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M-POLARS IN LATTICE-ORDERED GROUPS

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1. Introduction. Throughout this note G will denote a lattice-ordered group ("l-group"). If $S \subseteq G$ and if M is a convex l-subgroup of G, let $p(S, M) = \{x \in G \mid |x| \wedge |s| \in M \text{ for all } s \in S\}$. Then $p(S, M)$ will be called the *M-polar of S in G.* The definition of an M-polar extends the concept of a polar, that is, the case where $M = \{0\}$. Polars have been used extensively in the literature and this notc is devoted primarily to an investigation of those properties of polars which can be extended to M-polars.

In Lemma 3.1 it is shown that $p(S, M)$ is a convex l-subgroup of G and that S and the convex l-subgroup of G generated by S define the same M-polar. If S is a convex l-subgroup of G, then it is shown in Lemma 3.3 that $p(S, M) = p(S, S \cap M)$. Thus, without loss of generality, it may be assumed that $M \subseteq S$ and that S is a convex l-subgroup of G. It is shown (Theorem 3.10) that for a fixed convex l-subgroup M of G, the collection of all M-polars is a complete Boolean algebra. Also, it is shown (Theorem 3.14) that the collection of all M-summands is a subalgebra of this collection. These results generalize the theorems on polars and cardinal summands which were first proven by F. ŠIK in [9], and rediscovered by many others. P. CONRAD ([3], Theorem 3.5) used a mapping τ defined by $M\tau = M \cap S$ to establish a one to one correspondence between the prime subgroups of G not containing S and all proper prime subgroups of S, where S is a convex l-subgroup of G. In Theorem 3.5 the inverse of the mapping τ is extended to all convex l-subgroups of S and this extension is done with M-polars.

2. Notation and terminology. For the standard definitions and results concerning l-groups the reader is refered to [1] and [6]. A subgroup C of G is an *l-subgroup* provided that C is a sublattice of G, and C is a *convex* subgroup if $0 \leqq g \leqq c \in C$ and $g \in G$ imply that $g \in C$. A convex l-subgroup C of G is called a *prime* subgroup

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if whenever a and b belong to $G^+ = \{g \in G \mid g \geqq 0\}$ and not C, then $a \wedge b > 0$. Theorem 3.2 of [3] gives six equivalent definitions of a prime subgroup. A convex l-subgroup that is maximal with respect to not containing some g in G is called a *regular* subgroup. Each regular subgroup is prime ([3], Corollary to Theorem 3.1). Let Γ be an index set for the collection G_γ of regular subgroups of G. For each $\gamma \in \Gamma$ there exists a unique convex l-subgroup G^γ of G that covers G_γ. If g belongs to G^γ but not G_γ, then G_γ is said to be a *value* of g. By Zorn's lemma each $0 \neq g \in G$ has at least one value.

If $S \subseteq G$, then $\langle S \rangle$ ($[S]$) will denote the subsemigroup (subgroup) of G that is generated by S. If A and B are sets, then $A \backslash B$ will denote the set of elements in A but not in B, and $A \subset B$ denotes that A is a proper subset of B.

3. M-polars. If $S \subseteq G$ and if M is a convex l-subgroup of G, let $p^1(S, M) = p(S, M) = \{x \in G \mid |x| \wedge |s| \in M \text{ for all } s \in S\}$ and, by induction, let $p^n(S, M) = p(p^{n-1}(S, M), M)$ where $n > 1$. $p(S, M)$ will be called the *M-polar of S in G*. The $\{0\}$-polar of S will be denoted by $p(S)$ and will be called the *polar* of S. If $S = \{s\}$, then $p(S, M)$ will be denoted by $p(s, M)$ and we shall call $p(s, M)$ a *principal M-polar*. Clearly $p(S, M) = \bigcap\{p(s, M) \mid s \in S\}$. Let $S' = \{|s| \mid s \in S\}$ and if $X \subseteq G^+$, let $X_* = \{g \in G^+ \mid g \leqq x \text{ for some } x \in X\}$.

