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ON A CLASS OF SEMI-GROUP ALGEBRAS

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E. HEWITT and H. S. ZUCKERMAN [1] initiated the study of certain Banach algebras based on commutative semigroups. In this note an attempt is made to combine the approach of [1] with the functional-analysis approach of [2, 3]; besides this, two results are given that properly belong to the theory of Banach algebras; and finally, a few results in [1] are obtained by variant methods – the proof of Lemma 1 is called to the reader's attention.

Let H be an additive semi-group which, in order to avoid certain annoying complications, is reduced in the sense that

(1) $$2x = 2y\,, \quad 3x = 3y \Rightarrow x = y\,, \quad x, y \in H\,.$$

(This is not quite the standard formulation, cf. [1; 5.8].) Besides this, H is provided with a real functional $\omega \geqq 0$ such that

(2) $$\omega(x + y) \leqq \omega(x)\,\omega(y)\,, \quad x, y \in H\,.$$

(3) $$\omega(z) = 0\,, \quad \text{for at most one } z \text{ in } H\,.$$

(4) $$\lim_{n\to\infty} \omega(nx)^{1/n} \equiv \omega_\infty(x) > 0 \quad \text{if} \quad \omega(x) > 0\,.$$

The algebra $A \equiv A(H; \omega)$ is the space of complex functions on H (or on $H \sim \{z\}$) with finite norm $\|a\| \equiv \sum_H \omega(x)\,|a(x)| < \infty$. Multiplication in convolution is H, that is,

$$(a * b)(x) \equiv \sum\sum\{a(y_1)\,b(y_2) : y_1 + y_2 = x\}\,.$$

The axioms for a Banach algebra are verified as in [1; 2.4] and reduce to the latter when $\omega \equiv 1$. It will be convenient to write $\delta_x(y) = 0$ if $y \neq x$, $\delta_x(x) = 1$.

Elements x_1 and x_2 of H are called *strongly inverse* if

(5) $$2x_1 + x_2 = x_1\,, \quad 2x_2 + x_1 = x_2\,,$$

and

(6) $$\omega_\infty(x_1)\,\omega_\infty(x_2) = 1\,.$$

Theorem 1. *These properties are equivalent, for $a \in A$:*

(i) *a vanishes off the set of strongly invertible elements of H.*

(ii) *For each homomorphism T of A into the algebra of functions holomorphic in a connected domain of complex numbers, $T(a)$ is constant in D.*

Theorem 2. *In order that A admit a symmetric involution it is necessary that each element $\neq z$ of H be strongly invertible and* (in (6))

$$\omega(x_1)\,\omega_\infty(x_2) \leqq M\,\omega(x_2)\,\omega_\infty(x_1)\,. \tag{7}$$

Theorem 3. *If the Gelfand space of A is compact, A contains a unit — to be described in the proof,* [1; 8.14].

Lemma 1. *If $x \neq y$ in H then $\theta(x) \neq \theta(y)$ for some homomorphism θ of H into the complex disk $\{|\lambda| \leqq 1\}$,* [1; 5.6].

Proof. For any integer $N \geqq 2$

$$(\delta_x - \delta_y)^N = \sum_{k=0}^{N} \binom{N}{k} (-1)^k\, \delta_{ky+(N-k)x}\,,$$

so the norm of $(\delta_x - \delta_y)^N$ is an *integer*; if, then, $\delta_x - \delta_y$ is not nilpotent, its spectral radius is at least 1. In that case we have only to take a complex homomorphism with $\Phi(\delta_x) \neq \Phi(\delta_y)$; because $\|\Phi\| \leqq 1$ we can write $\theta(w) = \Phi(\delta_w)$, $w \in H$.

In the contrary case $(\delta_x - \delta_y)^N = 0$ for $N \geqq N_0$. But if p is a prime $\geqq N_0$ (and $p > 2$)

$$(\delta_x - \delta_y)^p \equiv \delta_{px} - \delta_{py} \ (\text{modulo } p)$$

so that $px = py$. It follows that $Nx = Ny$ for all $N \geqq N_1$, and then, via (1), $x = y$.

Lemma 2. *$A(H; \omega)$ is semi-simple.*

Proof. Let φ be any homomorphism of H into the complex numbers, subject to the inequality $|\varphi| \leqq \omega$. Then $a \to \varphi\,.\,a$, $a \in H$, is a homomorphism of A into $l_1(H)$. Here $\varphi\,.\,a$ is the ordinary product; note that $\varphi(z)\,.\,a(z) = 0$ under any convention. But $l_1(H)$ is semi-simple by Lemma 1 and [1; 3.5]. (It may be remarked that Theorem 3.4 of [1] is a straightforward consequence of Gelfand-Neumark theorem for commutative B^*-algebras, [4; p. 190].) The proof is complete as soon as it is established that for each x in H there is a homomorphism φ, as above, so that $|\varphi(x)| = \omega_\infty(x) > 0$. But this is an immediate consequence of the fact that $\omega_\infty(x)$ is the spectral radius of δ_x in the algebra A.

