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UNION AND SYMMETRY PRESERVING ENDOMORPHISMS OF THE SEMIGROUP OF ALL BINARY RELATIONS ON A SET*)

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The familiar theorem that every automorphism of a symmetric group of finite degree is inner has undergone several successive generalizations. In 1937, SCHREIER [8] extended it to the group of all permutations of any set X (*i.e.*, all one-to-one mappings of X onto X). In 1952, MALCEV [7] further extended it to the semigroup of all transformations of a set. (See also LJAPIN [5], p. 302.) In 1959, GLUSKIN [3] extended it in turn to the semigroup of all partial transformations of a set. In 1964, CRESTEY [2] extended it to the semigroup $\mathscr{B}_X$ of all binary relations on a set X, except that he imposed the hypothesis that the automorphism of $\mathscr{B}_X$ preserve finite unions. In 1965, ZARECKIĬ [9] proved it for the semigroup of all binary relations on X having domain and range both equal to X, with no restriction on the automorphism. Finally, in 1966, MAGILL [6] showed that every automorphism (without restriction) of $\mathscr{B}_X$ is inner. Independently, Gluskin [4] in 1967 obtained the same result for $\mathscr{B}_X$ and for two of its subsemigroups.

In the present paper we begin the study of the endomorphisms of $\mathscr{B}_X$, and determine all those that preserve arbitrary unions and map symmetric relations onto symmetric relations (Theorem 3 in §2). Theorem 1 in §1 determines all such endomorphisms that preserve the empty relation. In §3 we give a somewhat simplified proof of Magill's Theorem, and in §4 we consider the case X finite.

1. ENDOMORPHISMS PRESERVING UNIONS, SYMMETRY, AND THE EMPTY RELATION

A (binary) relation on a set X is just a subset of the cartesian product $X \times X$. The product $\alpha \circ \beta$ of two relations α and β on X is defined to be the relation

$$\alpha \circ \beta = \{(x, y) \in X \times X : \exists z \in X \quad \text{such that} \quad (x, z) \in \alpha \quad \text{and} \quad (z, y) \in \beta\} .$$

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This operation is associative, and hence the set $\mathscr{B}_X$ of all relations on X is a semigroup. Since $\mathscr{B}_X$ is the set of all subsets of $X \times X$, it is closed under set-theoretical union $\bigcup$ and intersection $\bigcap$. For the elementary properties of $\mathscr{B}_X$, the reader is referred to [1], §1.4. To avoid trivialities, we assume throughout that X has more than one element.

The converse α^{-1} of a relation α will be denoted by $\alpha\sigma$, and we shall regard σ as the transformation of $\mathscr{B}_X$ which takes α into $\alpha\sigma = \alpha^{-1}$. A relation is symmetric if and only if it is taken into itself by σ. By the domain of a relation α we mean the set

$$D(\alpha) = \{x \in X : (x, y) \in \alpha \text{ for some } y \in X\},$$

and the range of α is defined to be $R(\alpha) = D(\alpha\sigma)$.

An endomorphism of $\mathscr{B}_X$ is a transformation θ of $\mathscr{B}_X$ satisfying $(\alpha \circ \beta)\,\theta = \alpha\theta \circ \beta\theta$ (all α, β in $\mathscr{B}_X$). Any transformation θ of $\mathscr{B}_X$ will be said to *preserve unions* if, for any subset $\{\alpha_i : i \in I\}$ of $\mathscr{B}_X$,

$$(\bigcup\{\alpha_i : i \in I\})\,\theta = \bigcup\{\alpha_i\theta : i \in I\}.$$

We shall say that θ *preserves symmetry* if it maps symmetric relations into symmetric relations. We shall say that θ *preserves converses* if it commutes with σ. If θ preserves converses then it preserves symmetry. For if $\theta\sigma = \sigma\theta$, and α is symmetric, *i.e.*, $\alpha\sigma = \alpha$, then $(\alpha\theta)\,\sigma = (\alpha\sigma)\,\theta = \alpha\theta$, *i.e.*, $\alpha\theta$ is symmetric. Regarding the converse assertion, see Lemma 3 below and the remark following it.

Throughout this paper, the relation consisting of the single pair (x, y) in $X \times X$ will be denoted by $\Phi_{x,y}$. The empty relation will be denoted by $\emptyset$. Clearly

$$(1) \qquad \Phi_{x,y} \circ \Phi_{x',y'} = \begin{cases} \Phi_{x,y'} & \text{if } \; y = x', \\ \emptyset & \text{if } \; y \neq x'. \end{cases}$$

Let θ be any endomorphism of $\mathscr{B}_X$ such that $\emptyset\theta = \emptyset$. Let $\eta_{x,y} = \Phi_{x,y}\theta$. From (1) we obtain

$$(2) \qquad \eta_{x,y} \circ \eta_{x',y'} = \begin{cases} \eta_{x,y'} & \text{if } \; y = x', \\ \emptyset & \text{if } \; y \neq x'. \end{cases}$$

Any system $\{\eta_{x,y} : x, y \in X\}$ of elements of $\mathscr{B}_X$ (indexed by $X \times X$) satisfying (2) will be called a *matricial family*.

