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FOURIER L_2-TRANSFORM OF DISTRIBUTIONS

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LAURENT SCHWARTZ [1] has defined Fourier transform $\mathscr{F}$ of elements of the space $\mathscr{S}'$ dual to $\mathscr{S}$, the Fréchet space of "infinitely differentiable and rapidly decreasing functions". He has proved that $\mathscr{F}$ is an automorphism of $\mathscr{S}'$. Instead of $\mathscr{S}$ we define a sequence of Hilbert spaces $L_2 \supset L_2^1 \supset L_2^2 \supset \ldots \supset \mathscr{S}$ and their duals $L_2 \subset L_2^{-1} \subset \subset L_2^{-2} \subset \ldots \subset \mathscr{S}'$. Then it turns out that $\bigcup_{k=1}^{\infty} L_2^{-k} = \mathscr{S}'$ and Fourier transform $\mathscr{F}$ is a unitary automorphism on every L_2^{-k}, $k = 0, 1, 2, \ldots$ This procedure enables us also to define more rich spaces of operators of multiplication and convolution.

We make use of the following notation. Symbols R^n, C, L_1, L_2, are, respectively, the n-dimensional Euclidean space, the set of all complex numbers, the space of absolutely, and of square integrable functions $f : R^n \to C$. In R^n we use the inner product $(x, y) = \sum x_j y_j$, $x, y \in R^n$. By α we consistently denote a multiindex, i.e. an element $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$ whose components are non-negative integers. Given a multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then we write for brevity $|\alpha| = \sum_{j=1}^{n} \alpha_j$, $x^\alpha = \prod_{j=1}^{n} x_j^{\alpha_j}$, where $x \in R^n$, $\mathrm{D}^\alpha = \partial^{|\alpha|}/(\partial x)^\alpha$.

By $\mathscr{D}$ we denote the set of all infinitely differentiable functions $f : R^n \to C$ with compact support. We say that a function $f : R^n \to C$ has a generalized derivative g of order α, if for all $\varphi \in \mathscr{D}$ we have $\int_{R^n} f \, \mathrm{D}^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_{R^n} g\varphi \, \mathrm{d}x$. We denote such function g by $\mathrm{D}^\alpha f$. If a function f continuous on R^n has a continuous (classical) derivative $\partial f/\partial x_1$ on R^n and a function g has a generalized derivative $\partial g/\partial x_1$, then fg has the generalized derivative $\partial(fg)/\partial x_1 = (\partial f/\partial x_1)\, g + f(\partial g/\partial x_1)$.

We start with Fourier transform defined for functions from L_1 as follows

$$(\mathscr{F}f)(\xi) = \int_{R^n} f(x) \exp(-2\pi i x, \xi) \, \mathrm{d}x \,, \quad f \in L_1 \,.$$

From Plancherel's theory it is well-known that if we take a sequence of functions $f_k \in L_1 \cap L_2$, $k = 1, 2, \ldots$, converging to $f \in L_2$ in the topology of L_2, then there exists a unique limit $F \in L_2$ (in the topology of L_2) of the sequence $\mathscr{F}f_k$, $k = 1, 2, \ldots$, and this limit is independent of the choice of functions f_k, $k = 1, 2, \ldots$ If we denote $F = \mathscr{F}f$, then $\mathscr{F} : L_2 \to L_2$ is a unitary automorphism. Let us recall the formulae:

$$(1) \qquad f, \mathrm{D}^\alpha f \in L_2 \Rightarrow \mathscr{F}(\mathrm{D}^\alpha f) = (2\pi i x)^\alpha . \mathscr{F}f ,$$

$$f, x^\alpha f \in L_2 \Rightarrow \mathscr{F}((-2\pi i x)^\alpha f) = \mathrm{D}^\alpha \mathscr{F}f .$$

The mapping $\mathscr{F}$ and its inverse $\mathscr{F}^{-1}$ are related by the identity

$$(2) \qquad \mathscr{F}^{-1}f = \overline{\mathscr{F}\bar{f}}, \quad f \in L_2 .$$

As Fourier image of a real even, resp. real odd function is a real even, resp. pure imaginary odd function, the identity (2) implies that

$$(3) \qquad (\mathscr{F}^2 f)(x) = f(-x), \quad x \in R^n, \quad f \in L_2 ;$$

hence, the identities

$$(4) \qquad \mathscr{F}^4 = \mathscr{I}, \quad \mathscr{F}^{-1} = \mathscr{F}^3 ,$$

where $\mathscr{I}$ is the identity operator, are valid on L_2.

Definition 1. For every integer $k \geqq 0$ we define the linear space L_2^k as follows

$$L_2^k = \left\{ f : R^n \to C; \int_{R^n} x^{2\alpha} |\mathrm{D}^\beta f|^2 \, \mathrm{d}x < +\infty, \; |\alpha| + |\beta| \leqq k \right\}.$$

For brevity we introduce the operator $\mathrm{D}_k = (1 + \sum_{j=1}^{n} (2\pi i x_j + \partial/\partial x_j))^k$, where $k \geqq 0$ is an integer. (Performing the indicated k-th power we must be careful because the operators $2\pi i x_j$ and $\partial/\partial x_j$, $j = 1, 2, \ldots, n$, are not commutative.) Evidently, for given $f \in L_2^k$ we have $\mathrm{D}_k f \in L_2$.

