\bar{L}^2_{n,m}$, [4], determine every element $X \in \bar{L}^2_{n,m}$ by means of real numbers x^r_i, x^r_{ij}

(11) $$X = (x^r_i, x^r_{ij})\,, \quad r, s = 1, \ldots, n\,, \quad i, j = 1, \ldots, m\,,$$

and X is holonomic if and only if $x^r_{ij} = x^r_{ji}$. Further, let $Y \in \bar{L}^2_{p,n}$, $Y = (y^\xi_r, y^\xi_{rs})$, then the product $Z = YX$ has the following coordinates, [4],

(12) $$Z = (y^\xi_r x^r_i,\; y^\xi_{rs} x^r_i x^s_j + y^\xi_r x^r_{ij})\,, \quad \xi = 1, \ldots, p\,.$$

Proposition 10. *Let V, W be two manifolds and let $X \in \bar{J}^2(V, W)$, $\alpha X = v$, $\beta X = w$. Let h_1 or h_2 be a holonomic 2-frame on V or W at v or w respectively and let x^r_i, x^r_{ij} be the coordinates of X in these frames, i.e. the coordinates of $h_2^{-1} X h_1 \in \bar{L}^2_{n,m}$. Then*

(13) $$x^r_{[ij]} = \tfrac{1}{2}(x^r_{ij} - x^r_{ji})$$

are coordinates of a tensor.

Proof. Let $A \in L^2_n$, $B \in L^2_m$, $A = (a^{r'}_r, a^{r'}_{rs})$, $B = (b^i_{i'}, b^i_{i'j'})$, $a^{r'}_{[rs]} = 0$, $b^i_{[i'j']} = 0$ and let $A h_2^{-1} X h_1 B = (x^{r'}_{i'}, x^{r'}_{i'j'})$. Using (12), one finds easily $x^{r'}_{[i'j']} = a^{r'}_r x^r_{[ij]} b^i_{i'} b^j_{j'}$, QED.

Tensor (13) will be called the *difference tensor of semiholonomic 2-jet X* and will be denoted by $\Delta(X)$. It is $\Delta(X) \in T_w(W) \otimes \bigwedge^2 T^*_v(V)$ and $\Delta(X) = 0$ *if and only if X is holonomic.*

Suppose $m < n$. Let X be regular and let $T(X)$ be the m-dimensional subspace of $T_w(W)$ determined by $j^1_2 X$. Let $\psi : T_w(W) \to T_w(W)/T(X)$ be the canonical projection, then $\psi(\Delta(X)) \in (T_w(W)/T(X)) \otimes \bigwedge^2 T_v^*(V)$ will be called the *reduced difference tensor* of X. On the other hand, the *contact element* $k(X)$ determined by X is the set $Xh\bar{L}_m^2$ where h is a 2-frame at $v \in V$. The contact element $k(X)$ will be called *holonomic* if it contains a holonomic jet, cf. [9].

Proposition 11. *Let X be a regular semi-holonomic 2-jet, then its reduced difference tensor vanishes if and only if $k(X)$ is holonomic.*

Proof. Let $\varkappa = m + 1, \ldots, n$. One can choose such coordinates in $T_w(W)$ that $T(X)$ satisfies $x^\varkappa = 0$, i.e. $X = (x_j^i, 0; x_{ij}^r)$, $|x_i^j| \neq 0$. Suppose $x_{[ij]}^\varkappa = 0$. Consider the jet $X_0 = (x_j^i, x_{jk}^i) \in \bar{L}_m^2$ and let $X_0^{-1} = (a_j^i, a_{jk}^i)$ be its inverse. For $Y = XX_0^{-1} = (y_j^r, y_{jk}^r)$ we obtain $y_{ij}^k = 0$, $y_{[ij]}^\varkappa = x_{[ki]} a_i^k a_j^l = 0$, so that Y is holonomic. The converse assertion can be proved by the converse computation.