Lemma 1. *Let $\{\eta_{x,y} : x, y \in X\}$ be a matricial family of relations on a set X. If one member of the family is empty, then every member is empty. In the following, assume that at least one, and hence every, $\eta_{x,y}$ is not empty. Let D_x be the domain, and R_x the range, of $\eta_{x,x}$. Then the domain of $\eta_{x,y}$ is D_x, and its range is R_y; and, for all x, y in X, $D_x \cap R_y = \emptyset$ if and only if $x \neq y$.*

Proof. Suppose $\eta_{u,v} = \emptyset$ for some u, v in X. Then $\eta_{x,y} = \eta_{x,u} \circ \eta_{u,v} \circ \eta_{v,y} = \emptyset$ for every (x, y) in $X \times X$. From $\eta_{x,y} = \eta_{x,x} \circ \eta_{x,y}$ and $\eta_{x,x} = \eta_{x,y} \circ \eta_{y,x}$ we see that

$D(\eta_{x,y}) = D_x$, and the proof that $R(\eta_{x,y}) = R_y$ is similar. From $\eta_{x,x} \circ \eta_{x,x} = \eta_{x,x} \neq \emptyset$ and $\eta_{y,y} \circ \eta_{x,x} = \emptyset$ for $x \neq y$, we conclude that $R_x \cap D_x \neq \emptyset$ and that $R_y \cap D_x = \emptyset$ for $x \neq y$.

Lemma 2. *If θ is a union preserving transformation of $\mathscr{B}_X$, and if we set $\eta_{x,y} = \Phi_{x,y}\,\theta(x, y \in X)$, then, for any $\alpha \neq \emptyset$ in $\mathscr{B}_X$,*

$$\alpha\theta = \bigcup\{\eta_{x,y} : (x, y) \in \alpha\}\,. \tag{3}$$

Hence if two union preserving transformations of $\mathscr{B}_X$ agree on the one-element relations $\Phi_{x,y}$ and on $\emptyset$ then they are equal.

Proof. The proof is immediate from $\alpha = \bigcup\{\Phi_{x,y} : (x, y) \in \alpha\}$ and the hypothesis that θ preserves unions.

Lemma 3. *Let θ be an endomorphism of $\mathscr{B}_X$. If θ preserves converses, then it preserves symmetry. If θ preserves unions, symmetry, and the empty relation, then it preserves converses.*

Remark. It follows from Theorem 3 below that the hypothesis $\emptyset\theta = \emptyset$ in the second assertion can be omitted. We do not know if the hypothesis that θ preserve unions can likewise be omitted.

Proof. The first assertion has already been shown. Assume that θ preserves unions, symmetry, and $\emptyset$. Let $\eta_{x,y} = \Phi_{x,y}\theta$, and let $x \neq y$ in X. Since $\Phi_{x,y} \cup \Phi_{y,x}$ is symmetric, and θ preserves unions and symmetry, it follows that $\eta_{x,y} \cup \eta_{y,x} = (\Phi_{x,y} \cup \Phi_{y,x})\,\theta$ is symmetric. Let $(u, v) \in \eta_{x,y}$. Then $(v, u) \in \eta_{x,y} \cup \eta_{y,x}$. Since $u \in D_x$ and $v \in R_y$ by Lemma 1, and $D_x \cap R_y = \emptyset$ by the same lemma, we conclude that $(v, u) \notin \eta_{x,y}$, and hence $(v, u) \in \eta_{y,x}$. Thus $\eta_{x,y}\sigma \subseteq \eta_{y,x}$. Interchanging x and y, $\eta_{y,x}\sigma \subseteq \eta_{x,y}$, and hence $\eta_{y,x} = \eta_{y,x}\sigma\sigma \subseteq \eta_{x,y}\sigma$. Thus $\eta_{x,y}\sigma = \eta_{y,x}$, and

$$\Phi_{x,y}\theta\sigma = \eta_{x,y}\sigma = \eta_{y,x} = \Phi_{y,x}\theta = \Phi_{x,y}\sigma\theta\,.$$

Since θ and σ preserve unions and $\emptyset$, so do $\theta\sigma$ and $\sigma\theta$. By Lemma 2, $\theta\sigma = \sigma\theta$.

By a *partial equivalence* we mean a symmetric, transitive relation.

Theorem 1. *Let X be a set. Let E be any non-empty subset of X, let π be a partial equivalence on X with domain E, and let μ be a mapping of E onto X satisfying*

$$\pi \circ \mu = E \times X\,. \tag{4}$$

Define $\theta : \mathscr{B}_X \to \mathscr{B}_X$ by

$$\alpha\theta = \pi \cap (\mu \circ \alpha \circ \mu^{-1}) \quad (\textit{all } \alpha \textit{ in } \mathscr{B}_X)\,. \tag{5}$$

Then θ is a non-zero endomorphism of $\mathscr{B}_X$ which preserves unions and symmetry, and maps $\emptyset$ onto $\emptyset$.

Conversely, every such endomorphism of $\mathscr{B}_X$ is obtained in this way.

Remark 1. The zero endomorphism of $\mathscr{B}_X$ (which maps every element of $\mathscr{B}_X$ onto $\emptyset$) can be included in the formula (5) if we allow E to be empty, and do not require μ to be onto. If $E \neq \emptyset$, condition (4) forces μ to be onto.