Let us denote, in any but fixed manner, the addends of the operator D_1 by $\mathrm{A}_0, \mathrm{A}_1, \ldots, \mathrm{A}_{2n}$, i.e. $\mathrm{D}_1 = \sum_{j=0}^{2n} \mathrm{A}_j$. For every $f, g \in L_2^k$ we define

$$[\mathrm{D}_k f, \mathrm{D}_k g] = \sum_{j_1, \ldots, j_k = 0}^{2n} (\mathrm{A}_{j_1} \ldots \mathrm{A}_{j_k} f) \overline{(\mathrm{A}_{j_1} \ldots \mathrm{A}_{j_k} g)} .$$

Then from Hölder's inequality it follows that

$$f, g \in L_2^k \Rightarrow [\mathrm{D}_k f, \mathrm{D}_k g] \in L_1 .$$

This enables us to define an inner product in every L_2^k, $k = 1, 2, \ldots$, by

$$(f, g)_k = \int_{R^n} [\mathrm{D}_k f, \mathrm{D}_k g] \, \mathrm{d}x , \quad f, g \in L_2^k ,$$

which converts L_2^k, $k = 1, 2, \ldots$, into Hilbert spaces. We denote $\|\cdot\|_k$ the norm in L_2^k generated by inner product $(.,.)_k$. It is evident that $L_2 = L_2^0 \supset L_2^1 \supset L_2^2 \supset \ldots$ and that the identity operator $\mathscr{I} : L_2^k \to L_2^l$, $k \geqq l$ is continuous.

Let us demonstrate the completeness of L_2^k. Take a fundamental sequence $f_m \in L_2^k$, $m = 1, 2, \ldots$. Then for each pair of multiindices α, β, $|\alpha| + |\beta| \leqq k$, the sequence $x^\beta D^\alpha f_m$, $m = 1, 2, \ldots$ has a limit $f_{\alpha\beta}$ in L_2. Take $\Delta > 0$ then, using Hölder's inequality, we get

$$\int_{|x|<\Delta} |x^\beta f_{\alpha 0} - f_{\alpha\beta}| \, dx \leqq \int_{|x|<\Delta} |x^\beta f_{\alpha 0} - x^\beta D^\alpha f_m| \, dx + \int_{|x|<\Delta} |x^\beta D^\alpha f_m - f_{\alpha\beta}| \, dx \leqq$$

$$\leqq \left(\int_{|x|<\Delta} x^{2\beta} \, dx \right)^{1/2} \left(\int_{|x|<\Delta} |f_{\alpha 0} - D^\alpha f_m|^2 \, dx \right)^{1/2} +$$

$$+ (2\Delta)^{n/2} \int_{|x|<\Delta} |x^\beta D^\alpha f_m - f_{\alpha\beta}|^2 \, dx \to 0$$

as $m \to \infty$. Hence $f_{\alpha\beta} = x^\beta f_{\alpha 0}$ a.e. in R^n.

For a multiindex α, $|\alpha| \leqq k$, $\varphi \in \mathscr{D}$, we have (using the inner product in L_2): $(f_{\alpha 0}, \varphi) = \lim_{m\to\infty} (D^\alpha f_m, \varphi) = (-1)^{|\alpha|} \lim_{m\to\infty} (f_m, D^\alpha \varphi) = (-1)^{|\alpha|} (f_{00}, D^\alpha \varphi)$. This means according to the definition of generalized derivatives that $f_{\alpha 0} = D^\alpha f_{00}$. Thus $f_{00} \in L_2^k$ and $\|f_m - f_{00}\|_k \to 0$ as $m \to \infty$.

Theorem 1. *Fourier transform $\mathscr{F} : L_2^k \to L_2^k$, $k \geqq 0$, integer, is a unitary automorphism.*

Proof. The linearity of $\mathscr{F}$ is trivial, the one-to-one property of $\mathscr{F}$ is well known. From (1) it follows immediately that $\mathscr{F} L_2^k = L_2^k$. Thus, it remains to prove only, Parseval's equality

$$(f, g)_k = (\mathscr{F}f, \mathscr{F}g)_k , \quad f, g \in L_2^k . \tag{5}$$

By definition,

$$[D_k \mathscr{F} f, D_k \mathscr{F} g] = \sum_{j_1,\ldots,j_k=0}^{2n} (A_{j_1} \ldots A_{j_k} \mathscr{F} f) \overline{(A_{j_1} \ldots A_{j_k} \mathscr{F} g)} .$$

Let us distinguish 3 cases:

$$\left.\begin{array}{ll} \text{a)} & A_{j_k} = 1 \\ \text{b)} & A_{j_k} = 2\pi i x_j \\ \text{c)} & A_{j_k} = \dfrac{\partial}{\partial x_j} \end{array}\right\} \text{ then } A_{j_k} \mathscr{F} f = \left\{\begin{array}{l} \mathscr{F} A_{j_k} f \\ \mathscr{F} \dfrac{\partial}{\partial x_j} f \\ -\mathscr{F}(2\pi i x_j f) . \end{array}\right.$$

In the case c) the commuting of operators A_{j_k} and $\mathscr{F}$ in the term $\overline{A_{j_1} \dots A_{j_k} \mathscr{F} g}$ produces also the factor (-1) so that we need not take this change of signs into consideration. Having commuted $\mathscr{F}$ with all operators $A_{j_k}, A_{j_{k-1}}, \dots, A_{j_1}$, we see that the operators $2\pi i x_j$ and $\partial/\partial x_j$, $j = 1, 2, \dots, n$, only have commuted in the summation $(1 + \sum_{j=1}^{n} (2\pi i x_j + \partial/\partial x_j))$ which does not change the operator D_k.