6. Let M be a parallelizable manifold and let

$$(14) \qquad \omega_0^\alpha \quad \alpha, \beta, \ldots = 1, \ldots, r = \dim M\,,$$

be a basis of $T^*(M)$. Consider the trivial fibered manifold $E = M \times \mathbf{R}^m$ with base $\mathbf{R}^m$; the elements of $\mathbf{R}^m$ will be denoted by $(t^1, \ldots, t^m)$. Then

$$(15) \qquad \omega^\alpha = pr_1^* \omega_0^\alpha\,, \quad \mathrm{d}t^i = pr_2^*\, \mathrm{d}t^i \quad \text{(no danger of confusion)}$$

is a basis of $T^*(E)$. Every element $Y \in J^1E$, $\beta Y = z$ can be identified with a subspace of $T_z(E)$ determined by

$$(16) \qquad (\omega^\alpha)_z = A_i^\alpha (\mathrm{d}t^i)_z\,.$$

Thus, A_i^α are some functions on J^1E which introduce coordinates on J^1E in the following sense. Every $Y \in J^1E$ is uniquely determined by the point $\beta Y \in E$ and by real numbers $A_i^\alpha(Y)$ given by (16).

Let $X \in \bar{J}^2E$; $X = j_0^1 \varrho$ where ϱ is a local cross section of J^1E. The local cross section ϱ is uniquely determined by the local cross section $\beta\varrho$ of E and by the functions $A_i^\alpha(\varrho(t))$. We define $A_{ij}^\alpha(X)$ by

$$(17) \qquad \mathrm{d}A_i^\alpha(\varrho(0)) = A_{ij}^\alpha(X)\, \mathrm{d}t^j$$

and we put $A_i^\alpha(X) = A_i^\alpha(\varrho(0))$.

Now, we shall express the difference tensor $\Delta(X) \in T_z(E) \otimes \bigwedge^2 T_0^*(\mathbf{R}^m)$, $z = j_2^0 X$, by means of $A_i^\alpha(X)$ and $A_{ij}^\alpha(X)$. Choose some coordinates z^α in a coordinate neighbourhood U of the point $pr_1 z \in M$ and consider the coordinates z^α, t^i on $U \times \mathbf{R}^m$. In this coordinate system, the element $Y \in \bar{J}^2(U \times \mathbf{R}^m)$ has some coordinates $a_i^\alpha(Y)$, $a_{ij}^\alpha(Y)$ where a_i^α or a_{ij}^α can be considered as some functions on $\bar{J}^2(U \times \mathbf{R}^m)$ determined by $\mathrm{d}z^\alpha = a_i^\alpha\, \mathrm{d}t^i$ analogously to (16) or by $\mathrm{d}a_i^\alpha = a_{ij}^\alpha\, \mathrm{d}t^j$ analogously to (17) respectively. Since A_i^α, A_{ij}^α are given by $\omega^\alpha = A_i^\alpha\, \mathrm{d}t^i$, $\mathrm{d}A_i^\alpha = A_{ij}^\alpha\, \mathrm{d}t^j$, we can find the transformation formulae between both coordinate systems. Let $\omega^\alpha = B_\beta^\alpha\, \mathrm{d}z^\beta$, $\mathrm{d}B_\beta^\alpha = B_{\beta\gamma}^\alpha\, \mathrm{d}z^\gamma$ and let $\tilde{B}_\beta^\alpha B_\gamma^\beta = \delta_\gamma^\alpha$, then $a_i^\alpha = \tilde{B}_\beta^\alpha A_i^\beta$ and the differentiation gives $a_{ij}^\alpha\, \mathrm{d}t^j = \mathrm{d}a_i^\alpha =$

$= \mathrm{d}\tilde{B}^\alpha_\beta A^\beta_i + \tilde{B}^\alpha_\beta \,\mathrm{d}A^\beta_i$. Further, a standard computation yields $a^\alpha_{ij} = -\tilde{B}^\alpha_\beta B^\beta_{\gamma\delta} a^\gamma_i a^\delta_j + \tilde{B}^\alpha_\beta A^\beta_{ij}$. Now, let $K^\alpha_{\beta\gamma} = -K^\alpha_{\gamma\beta}$ be the functions determined by

$$\mathrm{d}\omega^\alpha = K^\alpha_{\beta\gamma}\omega^\beta \wedge \omega^\gamma \,, \tag{18}$$

then $K^\alpha_{\beta\gamma} = -B^\alpha_{[\delta\varepsilon]}\tilde{B}^\delta_\beta \tilde{B}^\varepsilon_\gamma$ and we get finally $a^\alpha_{[ij]} = \tilde{B}^\alpha_\delta K^\delta_{\beta\gamma} A^\beta_i A^\gamma_j + \tilde{B}^\alpha_\beta A^\beta_{ij}$. Thus, the coordinates of $\Delta(X)$ in the basis $(\mathrm{d}t^i)_0$ and in the basis dual to $(\omega^\alpha)_z$, $(\mathrm{d}t^i)_z$ are

$$A^\alpha_{[ij]}(X) + K^\alpha_{\beta\gamma} A^\beta_i(X)\, A^\gamma_i(X) \,. \tag{19}$$