Remark 2. Condition (4) is equivalent to: given t in X and y in E, there exists z in X such that $(y, z) \in \pi$ (hence $z \in E$) and $z\mu = t$.

Proof. Let E, π, and μ satisfy (4), and define θ by (5). The latter is equivalent to

$$\alpha\theta = \{(x, y) \in \pi : (x\mu, y\mu) \in \alpha\} . \tag{6}$$

For $(x, y) \in \mu \circ \alpha \circ \mu^{-1}$ if and only if there exist u, v in X such that

$$(x, u) \in \mu\,, \quad (u, v) \in \alpha\,, \quad (v, y) \in \mu^{-1}\,.$$

Since the first of these is equivalent to $x\mu = u$, and the third to $y\mu = v$, it follows that $(x, y) \in \mu \circ \alpha \circ \mu^{-1}$ if and only if $(x\mu, y\mu) \in \alpha$.

Let $\alpha, \beta \in \mathscr{B}_X$. We proceed to show that $\alpha\theta \circ \beta\theta = (\alpha \circ \beta)\,\theta$. Let $(x, y) \in \alpha\theta \circ \beta\theta$. Then there exists z in X such that $(x, z) \in \alpha\theta$ and $(z, y) \in \beta\theta$. Hence

$$(x, z) \in \pi\,, \quad (x\mu, z\mu) \in \alpha\,, \quad (z, y) \in \pi\,, \quad (z\mu, y\mu) \in \beta\,, \tag{7}$$

whence $(x, y) \in \pi$, $(x\mu, y\mu) \in \alpha \circ \beta$; that is, $(x, y) \in (\alpha \circ \beta)\,\theta$. Hence $\alpha\theta \circ \beta\theta \subseteq (\alpha \circ \beta)\,\theta$.

To show the converse inclusion, let $(x, y) \in (\alpha \circ \beta)\,\theta$. Then $(x, y) \in \pi$ and $(x\mu, y\mu) \in \alpha \circ \beta$. The latter implies $(x\mu, t) \in \alpha$ and $(t, y\mu) \in \beta$ for some t in X. By (4) and Remark 2, there exists z in E such that $(y, z) \in \pi$ and $z\mu = t$. We conclude that equations (7) hold, which imply $(x, z) \in \alpha\theta$ and $(z, y) \in \beta\theta$, hence $(x, y) \in \alpha\theta \circ \beta\theta$.

Since $\circ$ distributes over arbitrary unions ([1], Exercise 1(b), p. 15), it is clear from (5) that θ preserves unions. It is clear from (6) that if α is symmetric, so is $\alpha\theta$; thus θ preserves symmetry. If $\omega = X \times X$, then $\omega\theta = \pi \neq \emptyset$, and so θ is not the zero endomorphism. It is also clear from (5) that $\emptyset\theta = \emptyset$.

Conversely, let θ be a non-zero endomorphism of $\mathscr{B}_X$ which preserves unions and symmetry, and maps $\emptyset$ onto $\emptyset$. Let $\eta_{x,y} = \Phi_{x,y}\theta$. Since $\emptyset\theta = \emptyset$, equations (2) hold. If $\eta_{x,y} = \emptyset$ for every x, y in X, then θ is the zero endomorphism, by Lemma 2. Hence at least one $\eta_{x,y}$ is not empty, and, by Lemma 1, every $\eta_{x,y}$ is non-empty.

Since $\Phi_{x,x}$ is symmetric, and θ preserves symmetry, it follows that $\eta_{x,x}$ is symmetric, and so, in the notation of Lemma 1, $D_x = R_x$. Let us write $E_x = D_x = R_x$, and let $E = \bigcup\{E_x : x \in X\}$. By Lemma 1, the domain of $\eta_{x,y}$ is E_x, its range is E_y, $E_x \neq \emptyset$ for all x, and

$$E_x \cap E_y = \emptyset \quad \text{if} \quad x \neq y \quad (x, y \in X)\,. \tag{8}$$

By Lemma 3, θ preserves converses, and so

$$\eta_{x,y}\sigma = \Phi_{x,y}\theta\sigma = \Phi_{x,y}\sigma\theta = \Phi_{y,x}\theta = \eta_{y,x} . \tag{9}$$

Let

$$\pi = \bigcup\{\eta_{x,y} : (x, y) \in X \times X\} . \tag{10}$$

By (9), π is symmetric, and by (2) it is transitive. Hence it is a partial equivalence on X with domain

$$\begin{aligned}\{u \in X : (u, u) \in \pi\} &= \{u \in X : (u, u) \in \eta_{x,y} \text{ for some } (x, y) \text{ in } X \times X\} = \\ &= \{u \in X : u \in E_x \text{ for some } x \text{ in } X\} = \bigcup\{E_x : x \in X\} = E .\end{aligned}$$

Note that

$$\eta_{x,y} = \pi \cap (E_x \times E_y) . \tag{11}$$

For if $(u, v) \in \pi \cap (E_x \times E_y)$, then $(u, v) \in \eta_{x',y'}$ for some (x', y') in $X \times X$. Then $u \in E_x \cap E_{x'}$, and from (8) we conclude that $x = x'$. Similarly, $y = y'$, and hence $(u, v) \in \eta_{x,y}$. The converse inclusion is clear.