We have proved the identity

$$(6) \qquad [D_k \mathscr{F} f, D_k \mathscr{F} g] = [\mathscr{F} D_k f, \mathscr{F} D_k g], \quad f, g \in L_2^k .$$

Further, using Parseval's equality for L_2-functions, we get

$$(\mathscr{F} f, \mathscr{F} g)_k = \int_{R^n} [D_k \mathscr{F} f, D_k \mathscr{F} g] \, dx =$$

$$= \int_{R^n} [\mathscr{F} D_k f, \mathscr{F} D_k g] \, dx = \int_{R^n} [D_k f, D_k g] \, dx = (f, g)_k .$$

Definition 2. We denote L_2^{-k} the dual space of L_2^k, $k = 1, 2, \dots$ The norm of elements of L_2^{-k} we denote by $\|.\|_{-k}$. The elements of $\bigcup_{k=1}^{\infty} L_2^{-k}$ will be called distributions.

Proposition. *Assume $f \in L_2^{k+r}$, where $r = 1 + [\frac{1}{2}n]$. Then f has classical derivatives $D^\alpha f$ for all α, $|\alpha| \leqq k$, which are uniformly continuous on R^n. Moreover, if $|\beta| \leqq k - |\alpha|$ then*

$$\sup_{x \in R^n} (1 + 4\pi^2 |x|^2)^{|\beta|/2} |D^\alpha f(x)| \leqq \|f\|_{k+r} .$$

Proof. Let us take multiindices α, γ, $|\alpha| \leqq k$, $|\gamma| \leqq r$. Then according to (1) we have $\mathscr{F}(D^{\alpha+\gamma} f) = (2\pi i \xi)^\gamma \mathscr{F}(D^\alpha f) \in L_2$. Using Hölder's inequality we get

$$\int_{R^n} |\mathscr{F} D^\alpha f| \, d\xi = \int_{R^n} |\mathscr{F} D^\alpha f| (1 + |\xi|^2)^{(r-r)/2} \, d\xi \leqq$$

$$\leqq \left(\int_{R^n} |\mathscr{F} D^\alpha f|^2 (1 + |\xi|^2)^r \, d\xi \right)^{1/2} \left(\int_{R^n} (1 + |\xi|^2)^{-r} \, d\xi \right)^{1/2} < +\infty .$$

Hence $\mathscr{F}(D^\alpha f) \in L_1$. According to Parseval's equality we have

$$\int_{R^n} f \bar{\varphi} \, dx = \int_{R^n} \mathscr{F} f \overline{\mathscr{F} \varphi} \, d\xi = \int_{R^n} \hat{f}(\xi) \int_{R^n} \overline{\varphi(x)} \, e^{2\pi i (x, \xi)} \, dx \, d\xi =$$

$$= \int_{R^n} \left(\int_{R^n} \mathscr{F} f(\xi) \, e^{2\pi i (x, \xi)} \, d\xi \right) \overline{\varphi(x)} \, dx$$

for each $\varphi \in \mathscr{S}$. It is possible only when $f(x) = \int_{R^n} \mathscr{F} f(\xi) \, e^{2\pi i (x, \xi)} \, d\xi$ for almost all

$x \in R^n$. As for all α, $|\alpha| \leqq k$, $\mathscr{F}(D^\alpha f) = (2\pi i\xi)^\alpha\, \mathscr{F}f \in L_1$ we can differentiate f up to the k-th order. The uniform continuity is then evident.

Take now a multindex β, $|\beta| \leqq k - |\alpha|$, and put $g = (1 + 4\pi^2|x|^2)^{|\beta|/2}\, D^\alpha f(x)$. Then we can write

$$|g(x)| = |(\mathscr{F}\mathscr{F}^{-1}g)(x)| \leqq \|\mathscr{F}^{-1}g\|_{L_1} = \int_{R^n} |(\mathscr{F}^{-1}g)(x)|\,(1 + 4\pi^2|x|^2)^{(r-r)/2}\,dx \leqq$$

$$\leqq \left(\int_{R^n} |\mathscr{F}^{-1}g|^2 (1 + 4\pi^2|x|^2)^r\,dx\right)^{1/2} \left(\int_{R^n} (1 + 4\pi^2|x|^2)^{-r}\,dx\right)^{1/2} \leqq \|\mathscr{F}^{-1}g\|_r =$$

$$= \|g\|_r \leqq \|f\|_{r+k}\,.$$

Corollary. $\bigcap_{k>0} L_2^k = \mathscr{S}$, $\bigcup_{k>0} L_L^{-k} = \mathscr{S}'$. In fact from Proposition it follows that every $f \in \bigcap_{k>0} L_2^k$ has continuous derivatives of all orders and $s_{\alpha\beta}(f) = \sup_{x\in R^n} |x^\beta D^\alpha f(x)| < \infty$ holds for all multiindices α, β. The system of seminorms $s_{\alpha\beta}$ defines the topology of $\mathscr{S}$. Hence the inequalities $s_{\alpha\beta}(f) \leqq \|f\|_{|\alpha|+|\beta|+r}$, $f \in \mathscr{S}$, imply the second assertion of our Corollary.