7. Consider a principal fibre bundle $P(B, G)$ with projection π, $\dim G = r$, and denote by $\mathfrak{g}$ the Lie algebra of G. Let $\Gamma : P \to J^1P$ be a connection of the first order on P and let Ω be its curvature form, see [7]. Since Ω is a $\mathfrak{g}$-valued tensorial 2-form, $(\Omega)_u$ can be considered as a tensor of $\mathfrak{g} \otimes \bigwedge^2 T^*_x(B)$, $u \in P$, $x = \pi(u)$. Then the relation between the difference tensor of the prolonged connection $\Delta(\Gamma'(u)) \in T_u(P_x) \otimes \bigwedge^2 T^*_x(B)$ and $(\Omega)_u$ is described by

Proposition 12. *The difference tensor $\Delta(\Gamma'(u))$ coincides with the image of the curvature tensor $(\Omega)_u$ by the canonical mapping $u : G \to P_x$, $g \mapsto ug$.*

Proof. Since our problem is local, we may suppose that we consider the trivial fibre bundle $P = G \times \mathbf{R}^m$ in a neighbourhood of $0 \in \mathbf{R}^m$. Let e_α be a basis of $\mathfrak{g}$ and let ω^α_0 be the dual basis, then

$$\mathrm{d}\omega^\alpha_0 = \tfrac{1}{2}c^\alpha_{\beta\gamma}\omega^\beta_0 \wedge \omega^\gamma_0 \,. \tag{20}$$

Analogously to (15), we set

$$\omega^\alpha = pr^*_1\omega^\alpha_0 \,, \quad \mathrm{d}t^i = pr^*_2 \,\mathrm{d}t^i \tag{21}$$

and E_α, T_i will denote the dual basis to (21). It is easy to see that E_α is the fundamental vector field on P corresponding to $e_\alpha \in \mathfrak{g}$, see [7], p. 51. According to § 6, an arbitrary 1-distribution on P is given by $\omega^\alpha = \Gamma^\alpha_i \,\mathrm{d}t^i$ where Γ^α_i are some functions on P. This 1-distribution is invariant if and only if Γ^α_i are constant along each fibre. Thus, consider a connection Γ on P given by

$$\omega^\alpha = \Gamma^\alpha_i(t)\,\mathrm{d}t^i \,. \tag{22}$$

To construct the fundamental $\mathfrak{g}$-valued form ω of Γ, we must take the vertical component vX of a vector $X \in T(P)$ and then find that element of $\mathfrak{g}$ whose fundamental vector field contains vX. This implies that the fundamental form ω of (22) is

$$\omega = (\omega^\alpha - \Gamma^\alpha_i(t)\,\mathrm{d}t^i) \otimes e_\alpha \tag{23}$$

and its curvature form Ω is given by

$$\Omega = D\omega = (\tfrac{1}{2}c^\alpha_{\beta\gamma}\Gamma^\beta_i\Gamma^\gamma_j + \partial_{[j}\Gamma^\alpha_{i]})\,\mathrm{d}t^i \wedge \mathrm{d}t^j \otimes e_\alpha \,. \tag{24}$$

On the other hand, the difference tensor $\Delta(\Gamma'(u))$ can be constructed by Proposition 7 as follows. Let $\gamma(t)$, $\gamma(0) = u$ be a local cross section of P such that $j^1_0\,\gamma(t) =$

$= \Gamma(u)$, then $\Gamma'(u) = j_0^1 \Gamma(\gamma(t))$. Since Γ_i^α depends only on t, it is $\mathrm{d}\Gamma_i^\alpha(\gamma(t)) = \partial_j \Gamma_i^\alpha(t)\, \mathrm{d}t^j$. Applying (19) we obtain

$$\Delta(\Gamma'(u)) = (\partial_{[j}\Gamma^\alpha_{i]} + \tfrac{1}{2}c^\alpha_{\beta\gamma}\Gamma^\beta_i\Gamma^\gamma_j)\, \mathrm{d}t^i \wedge \mathrm{d}t^j \otimes E_\alpha\,. \tag{25}$$

Comparing (24) and (25) we get Proposition 12.