Let

$$\mu = \{(x, y) \in X \times X : x \in E_y\} . \tag{12}$$

Clearly μ is a (single-valued) mapping of E onto X.

Now let $\alpha \in \mathscr{B}_X$. Since θ is union preserving, Lemma 2 holds. From (3), (11), and (12), we see that $(u, v) \in \alpha\theta$ if and only if there exists (x, y) in $X \times X$ such that (in turn) each of the following equivalent assertions holds:

$$\begin{aligned}&(u, v) \in \eta_{x,y} \quad \text{and} \quad (x, y) \in \alpha ,\\ &(u, v) \in \pi , \quad u \in E_x , \quad v \in E_y , \quad \text{and} \quad (x, y) \in \alpha ,\\ &(u, v) \in \pi , \quad (u, x) \in \mu , \quad (x, y) \in \alpha , \quad \text{and} \quad (y, v) \in \mu^{-1} .\end{aligned}$$

Hence $(u, v) \in \alpha\theta$ if and only if both $(u, v) \in \pi$ and $(u, v) \in \mu \circ \alpha \circ \mu^{-1}$; that is, if and only if (5) holds.

Finally, we prove (4), as stated in Remark 2. Let $t \in X$, $y \in E$. By definition of E, $y \in E_x$ for some x in X. Since E_x is the domain of $\eta_{x,t}$, there exists z in X such that $(y, z) \in \eta_{x,t}$. By (11), $(y, z) \in \pi$ and $z \in E_t$.

Example 1. Let X be the set of positive integers, let E be the even integers, and let π be congruence modulo 4. Let $\mu : E \to X$ be defined by $n\mu^{-1} = E_n = \{4n - 2, 4n\}$, for every n in X. Then $\eta_{m,n}$ is the two-element relation consisting of the pairs $(4m - 2, 4n - 2)$ and $(4m, 4n)$.

In this section we remove the restriction that the endomorphism θ of $\mathscr{B}_X$ map $\emptyset$ onto $\emptyset$.

Lemma 4. *Let θ_1 and θ_2 be endomorphisms of $\mathscr{B}_X$, and let E_1 and E_2 be disjoint subsets of X, such that $\alpha\theta_i \subseteq E_i \times E_i$ $(i = 1, 2)$ for all α in $\mathscr{B}_X$. Then $\theta_1 \cup \theta_2$, defined by*

$$\alpha(\theta_1 \cup \theta_2) = \alpha\theta_1 \cup \alpha\theta_2 \quad (\textit{all } \alpha \textit{ in } \mathscr{B}_X)$$

is an endomoprhism of $\mathscr{B}_X$. The endomorphism $\theta_1 \cup \theta_2$ preserves unions or symmetry or the empty relation if and only if both θ_1 and θ_2 do the same, respectively.

Proof. Let $\alpha, \beta \in \mathscr{B}_X$. Then $\alpha\theta_2 \circ \beta\theta_1 \subseteq (E_2 \times E_2) \circ (E_1 \times E_1) = \emptyset$, and similarly, $\alpha\theta_1 \circ \beta\theta_2 = \emptyset$. Since $\circ$ distributes over $\cup$,

$$\alpha(\theta_1 \cup \theta_2) \circ \beta(\theta_1 \cup \theta_2) = (\alpha\theta_1 \cup \alpha\theta_2) \circ (\beta\theta_1 \cup \beta\theta_2) = (\alpha\theta_1 \circ \beta\theta_1) \cup (\alpha\theta_2 \circ \beta\theta_2) =$$
$$= (\alpha \circ \beta)\,\theta_1 \cup (\alpha \circ \beta)\,\theta_2 = (\alpha \circ \beta)(\theta_1 \cup \theta_2)\,.$$

The last assertion of the lemma is evident.

Lemma 5. *If $\{\eta_{x,y} : x, y \in X\}$ is a matricial family, and if we define $\theta : \mathscr{B}_X \to \mathscr{B}_X$ by $\emptyset\theta = \emptyset$ and*

$$(13) \qquad \alpha\theta = \bigcup\{\eta_{x,y} : (x, y) \in \alpha\}\,, \quad (\emptyset \neq \alpha \in \mathscr{B}_X)\,,$$

then θ is an endomorphism of $\mathscr{B}_X$ that preserves unions (and $\emptyset$). Moreover, θ preserves symmetry if and only if

$$(14) \qquad \eta_{x,y}\sigma = \eta_{y,x} \quad (\textit{all } x, y \in X)\,.$$

Proof. Let $\alpha, \beta \in \mathscr{B}_X$, and let $(u, v) \in \alpha\theta \circ \beta\theta$. Then there exists w in X such that $(u, w) \in \alpha\theta$ and $(w, v) \in \beta\theta$. This implies that there exist x, y, s, t in X such that

$$(x, y) \in \alpha\,, \quad (u, w) \in \eta_{x,y}\,, \quad (s, t) \in \beta\,, \quad (w, v) \in \eta_{s,t}\,.$$

Hence $(u, v) \in \eta_{x,y} \circ \eta_{s,t}$, which implies $y = s$, and then $(u, v) \in \eta_{x,t}$; and $y = s$ also implies $(x, t) \in \alpha \circ \beta$. We conclude that $(u, v) \in (\alpha \circ \beta)\,\theta$.