The relations $L_2^0 \supset L_2^1 \supset L_2^2 \supset \ldots$ and the evident inequality $\|f\|_k \geqq \|f\|_l$ for $f \in L_2^k$, $k \geqq l \geqq 0$, imply that $L_2^0 \subset L_2^{-1} \subset L_2^{-2} \subset \ldots$ Moreover, for (not necessary positive) integers p, q, $p \geqq q$, and $f \in L_2^p$ we have $\|f\|_p \geqq \|f\|_q$.

Let us show that the space $\mathscr{D}$ is dense in each L_2^k, k integer. Evidently we can assume $k < 0$. Be given a functional $F \in (L_2^k)'$. As $L_2^k \supset L_2^0$ an inclusion $(L_2^k)' \subset (L_2^0)' = L_2^0$ holds. The prime denotes the dual space. It means that F is a function. Assume that for every $\varphi \in \mathscr{D}$ we have $F\varphi = 0$. It would imply $0 = F\varphi = \int_{R^n} F(x)\,\varphi(x)\,dx = 0$ for every $\varphi \in \mathscr{D}$. Hence $F \equiv 0$. Thus, according to Hahn-Banach theorem the proposition is proved. As a corollary we see that for each pair of integers k, l, $k \geqq l$, L_2^k is dense in L_2^l.

Definition 3. Let $f : R^n \to C$ be measurable. Let an integer $k \geqq 0$ and a constant $A > 0$ exist such that for each $v \in L_2^k$ we have $vf \in L_1$ and

$$\left|\int_{R^n} v(x)\,f(x)\,dx\right| \leqq A\|v\|_k\,.$$

Then we identify the function f with the distribution $\int_{R^n} v(x)\,f(x)\,dx$.

Definition 4. Given an integer $k \geqq 0$, a multiindex α, $f \in L_2^{-k}$. Then we define the derivative $D^\alpha f$ as an element of $L_2^{-k-|\alpha|}$ by

$$(D^\alpha f)\,v = (-1)^{|\alpha|}\,f(D^\alpha v)\,, \quad v \in L_2^{k+|\alpha|}\,. \tag{7}$$

The evident inequality $\|D^\alpha f\|_{-k-|\alpha|} \leqq \|f\|_{-k}$ proves the continuity of differentiation-operator $D^\alpha : L_2^{-k} \to L_2^{-k-|\alpha|}$.

If $f \in L_2^{-k}$ is a function which has a generalized derivative $D^\alpha f \in L_2^{-k-|\alpha|}$ then this generalized derivative is identical with the distributive derivative introduced by Definition 4.

To show it denote for an instant by $\Delta^\alpha f$ the distributive derivative. Then for each $\varphi \in \mathscr{D}$ we have

$$(\Delta^\alpha f)\, \varphi = (-1)^{|\alpha|} f(\mathrm{D}^\alpha \varphi) = (-1)^{|\alpha|} \int_{R^n} f(x)\, \mathrm{D}^\alpha \varphi(x)\, \mathrm{d}x = \int_{R^n} \varphi(x)\, \mathrm{D}^\alpha f(x)\, \mathrm{d}x\,.$$

Hence $\Delta^\alpha f - \mathrm{D}^\alpha f \equiv 0$ on the linear subspace $\mathscr{D}$ which is dense in $L_2^{-k-|\alpha|}$. According to Hahn-Banach theorem $\Delta^\alpha f = \mathrm{D}^\alpha f$ on $L_2^{-k-|\alpha|}$.

Definition 5. Given integers $p, q, p \geqq q \geqq 0$. Then we denote by $\mathcal{O}_{p,q}$ a linear space of all functions $u : R^n \to C$ for which the mapping $v \to uv$ continuously maps L_2^{p-k} into L_2^{q-k}, for each $k = 0, 1, \ldots, q$. The space $\mathcal{O}_{p,q}$ is a normed space with the norm $\|u\|_{p,q} = \max_{k=0,1,\ldots,q} \sup_{\|v\|_{p-k} \leqq 1} \|uv\|_{q-k}$, $u \in \mathcal{O}_{p,q}$.

According to the continuity of identity-operator $\mathscr{I} : L_2^k \to L_2^l$, $k \geqq l$, integers, we can easily prove that $\mathcal{O}_{p,q} \subset \mathcal{O}_{p,q-1} \subset \ldots \subset \mathcal{O}_{p,0}$, $\mathcal{O}_{p,q} \subset \mathcal{O}_{r,q}$, $r \geqq p \geqq q \geqq 0$, $\mathcal{O}_{p+s,q+s} \subset \mathcal{O}_{p,q}$, $s = 0, 1, 2, \ldots$ Moreover, we have $\|u\|_{p,q} \geqq \|u\|_{p,s}$, $\|u\|_{p,q} \geqq \|u\|_{r,q}$, $r \geqq p \geqq q \geqq s \geqq 0$, $u \in \mathcal{O}_{p,q}$ and $\|u\|_{p+s,q+s} \geqq \|u\|_{p,q}$, $s \geqq 0$, $u \in \mathcal{O}_{p+s,q+s}$.

Lemma 1. *Let non-negative integers q, s and a function $u : R^n \to C$ be given. Let for every multiindex α, $|\alpha| \leqq q$, the continuous (classical) derivative $\mathrm{D}^\alpha u$ exist and fulfil an inequality*

$$\sup_{x \in R^n} |\mathrm{D}^\alpha u(x)| \, (1 + \sum_{j=1}^n |x_j|)^{-s-|\alpha|} < +\infty\,.$$

Then $u \in \mathcal{O}_{q+s,q}$.