8. Let H be a closed subgroup of G, let $F = G/H$ be the corresponding homogeneous space and let c be its natural centre which corresponds to subgroup H, $c = \{H\}$. Consider the associated fibre bundle $E(B, F, G, P)$ with standard fibre F, then the groupoid $\Phi = PP^{-1}$ associated with P is a groupoid of operators on E. A *space with connection (of the first order)* is a quintuple $\mathcal{S} = \mathcal{S}(B, \Phi, E, \sigma, C)$ where σ is a global cross section of E and C is a connection of the first order on Φ. If Γ is the representant of C on P, then $\mathcal{S}$ can also be considered as the quintuple $\mathcal{S}(B, P, E, \sigma, \Gamma)$. Cross section σ determines a reduction R of principal fibre bundle P to $H \subset G$ by

$$R = \{u \in P;\; u(c) = \sigma(\pi(u))\}\,, \tag{26}$$

the converse assertion being also true. Thus, the quintuple $\mathcal{S}(B, P, E, \sigma, \Gamma)$ is equivalent to $\mathcal{S}(B, P, H, R, \Gamma)$, which shows that our definition of a space with connection is equivalent to the corresponding definition by Švec, [12].

Let $\mu_0 : G \to F = G/H$ be the canonical projection and let $\mu_{0*} : \mathfrak{g} \to T_c(F) = \mathfrak{g}/\mathfrak{h}$ be its differential. Švec [12] defined the torsion form of $\mathcal{S}$ at $u \in R$ as the projection

$$\mu_{0*}(\Omega)_u \tag{27}$$

of the curvature form. Since (27) is a tensorial form, it can be considered as an element of $T_c(F) \otimes \bigwedge^2 T_x^*(B)$, $x = \pi(u)$. We shall use the following equivalent form of this concept. It is easy to see that $u_*\, \mu_{0*}(\Omega)_u$ is a 2-form with values in $T_{\sigma(x)}(E_x)$ which does not depend on the selection of $u \in R_x$. The corresponding tensor $\tau(x) \in T_{\sigma(x)}(E_x) \otimes \bigwedge^2 T_x^*(B)$ will be called the *torsion tensor of $\mathcal{S}$ at $x \in B$*. On the other hand, consider the development of σ by means of the prolonged connection $C'^{-1}(x)(\sigma) \in \bar{J}_x^2(B, E_x)$, $\beta C'^{-1}(x)(\sigma) = \sigma(x)$, which can be called the *second development of $\mathcal{S}$ at $x \in B$*, cf. § 2.

Proposition 13. *For every $x \in B$, it holds*

$$-\tau(x) = \Delta(C'^{-1}(x)(\sigma))\,.$$

Proof. First of all, we deduce a lemma. Let e be the unit of G. The associated fibre bundle $\tilde{P}(B, G, G, P)$ is canonically identified with P if $\{(u, e)\}$ is identified with u. Further, let $\bar{e}_\alpha$ be a vector field on G defined by $(\bar{e}_\alpha)_g = R_{g*}(e_\alpha)_e$ where R_g is the right translation determined by $g \in G$. In particular,

$$(\bar{e}_\alpha)_e = (e_\alpha)_e\,. \tag{28}$$

In what follows, we shall also use the fact that every 1-jet X of V into W, $\alpha X = v$, $\beta X = w$ is canonically identified with a tensor of $T_w(W) \otimes T_v^*(V)$.

Consider the trivial fibre bundle $P = G \times \mathbf{R}^m$. Let $u = (g, t) \in P$ and let $(\tilde{g}, t) \in \tilde{P} = P$, then $u^{-1}(\tilde{g}, t) = g^{-1}\tilde{g} \in G$. Further, consider on P the forms (21) and a connection Γ (22).

Lemma. *Let ε be the cross section of $\tilde{P}$ given by $\varepsilon(t) = (e, t)$. Let $(g, x) \in P$, then the development $\Gamma^{-1}(g, x)(\varepsilon) \in J_x^1(\mathbf{R}^m, G)$ is determined by the tensor*

$$-\Gamma_i^\alpha(x)\,\mathrm{d}t^i \otimes (\bar{e}_\alpha)_{g^{-1}} \in T_{g^{-1}}(G) \otimes T_x^*(\mathbf{R}^m)\,. \tag{29}$$