Conversely, let $(u, v) \in (\alpha \circ \beta)\,\theta$. Then $(u, v) \in \eta_{x,t}$ for some (x, t) in $\alpha \circ \beta$. The latter implies that there exists y in X such that $(x, y) \in \alpha$ and $(y, t) \in \beta$. Since $\eta_{x,t} = \eta_{x,y} \circ \eta_{y,t}$, the former implies that there exists w in X such that $(u, w) \in \eta_{x,y}$ and $(w, v) \in \eta_{y,t}$. Hence $(u, w) \in \alpha\theta$ and $(w, v) \in \beta\theta$, so that $(u, v) \in \alpha\theta \circ \beta\theta$.

We have thus shown that θ is an endomorphism of $\mathscr{B}_X$. Now let $\{\alpha_i : i \in I\}$ be any subset of $\mathscr{B}_X$. Then

$$(\bigcup_i \alpha_i)\,\theta = \bigcup_{x,y} \{\eta_{x,y} : (x, y) \in \bigcup_i \alpha_i\} = \bigcup_i \bigcup_{x,y} \{\eta_{x,y} : (x, y) \in \alpha_i\} = \bigcup_i (\alpha_i\theta)\,.$$

Hence θ preserves unions. That $\emptyset\theta = \emptyset$ is part of the definition of θ.

We note that $\Phi_{x,y}\,\theta = \eta_{x,y}$. If θ preserves symmetry, then it preserves converses, by Lemma 3, and (14) follows from (9). Conversely, assume (14), and let α be a symmetric element of $\mathscr{B}_X$. Let $(u, v) \in \alpha\theta$, so that $(u, v) \in \eta_{x,y}$ for some (x, y) in α. Then $(v, u) \in \in \eta_{x,y}\sigma = \eta_{y,x}$, and since $(y, x) \in \alpha$, we conclude that $(v, u) \in \alpha\theta$. Hence θ preserves symmetry.

Theorem 2. *Let X be a set. Let ζ be a partial equivalence on X with domain F. Let θ be an endomorphism of $\mathscr{B}_X$ mapping $\emptyset$ onto $\emptyset$ and having the property that there exists a subset E of X, disjoint from F, such that $\alpha\theta \subseteq E \times E$ for all α in $\mathscr{B}_X$. Define $\theta' : \mathscr{B}_X \to \mathscr{B}_X$ by*

$$\alpha\theta' = \alpha\theta \cup \zeta \quad (\textit{all } \alpha \textit{ in } \mathscr{B}_X)\,. \tag{15}$$

Then θ' is an endomorphism of $\mathscr{B}_X$ mapping $\emptyset$ onto ζ.

If θ preserves unions and symmetry, so does θ'. Conversely, every endomorphism of $\mathscr{B}_X$ that preserves unions and symmetry is obtained in this way.

Proof. If we define $\theta'' : \mathscr{B}_X \to \mathscr{B}_X$ by $\alpha\theta'' = \zeta$ for all α in $\mathscr{B}_X$, then (since $\zeta \circ \zeta = = \zeta$) θ'' is an endomorphism of $\mathscr{B}_X$. The transformation θ' of $\mathscr{B}_X$ is just $\theta \cup \theta''$ as defined in Lemma 4, and is an endomorphism of $\mathscr{B}_X$, since $E \cap F = \emptyset$. Moreover, $\emptyset\theta' = \emptyset\theta \cup \zeta = \emptyset \cup \zeta = \zeta$. That θ' preserves unions and symmetry if θ does, also follows from Lemma 4.

Conversely, let θ' be an endomorphism of $\mathscr{B}_X$ which preserves unions and symmetry. Let $\zeta = \emptyset\theta'$, and let $\eta'_{x,y} = \Phi_{x,y}\theta'$, $(x, y \in X)$. By Lemma 2,

$$\alpha\theta' = \bigcup\{\eta'_{x,y} : (x, y) \in \alpha\}\,, \quad \text{for all} \quad \alpha \neq \emptyset \quad \text{in} \quad \mathscr{B}_X\,. \tag{16}$$

Now $\zeta \circ \zeta = \zeta$, and $\zeta\sigma = \zeta$ since θ' preserves symmetry; hence ζ is a partial equivalence on X. Let F be the domain of ζ. If $\zeta = \emptyset$, there is nothing to prove. Consequently we may assume $\zeta \neq \emptyset$, and hence $F \neq \emptyset$.

Similarly, $\eta'_{x,x}$ is a partial equivalence on X, for each x in X. Let E'_x be the domain of $\eta'_{x,x}$. Since θ' preserves unions, it preserves inclusion, and so $\zeta \subseteq \eta'_{x,x}$ for all x in X. In particular, $F \subseteq E'_x$. Since θ' is an endomorphism of $\mathscr{B}_X$, we have:

$$\eta'_{x,y} \circ \eta'_{s,t} = \begin{cases} \eta'_{x,t} & \text{if} \quad y = s\,, \\ \zeta & \text{if} \quad y \neq s\,; \end{cases} \tag{17}$$

$$\eta'_{x,y} \circ \zeta = \zeta \circ \eta'_{x,y} = \zeta\,. \tag{18}$$

From $\eta'_{x,x} \circ \eta'_{x,y} = \eta'_{x,y}$ and $\eta'_{x,y} \circ \eta'_{y,x} = \eta'_{x,x}$, we see that the domain of $\eta'_{x,y}$ is E'_x, and similarly we can show that its range is E'_y.

