Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 2, 179–183

Persistent URL: http://dml.cz/dmlcz/100959

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of Czechoslovak Academy of Sciences at Prague
V. *20* (*95*), PRAHA 18. 6. 1970, No 2

TOLERANCE IN ALGEBRAIC STRUCTURES

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(Received January 8, 1968)

E. C. ZEEMAN [3] introduces the concept of tolerance on a set as a reflexive and symmetric relation. M. A. ARBIB [1, 2] applies this concept in the theory of automata, B. ZELINKA [4] in the theory of graphs. Here we shall introduce this concept into abstract algebra.

As mentioned above, the tolerance is a reflexive and symmetric relation on a set. If on a set M a tolerance ξ is given, we speak about the tolerance space (M, ξ).

Now let an algebraic structure $\mathfrak{A} = (A, \mathscr{F})$ be given. (By the symbol A we denote the set of elements of the algebraic structure, by the symbol $\mathscr{F}$ the set of operations on this set.) On the set A let a tolerance ξ be given. We say that $\mathfrak{A}$ is a ξ-tolerance algebraic structure, if and only if the following holds: Let $f \in \mathscr{F}$ and let f be an n-ary operation. If we have $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ of A such that $(x_i, y_i) \in \xi$ for $i = 1, \ldots, n$, then also

$$(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \xi .$$

We shall investigate the most important types of algebraic structures — groups, semigroups, rings, fields and lattices.

1. GROUPS

Theorem 1. *Let G be a group, let a tolerance ξ be given on its set of elements. If G is a ξ-tolerance semigroup with respect to its multiplication, it is also a ξ-tolerance group.*

Proof. The fact that G is a ξ-tolerance semigroup means that $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \in \xi$ implies $(x_1x_2, y_1y_2) \in \xi$ for arbitrary elements x_1, x_2, y_1, y_2 of G. To prove that G is a ξ-tolerance group it is necessary and sufficient to prove that $(x, y) \in \xi$ implies $(x^{-1}, y^{-1}) \in \xi$ for arbitrary elements x, y of G.

Thus let us have arbitrary two elements x, y of G and let $(x, y) \in \xi$. The unit element of the group G will be denoted by e. As ξ is reflexive, we have $(x^{-1}, x^{-1}) \in \xi$. From the relations $(x, y) \in \xi$, $(x^{-1}, x^{-1}) \in \xi$ we obtain $(xx^{-1}, yx^{-1}) \in \xi$, therefore $(e, yx^{-1}) \in \xi$. But there is also $(y^{-1}, y^{-1}) \in \xi$ and therefore $(y^{-1}e, y^{-1}yx^{-1}) \in \xi$ which means $(y^{-1}, x^{-1}) \in \xi$. As ξ is symmetric, also $(x^{-1}, y^{-1}) \in \xi$ holds.

Theorem 2. *Let G be a ξ-tolerance group, e its unit element. The set H of the elements $x \in G$ such that $(e, x) \in \xi$ is a normal subgroup of the group G.*

Proof. $x \in H$, $y \in H$ means that $(e, x) \in \xi$, $(e, y) \in \xi$. This implies $(e, xy) \in \xi$, therefore with any two elements of H also their product is contained in H. The reflexivity of ξ implies $(x^{-1}, x^{-1}) \in \xi$ and therefore this relation together with $(e, x) \in \xi$ implies $(x^{-1}, e) \in \xi$ or $(e, x^{-1}) \in \xi$, which means $x^{-1} \in H$. With any element of H also its inverse element is contained in H and H is therefore a subgroup of the group G. Now let $z \in G$. There is again $(z, z) \in \xi$, therefore $(e, x) \in \xi$ and $(z, z) \in \xi$ imply $(z, xz) \in \xi$. But there is also $(z^{-1}, z^{-1}) \in \xi$ and therefore $(z^{-1}, z^{-1}) \in \xi$ and $(z, xz) \in \xi$ imply $(e, z^{-1}xz) \in \xi$ or $z^{-1}xz \in H$ for any $x \in H$ and $z \in G$. The subgroup H is therefore a normal subgroup of G.

As it was already mentioned in [4], a tolerance ξ on a set M can be represented by a graph Ξ, the so-called graph of tolerance, whose vertex set is M and two vertices $x \in M$, $y \in M$ are joined by an edge in Ξ if and only if $(x, y) \in \xi$. We shall prove a theorem about the graph of tolerance of a ξ-tolerance group.