Proof. For $v \in L_2^{q+s-k}$, $k \leqq q$, we have to estimate

$$\|uv\|_{q-k}^2 = \int_{R^n} [D_{q-k}(uv), D_{q-k}(uv)]\, \mathrm{d}x = \sum_{j_1, \ldots, j_{q-k}=0}^{2n} \int_{R^n} |\mathrm{A}_{j_1} \ldots \mathrm{A}_{j_{q-k}}(uv)|^2\, \mathrm{d}x\,.$$

Performing the indicated operations we get $\mathrm{A}_{j_1} \ldots \mathrm{A}_{j_{q-k}}(uv) = \sum_{|\alpha|+|\beta|+|\gamma| \leqq q-k} a_{\alpha\beta\gamma} x^\alpha \,.$ $. \mathrm{D}^\beta u\, \mathrm{D}^\gamma v$, where the coefficients $a_{\alpha\beta\gamma}$ do not depend on the functions u, v. According the assumptions there is a constant $\varkappa_1 > 0$ such that $|x^\alpha \mathrm{D}^\beta u\, \mathrm{D}^\gamma v| \leqq \varkappa_1 (1 + \sum_{j=1}^n |x_j|)^{s+|\beta|} |x^\alpha \mathrm{D}^\gamma v|$. As $s + |\beta| + |\alpha| + |\gamma| \leqq s + q - k$ we can find another constant $\varkappa_2 > 0$ such that

$$\int_{R^n} |x^\alpha \mathrm{D}^\beta u\, \mathrm{D}^\gamma v|^2\, \mathrm{d}x \leqq \varkappa_2 \|v\|_{s+q-k}^2\,, \quad |\alpha| + |\beta| + |\gamma| \leqq q - k\,.$$

Then the existence of such constant $\varkappa_3 > 0$ that

$$\int_{R^n} |A_{j_1} \dots A_{j_{q-k}}(uv)|^2 \, dx \leqq \varkappa_3 \|v\|_{q+s-k}^2$$

for all integers j_m, $0 \leqq j_m \leqq 2n$, $m = 1, 2, \dots, q - k$, follows from Hölder's inequality. The proof is complete.

Corollary 1. *Every polynomial of degree k is an element of* $\mathcal{O}_{p,p-k}$, $p \geqq k$.

Corollary 2. $L_2^{q+r} \subset \mathcal{O}_{q,q}$, *where* $q = 0, 1, 2, \dots$, $r = 1 + [\frac{1}{2}n]$, *and the identity-operator* $\mathcal{I} : L_2^{q+r} \to \mathcal{O}_{q,q}$ *is continuous.*

Proof. It follows from the Proposition that the assumptions of Lemma 1 are fulfilled with $s = 0$.

From the proof of Lemma 1 it follows immediately the assertion: Let functions u_k, $k = 1, 2, \dots$, have continuous (classical) derivatives of all orders α, $|\alpha| \leqq q$, and let

$$\lim_{k\to\infty} \max_{|\alpha|\leqq q} \sup_{x\in R^n} |D^\alpha u_k(x)| \, (1 + \sum_{j=1}^{n} |x_j|)^{-s-|\alpha|} = 0 .$$

Then $\lim_{k\to\infty} \|u_k\|_{q+s,q} = 0$.

Remark. *Given* $p \geqq q \geqq 0$, $f \in \mathcal{O}_{pq}$. *Then for each multiindex* α, $|\alpha| \leqq q$, *the generalized derivative* $D^\alpha f$ *exists.*

Proof. Choose $v \in \mathcal{D}$ so that $v(x) = 1$ for $|x| \leqq 1$. Take α, $|\alpha| \leqq q$, and arbitrary $\varphi \in \mathcal{D}$. Then for $A > 0$ such that support $\varphi \subset \{x; |x| \leqq A\}$ we have $f(x) \, v(x/A) \in \in L_2^q$ and

$$\int_{R^n} f \, D^\alpha\varphi \, dx = \int_{R^n} f(x) \, v\left(\frac{x}{A}\right) D^\alpha\varphi(x) \, dx = (-1)^{|\alpha|} \int_{R^n} D^\alpha \left(f(x) \, v\left(\frac{x}{A}\right)\right) \varphi(x) \, dx .$$

Let $0 < A < B$ then for $\psi \in \mathcal{D}$, support $\psi \subset \{x; |x| \leqq A\}$ we have

$$\int_{R^n} D^\alpha \left(f(x) \, v\left(\frac{x}{A}\right)\right) \psi(x) \, dx = \int_{R^n} D^\alpha \left(f(x) \, v\left(\frac{x}{B}\right)\right) \psi(x) \, dx .$$

Hence $D^\alpha(f(x) \, v(x/A)) = D^\alpha(f(x) \, v(x/B))$ for almost all x, $|x| \leqq A$, and we can uniquely define a function $g : R^n \to C$ putting $g(x) = D^\alpha(f(x) \, v(x/A))$ for almost all x, $|x| \leqq A$, and all $A > 0$. Then evidently $g = D^\alpha f$.