Lemma 3. *Let x_1 be an element of H, $x_1 \neq z$, and without a strong inverse, cf.* (5), (6). *Then there is a homomorphism ψ of H into $[0, \infty)$ such that*

$$0 < \psi(x_1) < \omega_\infty(x_1)\,, \tag{8}$$

and

$$(9) \qquad 0 \leqq \psi \leqq \omega \quad \text{in} \quad H .$$

Proof. Let φ be the homomorphism constructed at the end of the previous proof, and θ any bounded homomorphism of H, as in Lemma 1. Then we can choose $\psi = |\varphi| \,.\, |\theta|$ if $0 < |\theta(x_1)| < 1$. This homomorphism θ exists unless the equation

$$(10) \qquad y + (n+1)\, x_1 = n x_1 \,, \quad y \in H \,, \quad n \geqq 1 \,,$$

has at least one solution, (and plainly if $0 < |\theta(x_1)| < 1$, $\theta^m(x_1) \to 0$ as $m \to \infty$) [3]. Using (1) we find that $x_2 = 2y + x_1$ is a solution of (5). Now let ψ be a homomorphism of H into $[0, \infty)$ meeting (9) and $\psi(x_2) = \omega_\infty(x_2) > 0$. Then $\psi(x_1)\, \psi(x_2) = 1$ so $\psi(x_1) = \omega_\infty(x_2)^{-1} < \omega_\infty(x_1)$. This completes the proof.

Proof of Theorem 1. If x_1 and x_2 are strongly inverse, and Φ is any complex homomorphism of A, with $\Phi(\delta_{x_1}) \neq 0$, then $\Phi(\delta_{x_1})\, \Phi(\delta_{x_2}) = 1$, $|\Phi(\delta_{x_i})| \leqq \omega_\infty(x_i)$, $i = 1, 2$. For any fixed λ in the domain D, $a \to (T(a))\,(\lambda)$ is a homomorphism, and so $T(\delta_{x_1})$ and $T(\delta_{x_2})$ are constant in D by the maximum principle.

To prove the reverse let ψ_1 and ψ_2 be homomorphisms of H into $[0, \infty)$, bounded by ω, and θ any complex homomorphism bounded in modulus by 1. Write

$$(T(a))\,(\lambda) = \sum_H \theta(x)\, \psi_1^\lambda(x)\, \psi_2(x)^{1-\lambda}\, a(x) \,,$$

for $a \in A$, $0 < \operatorname{Re} \lambda < 1$. If $T(a)$ is constant, then as in the proof of Lemma 2, $\psi_1^{1/2} \,.\, \psi_2^{1/2} \,.\, a = \psi_1^{3/4} \,.\, \psi_2^{1/4} \,.\, a$. By Lemma 3 a vanishes off the strongly invertible elements.

Proof of Theorem 2. Inasmuch as a real function holomorphic in D is constant, the first statement follows from Theorem 1. Since any involution of a semi-simple algebra is continuous, and $\omega_\infty(x_1)^{-1}\, \delta_{x_1}$ is plainly conjugate to $\omega_\infty(x_2)^{-1}\, \delta_{x_2}$, the inequality (7) is established.

Proof of Theorem 3. By assumption there are a number $\varepsilon > 0$ and elements $y_1, \ldots, y_m$ of H so that for any homomorphism ψ, as in (9)

$$(11) \qquad 0 \leqq \psi(y_i) \leqq \varepsilon \,, \quad 1 \leqq i \leqq m \Rightarrow \psi \equiv 0 \,.$$

We suppose that if any of the $m y_i$'s is omitted, the implication in (11) becomes false for any $\varepsilon > 0$ whatever. Thus there is a homomorphism $\varphi \leqq \omega$ of H with $\varphi(y_1) > \varepsilon$, $\varphi(y_i) = 0$, $2 \leqq i \leqq m$. If, now, $\varphi'(y_1) < \varepsilon^2(1 + \omega(y_1))^{-2}$, then $(\varphi'\varphi)^{1/2}$ meets all the requirements in (11), so $\varphi'\varphi \equiv 0$, $\varphi'(y_1) = 0$. Thus equation (10) with y_1 in place of x_1 is solvable, and $e_1 + y_1 = y_1$ for some idempotent e_1. (In the case of (10), the idempotent would be $x_1 + x_2$.) Plainly $\psi(e_1) < 1 \Rightarrow \psi(y_1) = 0$; by the same argument each y_i can be replaced in (11) by an idempotent. The unit of A is then the circle product [4; p. 16] of the idempotents δ_{y_i}, $1 \leqq i \leqq m$, for by what has gone before, this "large" idempotent is contained in no modular maximal ideal of A.

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