Theorem 3. *Let G be a ξ-tolerance group. The graph Ξ of the tolerance ξ consists of pairwise isomorphic connected components which are complete graphs.*

Proof. According to Theorem 2, the set H of all elements $x \in G$ such that $(e, x) \in \xi$ is a normal subgroup of the group G. Let $x \in H$, $y \in H$. This means that $(e, x) \in \xi$, $(e, y) \in \xi$. As ξ is a symmetric relation, we have also $(x, e) \in \xi$ which together with $(e, y) \in \xi$ implies $(x, y) \in \xi$. Therefore the subgraph of the graph Ξ generated by the set H is a complete graph. Let $z \in G$ and consider the class zH in the group G. Let $x' \in zH$, $y' \in zH$. This means that $x' = zx$, $y' = zy$, where $x \in H$, $y \in H$. As x and y are of H, there is $(x, y) \in \xi$. As ξ is reflexive, $(z, z) \in \xi$ and this together with $(x, y) \in \xi$ implies $(zx, zy) \in \xi$, thus $(x', y') \in \xi$. Therefore also the subgraph of the graph Ξ generated by the class zH is a complete graph. Now let us have two elements z_1, z_2 of G such that $z_1H \neq z_2H$. Let $x_1 \in z_1H$, $x_2 \in z_2H$. This means that $x_1 = z_1y_1$, $x_2 = z_2y_2$, where $y_1 \in H$, $y_2 \in H$. Assume that $(x_1, x_2) \in \xi$. This means $(z_1y_1, z_2y_2) \in \in \xi$. The relations $(z^{-1}, z^{-1}) \in \xi$, $(z_1y_1, z_2y_2) \in \xi$ imply $(y_1, z_1^{-1}z_2y_2) \in \xi$. This relation together with $(y^{-1}, y^{-1}) \in \xi$ implies $(e, z_1^{-1}z_2y_2y_1^{-1}) \in \xi$ and therefore $z_1^{-1}z_2y_2y_1^{-1} \in H$. As H is a subgroup of the group G and the elements y_1, y_2 belong to it, also the element $(z_1^{-1}z_2y_2y_1^{-1})\, y_1y_2^{-1} = z_1^{-1}z_2$ belongs to H. But then $z_2 \in z_1H$ (because $z_2 = z_1(z_1^{-1}z_2)$ and $z_1^{-1}z_2 \in H$) and therefore $z_1H = z_2H$, which is a contradiction with the assumption that $z_1H \neq z_2H$. Therefore elements of different classes of the group G according to H are not joined by edges in Ξ. Any class zH generates a connected component of the graph Ξ which is a complete graph. All connected components of Ξ have the same number of vertices, therefore they are pairwise isomorphic.

2. SEMIGROUPS

Theorem 4. *Let S be a ξ-tolerance semigroup, T its subsemigroup. The set ξT of elements x of S such that $(x, x') \in \xi$ where $x' \in T$, is a subsemigroup of S.*

Proof. Let $x \in \xi T$, $y \in \xi T$. This means that there exist $x' \in T$, $y' \in T$ so that $(x, x') \in \in \xi$, $(y, y') \in \xi$. These relations imply $(xy, x'y') \in \xi$. As T is a semigroup, there is $x'y' \in T$ and therefore $xy \in \xi T$ and ξT is also a semigroup.

Corollary 1. *Let p be an idempotent of a ξ-tolerance semigroup S. The set of the elements x such that $(p, x) \in \xi$ is a subsemigroup of the semigroup S.*

Theorem 5. *Let S be a ξ-tolerance semigroup, T its right (or left, or two-sided) ideal. The set ξT of the elements x of S such that $(x, x') \in \xi$ where $x' \in T$, is a right (or left, or two-sided, respectively) ideal of the semigroup S.*

Proof. Let $x \in \xi T$, let T be a left ideal of S. There exists $x' \in T$ such that $(x, x') \in \xi$. Now let $y \in S$. As the relation ξ is reflexive, we have $(y, y) \in \xi$. The relations $(x, x') \in \xi$, $(y, y) \in \xi$ imply $(xy, x'y) \in \xi$. But $x'y \in T$ because $x' \in T$ and T is a left ideal. Therefore $xy \in \xi T$ and ξT is also a left ideal of the semigroup S. Analogously for right and two-sided ideals.

Corollary 2. *Let o be a zero element of a ξ-tolerance semigroup S. The set of elements x such that $(o, x) \in \xi$ is a two-sided ideal of the semigroup S.*

Theorem 6. *Let S be a ξ-tolerance semigroup, let Ξ be the graph of the tolerance ξ. Let p be an idempotent of the semigroup S. Then for any positive integer n the set of elements whose distance from p in the graph Ξ is less than or equal to n is a subsemigroup of S. Also the set of vertices of the connected component of the graph Ξ containing p is a subsemigroup of S.*

Proof. Let $x \in S$, let the distance of elements x and p in the graph Ξ be a positive integer m. This means that there exist elements $y_1, \ldots, y_{m+1}$ of S such that $y_1 = p$, $y_{m+1} = x$ and $(y_i, y_{i+1}) \in \xi$ for $i = 1, \ldots, m$. Now let us have $m \leqq n$. The assertion of the theorem will be proved by induction. If $n = 1$, the assertion follows from Corollary 1. Let the assertion hold for $n = k - 1$. The set of elements whose distance from p in Ξ is less than or equal to k forms a subsemigroup T_{k-1} of the semigroup S.