We observe next that if $u \in F$ and $(u, v) \in \eta'_{x,y}$, then $(u, v) \in \zeta$. For, let z be any element of X such that $z \neq x$. Since $u \in F \subseteq E'_z$, and E'_z is the domain of the partial equivalence $\eta'_{z,z}$, it follows that $(u, u) \in \eta'_{z,z}$. From this and $(u, v) \in \eta'_{x,y}$, we conclude from (17) that $(u, v) \in \eta'_{z,z} \circ \eta'_{x,y} = \zeta$.

Similarly we can show that if $v \in F$ and $(u, v) \in \eta'_{x,y}$, then $(u, v) \in \zeta$. We conclude that if $(u, v) \in \eta'_{x,y}$, then either $(u, v) \in \zeta$ or $(u, v) \in E_x \times E_y$, where $E_x = E'_x \backslash F$. In other words, if we set

$$\eta_{x,y} = \eta'_{x,y} \cap (E_x \times E_y)\,, \tag{19}$$

then

$$\eta'_{x,y} = \eta_{x,y} \cup \zeta\,. \tag{20}$$

Here $\eta_{x,y}$ and ζ are disjoint in the strong sense that both their domains and their ranges are disjoint. Consequently

$$\eta'_{x,y} \circ \eta'_{s,t} = (\eta_{x,y} \circ \eta_{s,t}) \cup \zeta\,. \tag{21}$$

From (17) and (20) we conclude when $y = s$ that the left side of (21) is equal to $\eta_{x,t} \cup \zeta$; and from the strong disjointness we infer that $\eta_{x,y} \circ \eta_{y,t} = \eta_{x,t}$. If $y \neq s$, then the left side of (21) is equal to ζ, and we infer that $\eta_{x,y} \circ \eta_{s,t} = \emptyset$. Hence $\{\eta_{x,y} : : (x, y) \in X \times X\}$ is a matricial family.

By Lemma 5, the transformation θ of $\mathscr{B}_X$ defined by (13) and $\emptyset\theta = \emptyset$ is an endomorphism of $\mathscr{B}_X$. For every α in $\mathscr{B}_X$, $\alpha\theta \subseteq E \times E$, where $E = \bigcup\{E_x : x \in X\}$. Since each E_x is disjoint from F, so is E. (15) now follows from (13), (16), and (20) for $\alpha \neq \emptyset$; and it is clear for $\alpha = \emptyset$ since $\emptyset\theta' = \zeta$ and $\emptyset\theta = \emptyset$.

We combine Theorems 1 and 2 into the following, which is the main result of the paper.

Theorem 3. *Let X be a set, and let E and F be disjoint subsets of X. Let π and ζ be partial equivalences on X with domains E and F, respectively. Let μ be a mapping of E into X satisfying $\pi \circ \mu = E \times X$. Define $\theta : \mathscr{B}_X \to \mathscr{B}_X$ by*

$$\alpha\theta = [\pi \cap (\mu \circ \alpha \circ \mu^{-1})] \cup \zeta\,, \quad (\alpha \in \mathscr{B}_X)\,. \tag{22}$$

Then θ is an endomorphism of $\mathscr{B}_X$ preserving unions and symmetry; and, conversely, every such endomorphism of $\mathscr{B}_X$ is obtained in this way.

The purpose of this section is to give an alternative proof of Magill's Theorem that every automorphism of $\mathscr{B}_X$ is inner.

Let λ be a permutation of X, and let $\bar{\lambda} : \mathscr{B}_X \to \mathscr{B}_X$ be the corresponding inner automorphism of $\mathscr{B}_X$, that is,

$$\alpha\bar{\lambda} = \lambda^{-1} \circ \alpha \circ \lambda \quad (\alpha \in \mathscr{B}_X) . \tag{23}$$

We note that

$$\Phi_{x,y}\bar{\lambda} = \Phi_{x\lambda,y\lambda} \quad (x, y \in X) . \tag{24}$$

For $(u, v) \in \lambda^{-1} \circ \Phi_{x,y} \circ \lambda$ if and only if $(u, x) \in \lambda^{-1}$ and $(y, v) \in \lambda$; that is, if and only if $u = x\lambda$ and $v = y\lambda$.

Lemma 6. (1) *Let θ be an endomorphism of $\mathscr{B}_X$ which leaves fixed the empty relation $\emptyset$ and every one-element relation $\Phi_{x,y}$. Then θ is the identity automorphism of $\mathscr{B}_X$.*

(2) *If two endomorphisms θ_1 and θ_2 of $\mathscr{B}_X$ leave $\emptyset$ fixed and agree on all the $\Phi_{x,y}$, and one of them is an automorphism, then $\theta_1 = \theta_2$.*

Proof.