Definition 6. Given integers $p \geqq q \geqq 0$, $u \in \mathcal{O}_{p,q}$, $f \in L_2^{-q}$. Then we define uf as a distribution from L_2^{-p} by formula

$$(uf) \, v = f(uv) , \quad v \in L_2^p . \tag{8}$$

If $f \in L_2^{-p}$ is a function then the distribution uf, where $u \in \mathcal{O}_{p,q}$, is identical with the function uf. The mapping $(u, f) \to uf$ of $\mathcal{O}_{p,q} \times L_2^{-q}$ into L_2^{-p} is hypocontinuous, i.e. continuous in each variable locally uniformly with respect to the other one. In fact, $\|uf\|_{-p} = \sup_{\|v\|_p \leqq 1} (uf)\, v = \sup_{\|v\|_p \leqq 1} f(uv) \leqq \|f\|_{-q} \sup_{\|v\|_p \leqq 1} \|uv\|_q \leqq \|f\|_{-q} \|u\|_{p,q}$.

Lemma 2. *Given integers $p \geqq q \geqq 1$, $u \in \mathcal{O}_{p,q}$, $f \in L_2^{1-q}$. Let there exist a continuous derivative $\partial u/\partial x_1 \in \mathcal{O}_{p,q-1}$. Then*

$$\frac{\partial}{\partial x_1}(uf) = \frac{\partial u}{\partial x_1} f + u \frac{\partial f}{\partial x_1}. \tag{9}$$

Moreover, on both sides of (9) there are distributions from L_2^{-p}.

Proof. We have $\partial f/\partial x_1 \in L_2^{-q}$. Hence, the products uf, $(\partial u/\partial x_1) f$, $u(\partial f/\partial x_1)$ are defined and $uf \in L_2^{1-p}$ $(\partial u/\partial x_1) f \in L_2^{-p}$, $u(\partial f/\partial x_1) \in L_2^{-p}$. Let us take $v \in L_2^p$; then

$$\left(\frac{\partial u}{\partial x_1} f + u \frac{\partial f}{\partial x_1}\right) v = f\left(\frac{\partial u}{\partial x_1} v\right) + \frac{\partial f}{\partial x_1}(uv) = f\left(\frac{\partial u}{\partial x_1} v\right) - f\left(\frac{\partial}{\partial x_1}(uv)\right) =$$

$$= f\left(\frac{\partial u}{\partial x_1} v - \frac{\partial u}{\partial x_1} v - u \frac{\partial v}{\partial x_1}\right) = -uf\left(\frac{\partial v}{\partial x_1}\right) = \frac{\partial (uf)}{\partial x_1} v .$$

Theorem 2. *Given $f \in L_2^{-k}$, $k \geqq 0$, integer. Then for each multiindex α, $|\alpha| \leqq k$, there are a function $g_\alpha \in L_2$ and a polynomial P_α of degree $\leqq k - |\alpha|$ such that*

$$f = \sum_{|\alpha| \leqq k} P_\alpha \mathrm{D}^\alpha g .$$

Proof. According to Fréchet-Riesz theorem on the representation of linear functionals such element $g \in L_2^k$ exists that for every $v \in L_2^k$ we have

$$fv = (v, g)_k = \int_{R^n} [\mathrm{D}_k v, \mathrm{D}_k g]\, \mathrm{d}x = \sum_{j_1, \ldots, j_k = 0}^{2n} \int_{R^n} (\mathrm{A}_{j_1} \ldots \mathrm{A}_{j_k} v)\, \overline{(\mathrm{A}_{j_1} \ldots \mathrm{A}_{j_k} g)}\, \mathrm{d}x =$$

$$= \sum_{j_1, \ldots, j_k = 0}^{2n} \overline{(\mathrm{A}_{j_1} \ldots \mathrm{A}_{j_k} g)}\, (\mathrm{A}_{j_1} \ldots \mathrm{A}_{j_k} v) .$$

Let us choose a permutation $(j_1, \ldots, j_k)$ and for brevity denote $h = \overline{\mathrm{A}_{j_1} \ldots \mathrm{A}_{j_k} g}$. Evidently, $h \in L_2$. We now distinguish 3 cases:

$$\left.\begin{aligned} &\text{a)} \quad \mathrm{A}_{j_1} = 1 \\ &\text{b)} \quad \mathrm{A}_{j_1} = 2\pi i x_j \\ &\text{c)} \quad \mathrm{A}_{j_1} = \frac{\partial}{\partial x_j} \end{aligned}\right\} \quad \text{then} \quad h(\mathrm{A}_{j_1} \ldots \mathrm{A}_{j_k} v) = \begin{cases} h(\mathrm{A}_{j_2} \ldots \mathrm{A}_{j_k} v) \\ 2\pi i x_j h(\mathrm{A}_{j_2} \ldots \mathrm{A}_{j_k} v) \\ -\dfrac{\partial}{\partial x_j} h(\mathrm{A}_{j_2} \ldots \mathrm{A}_{j_k} v) . \end{cases}$$

Hence, according to Lemma 2,

$$h(A_{j_1} \dots A_{j_k} v) = \pm(A_{j_k} \dots A_{j_1} h)\, v = \Big(\sum_{|\alpha| \leqq k} Q_\alpha D^\alpha h \Big)\, v\,,$$

where Q_α are polynomials of degree $\leqq k - |\alpha|$.