The set T_k of the elements whose distance from p in Ξ is less than or equal to k is evidently the set of elements x such that $(x, y) \in \xi$ where $y \in T_{k-1}$, therefore according to Theorem 4 it is also a subsemigroup of the semigroup S, q. e. d. The set T of the connected component of the graph Ξ containing p is evidently $\bigcup_{i=1}^{\infty} T_i$. If we have two elements $x \in T$, $y \in T$, there is $x \in T_i$, $y \in T_j$, where i, j are some positive integers. If $k = \max(i, j)$, then obviously $x \in T_k$, $y \in T_k$. Therefore also $xy \in T_k \subset T$.

3. RINGS AND FIELDS

Theorem 7. *Let R be a ξ-tolerance ring, let O be its zero element. The set R_0 of elements $x \in R$ such that $(O, x) \in \xi$ is an ideal of the ring R.*

Proof. As O is the unit element of the additive group of the ring R, the set R_0 is according to Theorem 2 a normal subgroup of this group. And as O is at the same time the zero element of the multiplicative semigroup of the ring R, the set R_0 is according to Corollary 2 an ideal of this semigroup. Therefore R_0 is an ideal of the ring R.

Theorem 8. *Let R be a ξ-tolerance ring. The graph Ξ of the tolerance ξ consists of pairwise isomorphic connected components which are complete graphs.*

Proof is the same as that of Theorem 2.

Theorem 9. *Let T be a ξ-tolerance field. The graph Ξ of the tolerance ξ is either a complete graph with loops, or each of its components is formed by a single vertex with a loop.*

Proof. The unique ideals of a field are the field itself and its zero element. In the first case the graph from Theorem 8 has only one component and therefore it is a complete graph with loops. In the second case each component must consist of a single vertex (obviously with a loop).

Theorem 9 may be expressed also in the following way:

Let T be a ξ-tolerance field. Then ξ is either the universal relation (i.e. $(x, y) \in \xi$ for arbitrary two elements x, y), or the identity relation (i.e. $(x, y) \in \xi$ if and only if $x = y$).

Theorem 10. *Let R be a ξ-tolerance ring, let 0 be its zero element, 1 its unit element. Let $(0, 1) \in \xi$. Then the graph Ξ of the tolerance ξ is a complete graph with loops.*

Proof. According to Theorem 8 we have $1 \in R_0$ where R_0 is an ideal of the ring R. But the unique ideal of the ring R containing the unit element is the ring R itself.

Therefore the graph Ξ contains a single connected component and is a complete graph with loops.

Remark. In an algebraic structure we admit also partial operations, therefore a field is also an algebraic structure.

4. LATTICES

Theorem 11. *Let L be a ξ-tolerance lattice and let $x \in L$. The set $L(x)$ of the elements $y \in L$ such that $(x, y) \in \xi$ is a sublattice of the Lattice L.*

Proof. Let $(x, y) \in \xi$, $(x, z) \in \xi$. Directly from the definition it follows that $(x \vee x, y \vee z) \in \xi$, therefore $(x, y \vee z) \in \xi$ and at the same time $(x \wedge x, y \wedge z) \in \xi$, therefore $(x, y \wedge z) \in \xi$.

Assume that any of the lattices $L(x)$ for $x \in L$ has the least element and the greatest one. Denote the greatest element of $L(x)$ by $M(x)$ and the least element of $L(x)$ by $m(x)$.

Theorem 12. *The mapping M which to any element $x \in L$ assigns the element $M(x)$ is an isotone mapping of the lattice L into itself.*

Proof. Let $x \in L$, $y \in L$, $x \leqq y$. This means that $x \vee y = y$. As $(x, M(x)) \in \xi$, $(y, M(y)) \in \xi$, there is also $(x \vee y, M(x) \vee M(y)) \in \xi$. Therefore $M(x) \vee M(y) \in L(x \vee y)$ and thus $M(x) \vee M(y) \leqq M(x \vee y)$. But $x \vee y = y$, therefore $M(x) \vee M(y) \leqq M(y)$. As on the left-hand side we have a join, there must be also $M(x) \vee M(y) \geqq M(y)$ and therefore $M(x) \vee M(y) = M(y)$ which means that $M(x) \leqq M(y)$.

Theorem 13. *The mapping m which to any element $x \in L$ assigns the element $m(x)$ is an isotone mapping of the lattice L into itself.*

Proof is dual to that of Theorem 12.

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