Lemma 3.1. (1) *$p(S, M)$ is a convex l-subgroup of G, $M \subseteq p(S, M)$, and $S \subseteq p^2(S, M)$.*

(2) *If $T \subseteq G$ such that $S' \subseteq (T')_*$, then $p(T, M) \subseteq p(S, M)$.*

(3) *$\langle S' \rangle_*$ is a convex subsemigroup of G^+ that contains 0, hence $[\langle S' \rangle_*]$ is a convex l-subgroup of G. Moreover, $p(S, M) = p([\langle S' \rangle_*], M)$.*

Proof. (1) If $x, y \in p(S, M)$ and $s \in S$, then $0 \leqq |x - y| \wedge |s| \leqq (|x| + |y| + |x|) \wedge |s| \leqq (|x| \wedge |s|) + (|y| \wedge |s|) + (|x| \wedge |s|) \in M$. Since M is convex, it follows that $|x - y| \wedge |s| \in M$ and so $x - y \in p(S, M)$. If $z \in G$, $x \in p(S, M)$, and $s \in S$ such that $|z| \leqq |x|$, then $0 \leqq |z| \wedge |s| \leqq |x| \wedge |s| \in M$. Therefore $z \in p(S, M)$ and $p(S, M)$ is a convex l-subgroup of G ([3], Proposition 3.1). It follows from the definition of M-polar that $M \subseteq p(S, M)$ and that $S \subseteq p^2(S, M)$.

(2) Let $x \in p(T, M)$ and let $s \in S$. Then $|s| \leqq |t|$ for some t in T. Thus $0 \leqq |x| \wedge |s| \leqq |x| \wedge |t| \in M$ and so $x \in p(S, M)$.

(3) By the definition $\langle S' \rangle_*$ is a convex subset of G^+ and contains 0. If $x, y \in \langle S' \rangle_*$, then $0 \leqq x + y \leqq |s_1| + \ldots + |s_n| + |t_1| + \ldots + |t_m| \in \langle S' \rangle$, where $s_i, t_j \in S$. Thus $x + y \in \langle S' \rangle_*$. If T is a convex subsemigroup of G^+ that contains 0, then $[T]$ is a convex l-subgroup of G and $[T]^+ = T$ ([5], Theorem 2.1). By (2) $p([\langle S' \rangle_*], M) \subseteq p(S, M)$. Let $0 \leqq x \in p(S, M)$ and let $a \in \langle S' \rangle_*$. Then $0 \leqq a \leqq |s_1| + \ldots + |s_n| \in \langle S' \rangle$ where $s_i \in S$. Therefore $0 \leqq x \wedge a \leqq x \wedge (|s_1| + \ldots + |s_n|) \leqq (x \wedge |s_1|) + \ldots + (x \wedge |s_n|) \in M$. Thus $x \in p([\langle S' \rangle_*], M)$.

It will be assumed for the remainder of this note that S is a convex l-subgroup of G.

For $g \in G$, let $G(g) = \{x \in G \mid |x| \leqq n|g| \text{ for some positive integer } n\}$. Then $G(g)$ is the smallest convex l-subgroup of G containing g ([3], Proposition 3.4). Clearly $G(g) = G(|g|)$. The following is immediate from (3) of the lemma.

Corollary 3.2. *For each g in G, $p(g, M) = p(G(g), M)$.*

Lemma 3.3. (1) *If L is a convex l-subgroup of G such that $M \subseteq L$, then $p(S, M) \subseteq p(S, L)$. In particular, for any convex l-subgroup J of G, $p(S, M) \cap p(S, J) = p(S, M \cap J)$.*

(2) $p(S, M) = p(S, S \cap M) = p([S \cup M], M)$.

(3) $S \subseteq M$ *if and only if* $S \subseteq p(S, M)$ *if and only if* $p(S, M) = G$.

Proof. (1) Let $0 \leqq x \in p(S, M)$ and let $0 \leqq s \in S$. Then $x \wedge s \in M \subseteq L$ and so $x \in p(S, L)$. Thus it follows that $p(S, M \cap J) \subseteq p(S, M) \cap p(S, J)$. If $0 \leqq x \in p(S, M) \cap p(S, J)$ and if $0 \leqq s \in S$, then $x \wedge s \in M \cap J$ and so $x \in p(S, M \cap J)$.

(2) By (2) of Lemma 3.1, $p([S \cup M], M) \subseteq p(S, M)$. Let $0 \leqq x \in p(S, M)$ and let $0 \leqq s_1 + m_1 + \ldots + s_n + m_n \in [S \cup M]$. Then $0 \leqq x \wedge (s_1 + m_1 + \ldots + s_n + m_n) = x \wedge |s_1 + m_1 + \ldots + s_n + m_n| \leqq x \wedge (|s_1| + |m_1| + \ldots + |s_n| + |m_n| + |s_n| + \ldots + |m_1| + |s_1|) \leqq (x \wedge |s_1|) + (x \wedge |m_1|) + \ldots + (x \wedge |m_n|) + \ldots + (x \wedge |m_1|) + (x \wedge |s_1|) \in M$. Hence $x \in p([S \cup M], M)$. By (1), $p(S, S \cap M) \subseteq p(S, M)$. If $0 \leqq x \in p(S, M)$ and if $0 \leqq s \in S$, then $x \wedge s \in S \cap M$ and so $x \in p(S, S \cap M)$.