Proof of Lemma. Let $\Gamma(g, x) = j_x^1\,\varrho(t)$, $pr_1\,\varrho(t) = \gamma(t)$, $\gamma(x) = g$, then $j_x^1 e\,\gamma(t) \in J_x^1(\mathbf{R}^m, G)$ is determined by the tensor $\Gamma_i^\alpha(x)\,\mathrm{d}t^i \otimes (e_\alpha)_g$. Further we have $\Gamma^{-1}(g, x)(\varepsilon) = j_x^1\,\varrho^{-1}(t)\,(\varepsilon(t))$ by (8) so that $\Gamma^{-1}(g, x)(\varepsilon) = j_x^1\,\gamma^{-1}(t)$ and this jet is determined by (29) by virtue of the following assertion: Let $\varkappa : \mathbf{R} \to G$, $\varkappa(0) = g$ be a curve on G, then the coordinates of its tangent vector at 0 with respect to the basis $(e_\alpha)_g$ are opposite to the coordinates of the tangent vector of the curve $\varkappa^{-1}(s)$ at 0 with respect to the basis $(\bar{e}_\alpha)_{g^{-1}}$. Indeed, if we put $\varkappa(s) = g\,\tilde{\varkappa}(s)$, then $\varkappa^{-1}(s) = \tilde{\varkappa}^{-1}(s)\,g^{-1}$ and the curves $\tilde{\varkappa}(s)$ and $\tilde{\varkappa}^{-1}(s)$ pass through e for $s = 0$. But it is well-known that their tangent vectors are opposite.

Now, we are in a position to prove Proposition 13. Since our problem is local, we may consider the trivial fibre bundle $P = G \times \mathbf{R}^m$ in a neighbourhood of $0 \in \mathbf{R}^m$. Then $E = F \times \mathbf{R}^m$ and we may suppose that σ is the cross section $t \mapsto (c, t)$, so that the cross section ε of P belongs to the reduction R of P determined by σ. First of all, we shall find $\Delta(\Gamma'^{-1}(u)(\varepsilon))$, $u \in R$, $pr_2 u = 0$. Let $\gamma(t)$ be a local cross section of P such that $j_0^1\,\gamma(t) = \Gamma(u)$, then Corollary 2 gives $\Gamma'^{-1}(u)(\varepsilon) = j_0^1\,\varphi(t)$ where $\varphi(t) = \Gamma^{-1}(\gamma(t))(\varepsilon)$ is a local cross section of $J^1(\mathbf{R}^m, G)$. Denote by ϖ_0^α the basis of $T^*(G)$ dual to $\bar{e}_\alpha$, then ϖ_0^α are Maurer-Cartan forms of the Lie group anti-isomorphic to G, which is why they satisfy

$$\mathrm{d}\varpi_0^\alpha = -\tfrac{1}{2}c_{\beta\gamma}^\alpha\varpi_0^\beta \wedge \varpi_0^\gamma\,. \tag{30}$$

On $G \times \mathbf{R}^m$, consider the forms

$$\varpi^\alpha = pr_1^*\varpi_0^\alpha\,, \quad \mathrm{d}t^i = pr_2^*\,\mathrm{d}t^i. \tag{31}$$

Since $J^1(\mathbf{R}^m, G)$ is canonically identified with $J^1(G \times \mathbf{R}^m)$, the section $\varphi(t)$ is determined by the functions $-\Gamma_i^\alpha(t)$, see § 6. Now, (18), (19) and (30) give

$$\Delta(j_0^1\,\varphi(t)) = [-\partial_{[j}\Gamma_{i]}^\alpha - \tfrac{1}{2}c_{\beta\gamma}^\alpha\Gamma_i^\beta\Gamma_j^\gamma]\,\mathrm{d}t^i \wedge \mathrm{d}t^j \otimes (\bar{e}_\alpha)_e\,. \tag{32}$$

Further, let $\mu : \tilde{P} \to E$ be the projection corresponding to $\mu_0 : G \to F$, i.e. $\mu(g, x) = (\mu_0 g, x)$, then it is easy to see that

$$\mu_{0*}\Delta(\Gamma'^{-1}(u)(\varepsilon)) = \Delta(\Gamma'^{-1}(u)(\mu\varepsilon)) = \Delta(\Gamma'^{-1}(u)(\sigma))$$

and the comparison of (24), (28) and (32) proves Proposition 13.

