Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 19 (1969), No. 3, 515–517

Persistent URL: http://dml.cz/dmlcz/100918

Czechoslovak Mathematical Journal, 19 (94) 1969, Praha

REMARK ON INVOLUTIVE SUBSPACES AND REGULAR BASES

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(Received November 27, 1968)

Let us consider a vector space V of dimension n over the field $\mathbf{T}$ (of real or complex numbers) and let V^* be its dual. Further let $(v_1, \ldots, v_n)$ be a basis of V and $(v^1, \ldots, v^n)$ its dual basis in V^*. For every integer $l \geqq 1$ and $v \in V$ we introduce the mapping

$$\delta_v : S^l V^* \to S^{l-1} V^*$$

$S^l V^*$ has the basis consisting of the elements of the form $v^{j_1} \circ \ldots \circ v^{j_l}$, where $(j_1, \ldots \ldots, j_l)$ takes values in the set $\Xi(l)$ of all l-tuples of integers such that $1 \leqq j_1 \leqq \ldots \ldots \leqq j_l \leqq n$. It is clearly sufficient to define δ_v only for the elements of this basis. We set

$$\delta_v(v^{j_1} \circ \ldots \circ v^{j_l}) = \sum_{k=1}^{l} \langle v, v^{j_k} \rangle \, v^{j_1} \circ \ldots \circ \hat{v}^{j_k} \circ \ldots \circ v^{j_l} .$$

It is easy to show that δ_v does not depend on the choice of basis in V.

Lemma 1. *For any $v, v' \in V$ there is $\delta_v \delta_{v'} = \delta_{v'} \delta_v$.*

Proof is easy.

Lemma 2. *Let $f \in S^l V^*$, $\delta_{v_{r+1}} f = \ldots = \delta_{v_n} f = 0$. Then $f \in S^l V_r^* \subset S^l V^*$, where V_r^* is the subspace spanned by the vectors $v^1, \ldots, v^r$.*

Proof. f can be expressed in the form $f = \sum_{(j_1, \ldots, j_l) \in \Xi(l)} a_{j_1 \ldots j_l} (v^{j_1} \circ \ldots \circ v^{j_l})$. It follows from the condition $\delta_{v_n} f = 0$ that if one of the indices $j_1, \ldots, j_l$ equals n then $a_{j_1 \ldots j_l} = 0$ and thus $f \in S^l V_{n-1}^*$. The remainder of the proof is easy.

Let W be a vector space of dimension m over $\mathbf{T}$. For $l \geqq 1$ and $v \in V$ we introduce a mapping $W \otimes S^l V^* \to W \otimes S^{l-1} V^*$ that we shall also denote by δ_v. Let $(w_1, \ldots \ldots, w_m)$ be a basis of W. $W \otimes S^{l-1} V^*$ has the basis consisting of the elements $w_k \otimes \otimes (v^{j_1} \circ \ldots \circ v^{j_l})$, $1 \leqq k \leqq m$, $(j_1, \ldots, j_l) \in \Xi(l)$. It is sufficient to define δ_v for these elements. We set

$$\delta_v(w_k \otimes (v^{j_1} \circ \ldots \circ v^{j_l})) = w_k \otimes (\delta_v(v^{j_1} \circ \ldots \circ v^{j_l}))$$

We can again easily find out that δ_v does not depend on the choice of bases.

Now let $g^k \subset W \otimes S^k V^*$ be a subspace and let $pg^k \subset W \otimes S^{k+1} V^*$ be its prolongation defined by

$$pg^k = (g^k \otimes V^*) \cap (W \otimes S^{k+1} V^*)$$

(see [1]). For any subset $M \subset V$ we denote $g^k_M = \{f \in g^k;\ \delta_v f = 0 \text{ for all } v \in M\}$. Obviously if $V^c_r \subset V$ denotes the subspace spanned by $v_{r+1}, \ldots, v_n$, then $g^k_{V_r{}^c} = g^k_{v_{r+1}} \cap \ldots \cap g^k_{v_n}$.

Lemma 3. *Let $V' \subset V$ be a subspace. There is $p(g^k_{V'}) = (pg^k)_{V'}$.*

Proof. Let us take such basis $(v_1, \ldots, v_n)$ of V that $V' = V^c_r$ for some $1 \leqq r \leqq n$. It can be easily seen that $p(g^k_{V_r{}^c}) = p(g^k_{v_{r+1}}) \cap \ldots \cap p(g^k_{v_n})$ and therefore it is sufficient to prove for all $1 \leqq i \leqq n$ the equality $p(g^k_{v_i}) = (pg^k)_{v_i}$. So that not to complicate the notation we shall do the proof for $i = n$.

a) Let $f \in (pg^k)_{v_n} = [(g^k \otimes V^*) \cap (W \otimes S^{k+1} V^*]_{v_n}$. As $f \in g^k \otimes V^*$ there is $f = f_1 \otimes v^1 + \ldots + f_n \otimes v^n$, where $f_1, \ldots, f_n \in g^k$. But because of $\delta_{v_n} f = 0$ owing to lemma 2 there must be $f \in W \otimes S^{k+1} V^*_{n-1}$, i.e. $f_n = 0$; $f_1, \ldots, f_{n-1} \in g^k \cap (W \otimes S^k V^*_{n-1})$. Therefore we have $f = f_1 \otimes v^1 + \ldots + f_{n-1} \otimes v^{n-1}$, where $f_1, \ldots, f_{n-1} \in g^k_{v_n}$ and thus $f \in p(g^k_{v_n})$.

b) Let $f \in p(g^k_{v_n}) = (g^k_{v_n} \otimes V^*) \cap (W \otimes S^{k+1} V^*)$. f can again be expressed in the form $f = f_1 \otimes v^1 + \ldots + f_n \otimes v^n$, where $f_1, \ldots, f_n \in g^k_{v_n}$. But as $g^k_{v_n} \subset W \otimes S^k V^*_{n-1}$ and $f \in W \otimes S^{k+1} V^*$, it is obviously $f_n = 0$. Thus $\delta_{v_n} f = 0$, i.e. $f \in (pg^k)_{v_n}$ and this finishes the proof.