The proof at (3) is straightforward and will be omitted. *For the remainder of this note it will be assumed that M is a convex l-subgroup of S.*

Lemma 3.4. (1) $M = p(S, M) \cap S = p(S, M) \cap p^2(S, M)$.

(2) *If L is a convex l-subgroup of G such that $L \cap S \subseteq M$, then $L \subseteq p(S, M)$. Thus $p(S, M)$ is the largest convex l-subgroup of G whose intersection with S is contained in M.*

(3) $p(S, M) = p^3(S, M)$.

Proof. (1) By assumption $M \subseteq S$ and by (1) of Lemma 3.1, $M \subseteq p(S, M)$ and $S \subseteq p^2(S, M)$. Thus $M \subseteq p(S, M) \cap S \subseteq p(S, M) \cap p^2(S, M)$. If $0 \leqq x \in p(S, M) \cap p^2(S, M)$ then $x \in p(S, M)^+$ and $x \in p(p(S, M), M)$. Therefore $x = x \wedge x \in M$.

(2) If $0 \leqq x \in L$ and if $0 \leqq s \in S$, then $x \wedge s \in L \cap S \subseteq M$. Hence $x \in p(S, M)$. The remainder of (2) follows from (1).

(3) From (1) of Lemma 3.1, $S \subseteq p^2(S, M)$ and so by (2) at the same lemma, $p(S, M) \supseteq p(p^2(S, M), M) = p^3(S, M)$. $p(S, M) \cap p^2(S, M) \subseteq M$ implies by (2) that $p(S, M) \subseteq p^3(S, M)$.

Let $\mathscr{S} = \{J \mid J \text{ is a convex } l\text{-subgroup of } S\}$ and let $\mathscr{I} = \{p(S, J) \mid J \in \mathscr{S}\}$. Define a mapping σ from $\mathscr{S}$ into $\mathscr{I}$ by $J\sigma = p(S, J)$.

Theorem 3.5. *σ is a one to one inclusion preserving mapping of $\mathcal{S}$ onto $\mathcal{J}$ such that for $J, M \in \mathcal{S}$, $(J \cap M)\,\sigma = J\sigma \cap M\sigma$. σ^{-1} is given by $p(S, J)\,\sigma^{-1} = p(S, J) \cap S$. If L is a prime subgroup of G that does not contain S, then $L = p(S, S \cap L) = (S \cap L)\,\sigma$. J is a prime (regular) subgroup of S if and only if $p(S, J)$ is a prime (regular) subgroup of G. Moreover, if $s \in S$, then J is a value of s in S if and only if $p(S, J)$ in a value of s in G. Finally, if $S = G(g)$, then M is a maximal convex l-subgroup of S if and only if $p(S, M)$ is a value of g in G.*

Proof. Clearly σ is a function. By (1) of Lemma 3.3, σ is inclusion preserving. It follows from (1) and (2) at Lemma 3.4 that σ is one to one and by the definition of $\mathcal{J}$, σ is onto. (1) of Lemma 3.3 shows that σ distributes over finite intersections and (1) of Lemma 3.4 shows that $p(S, J)\,\sigma^{-1} = J = p(S, J) \cap S$.

Suppose that $L$is a prime subgroup of G that does not contain S. By (2) of Lemma 3.4, $L \subseteq p(S, S \cap L)$. Suppose (by way of contradiction) that there exists $0 < x \in p(S, S \cap L)\backslash L$. Let $0 < s \in S\backslash L$. Then $x \wedge s \in S \cap L \subseteq L$, but this is a contradiction as $L$is a prime subgroup of G ([3], Theorem 3.2).

The proof of the remainder of this theorem is analogous to the proof of Theorem 3.5 in [3] and will be omitted.

If X is a subset of $S(G)$, then $N_s(X)$ $(N(X))$ will denote the normalizer of X in $S(G)$.