Corollary. *The space L_2^{-k}, $k \geqq 0$, consists entirely of such elements which we get by formal differentiation of elements of L_2 and multiplication by functions $2\pi i x_j$, $j = 1, 2, \dots, n$. At the same time the sum of these operations applied on any element of L_2 may not exceed k.*

Lemma 3. *Given integers p, q, $p \geqq q \geqq 0$. Then $\mathcal{O}_{p,q} \subset L_2^{q-p-r}$, where $r = 1 + [\frac{1}{2}n]$, and the identity-operator $\mathcal{I} : \mathcal{O}_{p,q} \to L_2^{q-p-r}$ is continuous.*

Proof. Let us take $f \in \mathcal{O}_{p,q}$ and denote $g(x) = (1 + (x, x))^{-(p+r)/2}$. Then $g \in L_2^p$ and $1/g(x) \in \mathcal{O}_{s+p+r,s}$, $s = 0, 1, \dots$. Hence $fg \in L_2^q$ and $f = (1/g)\, fg \in L_2^{q-p-r}$.

Using the hypocontinuity of multiplication we see that there exists a constant $A > 0$, which does not depend on f, such that $\|f\|_{q-p-r} = \|(1/g)\, fg\|_{q-p-r} \leqq \leqq A\|fg\|_q \leqq A\|f\|_{p,q} \cdot \|g\|_p$.

Lemma 4. *Given integer $k \geqq 0$, $f, g \in L_2^k$. Then according to Fréchet-Riesz theorem there are unique elements $\varphi, \psi \in L_2^k$ such that $fv = (v, \varphi)_k$, $gv = (v, \psi)_k$, $v \in L_2^k$. If we denote $(f, g)_{-k} = (\psi, \varphi)_k$, we get an inner product defined in L_2^{-k}. The norm induced by this inner product is identical with the norm $\|\,.\,\|_{-k}$.*

Proof. The mapping $f, g \to (f, g)_{-k}$ has evidently all properties of an inner product. We only show the equality of norms. Actually,

$$\|f\|_{-k} = \sup_{\|v\|_k \leqq 1} fv = \sup_{\|v\|_k \leqq 1} (v, \varphi)_k = \left(\frac{\varphi}{\|\varphi\|_k}, \varphi \right)_k = \|\varphi\|_k\,.$$

Since $(f, f)_{-k} = (\varphi, \varphi)_k$, the proof is completed.

Definition 7. *Given integer $k \geqq 0$, $f \in L_2^{-k}$.* Then we define the Fourier image $\mathcal{F}f$ as an element of L_2^{-k} *by* $(\mathcal{F}f)\, v = f(\mathcal{F}v)$, $v \in L_2^k$.

Remark. If a distribution $f \in L_2^{-k}$ is a function from L_2, then the Fourier image $\mathcal{F}f$ defined by Definition 7 coincides with the classially defined Fourier image. This follows from the well-known theorem

$$\int_{R^n} (\mathcal{F}f)\, v \,\mathrm{d}x = \int_{R^n} f \,.\, \mathcal{F}v \,\mathrm{d}x\,, \quad f, v \in L_2\,.$$

Now we are prepared to drop the assumption $k \geqq 0$ in Theorem 1. Actually, we have

Theorem 1a. *Fourier transform $\mathscr{F} : L_2^k \to L_2^k$, k integer, is a unitary automorphism.*

Proof. Let $k < 0$. The equality $\mathscr{F}L_2^k = L_2^k$ is an immediate consequence of Theorem 1 and Definition 7. Let us prove the invariance of inner product. Take $f, g \in L_2^k$; then according to Lemma 4 there are elements $\varphi, \psi \in L_2^{-k}$ such that $fv = (v, \varphi)_{-k}$, $gv = (v, \psi)_{-k}$, $v \in L_2^{-k}$, $(f, g)_k = (\psi, \varphi)_{-k}$. For every $v \in L_2^{-k}$ we have $(\mathscr{F}f)\, v = f(\mathscr{F}v) = (\mathscr{F}v, \varphi)_{-k} = (v, \mathscr{F}^{-1}\varphi)_{-k}$; similarly, $(\mathscr{F}g)\, v = (v, \mathscr{F}^{-1}\psi)_{-k}$. Hence, as a consequence of Theorem 1 we have $(\mathscr{F}f, \mathscr{F}g)_k = (\mathscr{F}^{-1}\psi, \mathscr{F}^{-1}\varphi)_{-k} = (\psi, \varphi)_{-k} = (f, g)_k$.

Theorem 3. *Given integer k, $f \in L_2^k$ and multiindex α. Then*

$$\text{(1a)} \qquad \mathscr{F}(\mathrm{D}^\alpha f) = (2\pi i x)^\alpha\, \mathscr{F}f$$
$$\mathscr{F}((-2\pi i x)^\alpha f) = \mathrm{D}^\alpha(\mathscr{F}f)\,.$$

Proof. On both sides of (1a) there are elements of $L_2^{k-|\alpha|}$. If $k - |\alpha| \geqq 0$, then the statement of Theorem 3 is the well-known result. Thus, let $k - |\alpha| < 0$, $v \in L_2^{|\alpha|-k}$. We get

$$\mathscr{F}(\mathrm{D}^\alpha f)\, v = (\mathrm{D}^\alpha f)\,(\mathscr{F}v) = (-1)^{|\alpha|} f(\mathrm{D}^\alpha \mathscr{F}v) = (-1)^{|\alpha|} f(\mathscr{F}((-2\pi i x)^\alpha v)) =$$
$$= (\mathscr{F}f)\,((2\pi i x)^\alpha v) = ((2\pi i x)^\alpha\, \mathscr{F}f)\, v\,,$$
$$\mathscr{F}((-2\pi i x)^\alpha f)\, v = ((-2\pi i x)^\alpha f)\,(\mathscr{F}v) = f((-2\pi i x)^\alpha\, \mathscr{F}v) =$$
$$= (-1)^{|\alpha|} f(\mathscr{F}\mathrm{D}^\alpha v) = (-1)^{|\alpha|}\,(\mathscr{F}f)\,(\mathrm{D}^\alpha v) = \mathrm{D}^\alpha(\mathscr{F}f)\, v\,.$$