Let now g^k be a subspace of $W \otimes S^k V^*_r$. Let us set as usual $pg^k = (g^k \otimes V^*) \cap (W \otimes S^{k+1} V^*)$ and moreover $p_1 g^k = (g^k \otimes V^*_r) \cap (W \otimes S^{k+1} V^*)$. We have

Lemma 4. *There is $p_1 g^k = pg^k$.*

Proof. Obviously $p_1 g^k \subset pg^k$ and therefore it remains to prove the converse inclusion. We have again $f = f_1 \otimes v^1 + \ldots + f_n \otimes v^n$, where $f_1, \ldots, f_n \in g^k$. Regarding the inclusion $g^k \subset W \otimes S^k V^*_r$ and the fact that $f \in W \otimes S^{k+1} V^*$, it is clear that $f_{r+1} = \ldots = f_n = 0$ and therefore $f \in p_1 g^k$.

Further let us suppose that $g^k \subset W \otimes S^k V^*$ is an involutive subspace (see [1]). Let $V' \subset V$ be a subspace, $\dim V' = r$. We shall seek for a regular basis of V such that its first r vectors lie in V'. Let $(v''_1, \ldots, v''_n)$ be a basis of V such that the vectors $v''_1, \ldots, v''_r$ span V'. The set of all regular bases of an involutive subspace is dense in the Stiefel manifold of all bases (see [2], § 6, p. 31). Therefore we can find a regular basis $(v'_1, \ldots, v'_n)$ of V such that $(v''_1, \ldots, v''_r, v'_{r+1}, \ldots, v'_n)$ is a basis of V. For the sake of simplicity we shall denote this last basis by $(v_1, \ldots, v_n)$.

For any $m \geqq k$ let $g^m \subset W \otimes S^m V^*$ be the subspace defined by $g^m = p^{m-k} g^k$.

As $(v'_1, \ldots, v'_n)$ is a regular basis, the following mappings

$$\begin{aligned} \delta_{v_n} &: g^{m+1} \to g^m \\ \delta_{v_{n-1}} &: g^{m+1}_{V_{n-1}c} \to g^m_{V_{n-1}c} \\ &\vdots \\ \delta_{v_{r+1}} &: g^m_{V_{r+1}c} \to g^m_{V_{r+1}c} \end{aligned}$$

are surjective for all $m \geqq k$. According to lemma 2 there is $g^k_{V_rc} \subset W \otimes S^k V_r^*$. Moreover with respect to the fixed basis $(v_1, \ldots, v_n)$, V_r^* is canonically isomorphic with the dual of V_r (V_r is spanned by $v_1, \ldots, v_r$). Finally according to lemma 4 it makes no difference if we prolong $g^k_{V_rc}$ as a subspace of $W \otimes S^k V^*$ or as a subspace of $W \otimes V_r^*$. According to the well-known prolongation theorem there exists $k_0 \geqq k$ such that $p^{k_0-k}(g^k_{V_rc})$ is involutive. In other words there exists a regular basis $(\bar{v}_1, \ldots$ $\ldots, \bar{v}_r)$ of V_r for $p^{k_0-k}(g^k_{V_rc})$, i.e. such basis that the following mappings

$$\begin{aligned} \delta_{\bar{v}_r} &: p^{m+1}(g^k_{V_rc}) \to p^m(g^k_{V_rc}) \\ \delta_{\bar{v}_{r-1}} &: (p^{m+1}(g^k_{V_rc}))_{\{\bar{v}_r\}} \to (p^m(g^k_{V_rc}))_{\{\bar{v}_r\}} \\ &\vdots \\ \delta_{\bar{v}_1} &: (p^{m+1}(g^k_{V_rc}))_{\{\bar{v}_2,\ldots,\bar{v}_r\}} \to (p^m(g^k_{V_rc}))_{\{\bar{v}_2,\ldots,\bar{v}_r\}} \end{aligned}$$

are surjective for all $m \geqq k_0 - k$. Here $\{\bar{v}_i, \ldots, \bar{v}_r\}$ denotes the subspace of V_r spanned by the vectors $\bar{v}_i, \ldots, \bar{v}_r$. But according to lemma 3 it follows from the last assertion, that the following mappings

$$\begin{aligned} \delta_{\bar{v}_r} &: g^{m+1}_{V_rc} \to g^m_{V_rc} \\ \delta_{\bar{v}_{r-1}} &: g^{m+1}_{V_{r-1}c} \to g^m_{V_{r-1}c} \\ &\vdots \\ \delta_{\bar{v}_1} &: g^{m+1}_{V_1c} \to g^m_{V_1c} \end{aligned}$$

are surjective for all $m \geqq k_0$. And from this fact it follows immediately that $(\bar{v}_1, \ldots$ $\ldots, \bar{v}_r, v_{r+1}, \ldots, v_n)$ is a regular basis for g^{k_0}. Thus we have proved the following

Theorem. *Let $g^k \subset W \otimes S^k V^*$ be an involutive subspace. Let $V' \subset V$ be a subspace of dimension r. Then there exist $k_0 \geqq k$ and a basis $(v_1, \ldots, v_n)$ of V such that*

a) *$(v_1, \ldots, v_n)$ is a regular basis for $g^{k_0} \subset W \otimes S^{k_0} V^*$,*

b) *$v_1, \ldots, v_r \in V'$.*

References

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