(1) Let $\alpha \in \mathscr{B}_X$. Then, for any x, y in X,

$$\Phi_{x,x} \circ \alpha \circ \Phi_{y,y} = \begin{cases} \Phi_{x,y} & \text{if } (x, y) \in \alpha , \\ \emptyset & \text{if } (x, y) \notin \alpha . \end{cases}$$

Applying θ and using the hypothesis that $\Phi_{x,y}\theta = \Phi_{x,y}$, etc., and $\emptyset\theta = \emptyset$, we get

$$\Phi_{x,x} \circ \alpha\theta \circ \Phi_{y,y} = \begin{cases} \Phi_{x,y} & \text{if } (x, y) \in \alpha , \\ \emptyset & \text{if } (x, y) \notin \alpha . \end{cases}$$

But

$$\Phi_{x,x} \circ \alpha\theta \circ \Phi_{y,y} = \begin{cases} \Phi_{x,y} & \text{if } (x, y) \in \alpha\theta , \\ \emptyset & \text{if } (x, y) \notin \alpha\theta . \end{cases}$$

Hence $(x, y) \in \alpha$ if and only if $(x, y) \in \alpha\theta$, that is, $\alpha\theta = \alpha$.

(2) Suppose θ_1 is an automorphism. Then $\theta_2\theta_1^{-1}$ leaves $\emptyset$ and every $\Phi_{x,y}$ fixed, so $\theta_2\theta_1^{-1}$ is the identity automorphism of $\mathscr{B}_X$, by (1), whence $\theta_2 = \theta_1$.

For subsets E, F of X, let $\Phi_{E,F}$ denote the relation $E \times F$ on X. We write $\Phi_{E,x}$ for $\Phi_{E,\{x\}}$, etc. Note that $\Phi_{E,F} = \emptyset$ if either $E = \emptyset$ or $F = \emptyset$. As pointed out by

Crestey [2], $\mathscr{B}_X$ has a 0-minimal ideal M which is contained in all non-zero ideals of $\mathscr{B}_X$, namely $M = \{\Phi_{E,F} : E \subseteq X,\ F \subseteq X\}$. For if $\emptyset \neq \alpha \in \mathscr{B}_X$ then

$$\alpha \circ \Phi_{E,F} = \Phi_{E',F}\,, \quad \text{where} \quad E' = \{x \in X : (x, y) \in \alpha \text{ for some } y \text{ in } E\}\,,$$

$$\Phi_{E,F} \circ \alpha = \Phi_{E,F'}\,, \quad \text{where} \quad F' = \{y \in X : (x, y) \in \alpha \text{ for some } x \text{ in } F\}\,.$$

Thus M is an ideal of $\mathscr{B}_X$. If A is any non-zero ideal of $\mathscr{B}_X$, and $\emptyset \neq \alpha \in A$, and if $(x, y) \in \alpha$, then A contains the element $\Phi_{E,x} \circ \alpha \circ \Phi_{y,F} = \Phi_{E,F}$, for any $E \subseteq X$, $F \subseteq X$.

Let N be the set of right non-zero-divisors in M, *i.e.*,

$$N = \{\alpha \in M : \gamma \circ \alpha = \emptyset \text{ implies } \gamma = \emptyset\ (\gamma \in \mathscr{B}_X)\}\,.$$

Clearly $\Phi_{E,F}$ is a right non-zero-divisor if and only if $E = X$ and $F \neq \emptyset$, and consequently $N = \{\Phi_{X,F} : F \subseteq X,\ F \neq \emptyset\}$.

We define a relation $\leqq$ on N as follows:

$$\alpha \leqq \beta \quad \text{if and only if} \quad \beta \circ \gamma = \emptyset \quad \text{implies} \quad \alpha \circ \gamma = \emptyset(\alpha, \beta \in N;\ \gamma \in \mathscr{B}_X)\,.$$

Note that $\Phi_{X,E} \leqq \Phi_{X,F}$ if and only if $E \subseteq F$, so $\leqq$ is a partial order on N.

Now let θ be any automorphism of $\mathscr{B}_X$. Clearly $M\theta = M$ and $N\theta = N$. Since the relation $\leqq$ on N is defined in terms of $\circ$, it is preserved by θ. Thus θ maps the set of minimal elements of N onto itself; but this is just the set $\{\Phi_{X,y} : y \in X\}$. Similarly, θ maps the set $\{\Phi_{x,X} : x \in X\}$ onto itself. Since the same holds for θ^{-1}, we conclude that there exist permutations λ and μ of X such that $\Phi_{x,X}\theta = \Phi_{x\lambda,X}$ and $\Phi_{X,y}\theta = \Phi_{X,y\mu}$. Applying θ to $\Phi_{x,y} = \Phi_{x,X} \circ \Phi_{X,y}$, we obtain $\Phi_{x,y}\theta = \Phi_{x\lambda,X} \circ \Phi_{X,y\mu} = \Phi_{x\lambda,y\mu}$. Applying θ to $\Phi_{x,x} = \Phi_{x,x} \circ \Phi_{x,x}$ we obtain $\Phi_{x\lambda,x\mu} = \Phi_{x\lambda,x\mu} \circ \Phi_{x\lambda,x\mu}$. Since $\Phi_{x\lambda,x\mu} \neq \emptyset$ for every x in X, we conclude that $x\lambda = x\mu$, and hence that $\lambda = \mu$. We have thus shown that there is a permutation λ of X such that $\Phi_{x,y}\theta = \Phi_{x\lambda,y\lambda}$.