Theorem 3.6. *$N_s(M) = S \cap N(p(S, M))$. Thus M is normal in S if and only if $p(S, M)$ is normal in $[S \cup p(S, M)]$. In particular for any γ in Γ, the following are equivalent.*

(1) *G_γ is normal in G^γ.*

(2) *$G_\gamma \cap G(g)$ is normal in $G(g)$ for all $g \in G^\gamma\backslash G_\gamma$.*

(3) *$G_\gamma \cap G(g)$ is normal in $G(g)$ for some $g \in G^\gamma\backslash G_\gamma$. This is the case if G_γ is the only value of some g in G.*

Proof. If $x \in S \cap N(p(S, M))$, then $x + M - x = x + p(S, M) \cap S - x = (x + p(S, M) - x) \cap (x + S - x) = p(S, M) \cap S = M$. Thus $x \in N_s(M)$. Conversely if $x \in N_s(M)$, then $M = x + M - x = x + p(S, M) \cap S - x = (x + p(S, M) - x) \cap S$. By (2) of Lemma 3.4, $x + p(S, M) - x \subseteq p(S, M)$. Therefore $x \in S \cap N(p(S, M))$.

If M is normal in S, then $S \subseteq N(p(S, M))$. Hence $[S \cup p(S, M)] \subseteq N(p(S, M))$. Conversely if $[S \cup p(S, M)] \subseteq N(p(S, M))$, then $N_s(M) = S \cap N(p(S, M)) = S$.

Next suppose that (1) is true, let $g \in G^\gamma\backslash G_\gamma$, and let $S = G(g)$. Then $N_s(G_\gamma \cap G(g)) = G(g) \cap N(G_\gamma) = (G(g) \cap G^\gamma) \cap N(G_\gamma) = G(g) \cap G^\gamma = G(g)$. Thus (2) is true. (2) implies (3) is trivial. Suppose that (3) is true. Then since $[G_\gamma \cup G(g)]$ is a convex l-subgroup of G that properly contains G_γ, it follows that $G^\gamma \subseteq [G_\gamma \cup G(g)] \subseteq N(G_\gamma)$. If G_γ is the only value of some g in G, then $G_\gamma \cap G(g)$ is the largest convex l-subgroup of $G(g)$ and hence normal in $G(g)$. This last assertion was proven in [2] (Proposition 2.4) by P. Conrad.

The next theorem is a generalization of Theorem 2.3 in [4].

Theorem 3.7. *For $M \subset S$, the following are equivalent.*

(a) *M is prime in $p^2(S, M)$.*
(b) *M is prime in S.*
(c) *$p(S, M)$ is prime in G.*
(d) *$p(S, M) = p(s, M)$ for each $0 < s \in S \backslash M$.*
(e) *$p(S, M)$ is a maximal M-polar.*
(f) *$p^2(S, M)$ is a minimal M-polar.*
(g) *$p^2(S, M)$ is a maximal convex l-subgroup of G with respect to the property that M is prime in $p^2(S, M)$.*

Proof. (a) implies (b). This follows from the definition of prime and the fact that $S \subseteq p^2(S, M)$.

(b) implies (c). This follows from Theorem 3.5.

(c) implies (d). By (2) of Lemma 3.1, $p(S, M) \subseteq p(s, M)$ for each $0 < s \in S \backslash M$. Suppose (by way of contradiction) that there exists $0 < x \in p(s, M) \backslash p(S, M)$ for some $0 < s \in S \backslash M$. Then $s \notin p(S, M)$, for otherwise, $s \in S \cap p(S, M) = M$. Therefore $x \wedge s \notin p(S, M)$ as $p(S, M)$ is prime ([3], Theorem 3.2), but this is a contradiction as $x \wedge s \in M \subseteq p(S, M)$.

(d) implies (e). Suppose that $p(S, M) \subseteq p(D, M) \subset G$, where D is a convex l-subgroup of G that contains M. $p(D, M) \subset G$ implies that $M \subset D$. If $D \subseteq p(S, M)$, then $D = D \cap p(S, M) \subseteq D \cap p(D, M) = M$, a contradiction. Let $0 < d \in D \backslash p(S, M)$. $d \notin p(S, M)$ implies that there exists $0 < s \in S$ such that $d \wedge s \notin M$ and hence $d \wedge s \in D \cap (S \backslash M)$. By (2) of Lemma 3.1, $p(D, M) \subseteq p(s \wedge d, M)$ and by (d), $p(S, M) = p(s \wedge d, M)$. Therefore $p(D, M) = p(S, M)$.