From Proposition 13 and from § 5 we deduce immediately

Theorem 1. *The torsion form of a space with connection $\mathcal{S}(B, \Phi, E, \sigma, C)$ vanishes at a point $x \in B$ if and only if the second development $C'^{-1}(x)(\sigma)$ of $\mathcal{S}$ is a holonomic jet.*

9. A space with connection $\mathcal{S}(B, \Phi, E, \sigma, C)$ will be called a *manifold with connection*, if $\dim B < \dim F$ and $C^{-1}(x)(\sigma)$ is regular for every $x \in B$. It is easy to see that a manifold with connection is locally equivalent to a submanifold of a space with Cartan connection, [2].

Let $K_x = T(C^{-1}(x)(\sigma)) \subset T_{\sigma(x)}(E_x)$ be the subspace determined by $C^{-1}(x)(\sigma)$, cf. § 5, and let $\psi_x : T_{\sigma(x)}(E_x) \to T_{\sigma(x)}(E_x)/K_x$ be the canonical projection. The form

$$\psi_x(\tau(x)) \in (T_{\sigma(x)}(E_x)/K_x) \otimes \Lambda^2 T_x^*(B) \tag{33}$$

will be called the reduced torsion form of $\mathcal{S}$ at $x \in B$. From Proposition 11 and 13 we get directly

Theorem 2. *The reduced torsion form of a manifold with connection $\mathcal{S}(B, \Phi, E, \sigma, C)$ vanishes at a point $x \in B$ if and only if the contact element $k(C'^{-1}(x)(\sigma))$ determined by the second development of $\mathcal{S}$ is holonomic.*

We shall also write the coordinate form of this condition. Let U be an open subset of B, let $\varphi : U \to R$ be a cross section and let $\omega_0 = \varphi^*\omega$ be the restriction of ω to φ. Analogously we put $\Omega_0 = \varphi^*\Omega$. Let the vectors

$$(e_\lambda)_e, \quad \lambda = n + 1, \ldots, r, \quad n = \dim F$$

belong to $\mathfrak{h}$. Then

$$\omega_0 = \omega_0^s \otimes e_s + \omega_0^\lambda \otimes e_\lambda, \quad s = 1, \ldots, n$$

and there are $n - m$ linear relations among ω_0^s. We may suppose that ω_0^i are independent, then the linear relations can be written in the form

$$\omega_0^\varkappa = A_i^\varkappa \omega_0^i, \quad i = 1, \ldots, m = \dim B, \quad \varkappa = m + 1, \ldots, n. \tag{34}$$

Let $\Omega_0 = \Omega_0^s \otimes e_s + \Omega_0^\lambda \otimes e_\lambda$, then we deduce from our previous considerations that the reduced torsion form of $\mathcal{S}$ vanishes over U if an only if the analogous relations to (34) are satisfied for Ω_0, i.e.

$$\Omega_0^\varkappa = A_i^\varkappa \Omega_0^i.$$

10. We conclude with a small concrete example. Let $\mathcal{P}$ be a surface with projective connection according to Švec [11], then the reduced torsion form of $\mathcal{P}$ vanishes if and only if $\mathcal{P}$ is without torsion in the Švec's terminology. Further, consider a congruence $\mathscr{L}$ with projective connection, [11], then we have defined the reduced torsion form of $\mathscr{L}$. We can state a simple

Proposition 14. *Let $\mathscr{L}$ be a non-parabolic congruence with projective connection and let $\mathscr{F}_1$, $\mathscr{F}_2$ be its focal surfaces, then the reduced torsion form of $\mathscr{L}$ vanishes if and only if both $\mathscr{F}_1$ and $\mathscr{F}_2$ are without torsion.*

Proof. Take the frame field of the first order of $\mathscr{L}$, see [11], p. 74, then A_1, A_2 are foci of $\mathscr{L}$ and it holds (we write $\omega_1^3 = \omega^1$, $\omega_2^4 = \omega^2$)

$$\mathrm{d}A_1 = \omega_1^1 A_1 + \omega_1^2 A_2 + \omega^1 A_3 ,$$

$$\mathrm{d}A_2 = \omega_2^1 A_1 + \omega_2^2 A_2 \qquad + \omega^2 A_4 ,$$

as well as

$$\mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j + R_i^j \omega^1 \wedge \omega^2 .$$

The components of the reduced torsion form of $\mathscr{L}$ are $R_2^3\omega^1 \wedge \omega^2$, $R_1^4\omega^1 \wedge \omega^2$, while the reduced torsion form of $\{A_1\}$ or $\{A_2\}$ is $R_1^4\omega^1 \wedge \omega^2$ or $R_2^3\omega^1 \wedge \omega^2$ respectively, which proves our assertion.

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