By (24), θ agrees with the inner autmorphism $\bar{\lambda}$ of $\mathscr{B}_X$ on $\{\Phi_{x,y} : x, y \in X\}$, and so $\theta = \bar{\lambda}$ by Lemma 6. Hence θ is inner, which concludes the proof of Magill's Theorem.

4. ENDOMORPHISMS OF $\mathscr{B}_X$ WHEN X IS FINITE

In Theorem 3, if $E \neq \emptyset$, then μ must map E onto X. If X is finite, $E \neq \emptyset$ implies that $E = X$, that μ is a permutation of X, and that $F = \emptyset$, hence $\zeta = \emptyset$. Furthermore, the condition $\pi \circ \mu = E \times X = X \times X$, with μ a permutation of X, requires that $\pi = X \times X$. (22) thus reduces to $\alpha\theta = \mu \circ \alpha \circ \mu^{-1}$; that is, θ is an inner automorphism. On the other hand, if $E = \emptyset$, then $\pi = \emptyset$, and (22) reduces to $\alpha\theta = \zeta$. Thus the following is an immediate consequence of Theorem 3.

Theorem 4. *Let X be a finite set. Every endomorphism θ of $\mathscr{B}_X$ which preserves* *unions and symmetry is either an inner automorphism of $\mathscr{B}_X$, or else maps $\mathscr{B}_X$ onto a single partial equivalence ζ.*

We conclude with a theorem in which the properties of preserving unions and symmetry are not in the hypothesis, but (for finite X) are consequences of the conclusion. Since the two hypotheses of Theorem 5 are satisfied by any automorphism of $\mathscr{B}_X$, we get a simple proof of Magill's Theorem (for finite X), depending only on Lemma 1 and Lemma 6.

Theorem 4 shows that Theorem 5 would be false if we omitted the hypothesis (i). We give an example to show that we likewise cannot omit the hypothesis (ii).

Example 2. Let $\theta : \mathscr{B}_X \to \mathscr{B}_X$ map every unit $\bar{\lambda}$ of $\mathscr{B}_X$ onto itself (λ a permutation of X), and let θ map every non-unit of $\mathscr{B}_X$ onto $\emptyset$. (This works for X finite because the set of non-units of $\mathscr{B}_X$ is an ideal of $\mathscr{B}_X$, which is not so if X is infinite.) It is clear that θ is an endomorphism of $\mathscr{B}_X$. This also affords us an example of an endomorphism preserving $\emptyset$ and symmetry (in fact, converses), but not unions.

Theorem 5. *Let X be a finite set. Let θ be an endomorphism of $\mathscr{B}_X$ satisfying* (i) $\emptyset\theta = \emptyset$, *and* (ii) $\Phi_{x,y}\theta \neq \emptyset$ *for at least one pair (x, y) in $X \times X$. Then θ is an inner automorphism of $\mathscr{B}_X$.*

Proof. Let $\eta_{x,y} = \Phi_{x,y}\theta$. By (i), $\{\eta_{x,y} : (x, y) \in X \times X\}$ is a matricial family. By (ii), $\eta_{x,y} \neq \emptyset$ for at least one pair (x, y), and, by Lemma 1, $\eta_{x,y} \neq \emptyset$ for every pair (x, y). Let D_x be the domain and R_x the range of $\eta_{x,x}$. By Lemma 1, the domain of $\eta_{x,y}$ is D_x and its range is R_y, and $D_x \cap R_y = \emptyset$ if and only if $x \neq y$. In particular, $E_x = D_x \cap R_x \neq \emptyset$. On the other hand, if $x \neq y$, $E_x \cap E_y \subseteq D_x \cap R_y = \emptyset$.

Since X is finite, these facts imply that each E_x is a one-element set. Denoting the element of E_x by $x\lambda$, it follows that λ is a permutation of X. (λ is the inverse of the mapping μ defined by (12).)

It follows that R_x and D_x are also one-element sets. For suppose that R_x contained some element other than $x\lambda$. Since λ is a permutation, this additional element is $y\lambda$ for some $y \neq x$ in X. But $y\lambda \in E_y \subseteq D_y$, contradicting $R_x \cap D_y = \emptyset$. Thus $R_x = E_x = \{x\lambda\}$. Similarly, $D_x = E_x$ for each x in X. Since $\eta_{x,y} \subseteq D_x \times R_y$, it follows that $\eta_{x,y} = \Phi_{x\lambda,y\lambda}$. Thus $\Phi_{x,y}\theta = \Phi_{x\lambda,y\lambda}$.

Comparing with (24), θ and $\bar{\lambda}$ have the same effect on the one-element relations $\Phi_{x,y}$, and both map $\emptyset$ onto $\emptyset$. By Lemma 6, they coincide. Thus the given endomorphism θ is equal to the inner automorphism $\bar{\lambda}$ of $\mathscr{B}_X$.

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