Definition 8. Given integers p, q, $p \geqq q \geqq 0$. Then we define $\mathcal{O}_{p,q}^* = \{f \in \bigcup_{k>0} L_2^{-k} : \mathscr{F}f \in \mathcal{O}_{p,q}\}$. If we define a norm $\|f\|_{p,q}^* = \|\mathscr{F}f\|_{p,q}$ for $f \in \mathcal{O}_{p,q}^*$ then $\mathcal{O}_{p,q}^*$ turns into a normed linear space.

Example. $x_1^{p-q} \in \mathcal{O}_{p,q}$, $p \geqq q \geqq 0$, has Fourier image $\mathscr{F}x_1^{p-q} = (-2\pi i)^{q-p} \,.\, (\partial^{p-q}/\partial x_1^{p-q})\, \delta_0 \in \mathcal{O}_{p,q}$ which is not a function. Hence for each p, q, $p \geqq q \geqq 0$, $\mathcal{O}_{p,q}^* \neq \mathcal{O}_{p,q}$ holds.

Let $f \in \mathcal{O}_{p,q}^*$. According to Definition 8 $\mathscr{F}f \in \mathcal{O}_{p,q}$ is a function. For every $x \in R^n$ $(\mathscr{F}^{-1}f)(x) = (\mathscr{F}f)(-x)$ holds. Hence $\mathscr{F}^{-1}f \in \mathcal{O}_{p,q}$. Thus, we could also define $\mathcal{O}_{p,q}^*$ by the formula $\mathcal{O}_{p,q}^* = \mathscr{F}\mathcal{O}_{p,q}$.

We know that for each pair p, q, $p \geqq q \geqq 0$, $L_2^{q+r} \subset \mathcal{O}_{p,q} \subset L_2^{q-p-r}$, where $r = 1 + [\frac{1}{2}n]$, holds. Then from Theorem 1a the inclusions $L_2^{q+r} \subset \mathcal{O}_{p,q}^* \subset L_2^{q-p-r}$ follow. The identity-operator corresponding to each inclusion is continuous. Moreover, $\mathscr{S}(R^n) \subset \mathcal{O}_{p,q}^*$ and henceforth for each integer k the space $\mathcal{O}_{p,q}^* \cap L_2^k$ is dense in L_2^k.

Definition 9. Given integers p, q, $p \geqq q \geqq 0$, $f \in \mathcal{O}_{p,q}^*$, $g \in L_2^{-q}$. Then the distribution $\mathscr{F}^{-1}(\mathscr{F}f \,.\, \mathscr{F}g) \in L_2^{-p}$ is called the convolution of distributions f, g and denoted by $f * g$.

For $f \in \mathcal{O}^*_{p,q}$, $g \in L_2^{-q}$ we have

$$\|f * g\|_{-p} = \sup_{\|v\|_p \leqq 1} (f * g)\, v = \sup_{\|v\|_p \leqq 1} (f * g)\, \mathscr{F} v = \sup_{\|v\|_p \leqq 1} \mathscr{F} g(v \,.\, \mathscr{F} f) \leqq$$
$$\leqq \|\mathscr{F} g\|_{-q} \sup_{\|v\|_p \leqq 1} \|v \,.\, \mathscr{F} f\|_q \leqq \|\mathscr{F} g\|_{-q} \|\mathscr{F} f\|_{p,q} = \|f\|^*_{p,q} \,.\, \|g\|_{-q} \,.$$

Hence the mapping $(f, g) \to f * g$ of Cartesian product $\mathcal{O}^*_{p,q} \times L_2^{-q}$ into L_2^{-p} is hypocontinuous.

Theorem 4. *Given integers p, q, $p \geqq q \geqq 0$, $f \in \mathcal{O}^*_{p,q}$, $g \in L_2^{-q}$, $h \in \mathcal{O}_{p,q}$. Then*

$$\mathscr{F}(f * g) = \mathscr{F} f \,.\, \mathscr{F} g \,, \tag{10}$$

$$\mathscr{F}(hg) = \mathscr{F} h * \mathscr{F} g \,. \tag{11}$$

Proof. Formula (10) is an immediate consequence of Definition 9. To prove (11), let us take $v \in L_2^p$; then

$$\mathscr{F}(hg)\, v = (hg)\, \mathscr{F} v = g(h \mathscr{F} v) = g(\mathscr{F}^4(h \mathscr{F} v)) = g(\mathscr{F}^2(\mathscr{F}^2 h \mathscr{F}^{-1} v)) =$$
$$= \mathscr{F}^2 g(\mathscr{F}^2 h \mathscr{F}^{-1} v) = (\mathscr{F}^2 h \mathscr{F}^2 g)\, \mathscr{F}^{-1} v =$$
$$= \mathscr{F}^{-1}(\mathscr{F}^2 h \mathscr{F}^2 g)\, v = (\mathscr{F} h * \mathscr{F} g)\, v \,.$$

Remark. From the identity (4) it follows immediately that formulae (10), (11) are also valid if the operator $\mathscr{F}$ is replaced by $\mathscr{F}^{-1}$.

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