(e) implies (f). Suppose that $M \subset p(D, M) \subseteq p^2(S, M)$, where D is a convex l-subgroup of G that contains M. By (2) of Lemma 3.1 and (3) of Lemma 3.4, $p^2(D, M) \supseteq p^3(S, M) = p(S, M)$ and since $M \subset p(D, M)$, $G = p(M, M) \supset p^2(D, M)$. Since $p(S, M)$ is maximal, it follows that $p(S, M) = p^2(D, M)$. Therefore $p(D, M) = p^2(S, M)$.

(f) implies (g). Suppose (by way of contradiction) that M is not prime in $p^2(S, M)$. Then there exists $0 < x, y \in p^2(S, M) \backslash M$ such that $x \wedge y = 0$. $x \in p^2(S, M)$ implies that $p(x, M) \supseteq p^3(S, M) = p(S, M)$ and so $p^2(x, M) \subseteq p^2(S, M)$. Since $p^2(S, M)$, is assumed to be minimal and $x \in p^2(x. M) \backslash M$, it follows that $p^2(x, M) = p^2(S, M)$. Hence $p(x, M) = p(S, M)$. $y \wedge x = 0$ implies that $y \in p(S, M)$. Since $y \in p^2(S, M)$, it follows that $y \in p(S, M) \cap p^2(S, M) = M$, a contradiction. Thus M is a prime subgroup of $p^2(S, M)$. Suppose that B is a convex l-subgroup of G such that $p^2(S, M) \subseteq B$ and such that M is prime in B. Let $0 < s \in S \backslash M \subseteq B \backslash M$. Since it has been shown that (b) implies (d), it follows that $p(B, M) = p(s, M) = p(S, M)$. Therefore $B \subseteq p^2(B, M) = p^2(S, M)$.

(g) implies (a) is immediate.

Corollary 3.8. *If M is a proper prime subgroup of S, then the following are equivalent.*

(a) *M is prime in G.*
(b) $p^2(S, M) = G$.
(c) $p(S, M) = M$.

This corollary follows from the theorem and Theorem 3.5. A convex l-subgroup C of G is said to be *closed* if whenever $\{g_\alpha \mid \alpha \in A\} \subseteq C$ such that $\bigvee g_\alpha$ exists implies that $\bigvee g_\alpha \in C$. It is well known that polars are closed subgroups.

Lemma 3.9. (1) *M is closed if and only if $p(S, M)$ and $p^2(S, M)$ are closed.*

(2) *For each $\lambda \in A$ let S_λ be a convex l-subgroup of G such that $M \subseteq S_\lambda$. Then $\bigcap p(S_\lambda, M) = p([\bigcup S_\lambda], M)$.*

(3) *If T is a convex l-subgroup of G that contains M, then $p^2(S \cap T, M) = p^2(S, M) \cap p^2(T, M)$.*

Proof. (1) To show that a convex l-subgroup is closed, it suffices to consider positive elements. Suppose that M is closed and let $\{g_\alpha \mid \alpha \in A\} \subseteq p(S, M)^+$ such that $\bigvee g_\alpha$ exists. If $0 \leqq s \in S$, then $g_\alpha \wedge s \in M$ for each $\alpha \in A$, hence $(\bigvee g_\alpha) \wedge s = \bigvee(g_\alpha \wedge s) \in M$ ([1], p. 221) since M is closed. By a similar argument it follows that $p^2(S, M)$ is closed. The converse is trivial as the intersection of closed subgroups is closed and $p(S, M) \cap p^2(S, M) = M$.

(2) For each $\alpha \in A$ it follows by (2) of Lemma 3.1 that $p(S_\alpha, M) \supseteq p([\bigcup S_\lambda], M)$, hence $\bigcap p(S_\lambda, M) \supseteq p([\bigcup S_\lambda], M)$. Conversely for each $\alpha \in A$, $(\bigcap p(S_\lambda, M)) \cap S_\alpha \subseteq p(S_\alpha, M) \cap S_\alpha \subseteq M$, hence $\bigvee_\alpha((\bigcap p(S_\lambda, M)) \cap S_\alpha) = (\bigcap p(S_\lambda, M)) \cap ([\bigcup S_\lambda]) \subseteq M$ ([7], Theorem 2). Therefore by (2) of Lemma 3.4, $\bigcap p(S_\lambda, M) \subseteq p([\bigcup S_\lambda], M)$.

(3) From (2) of Lemma 3.1 it follows that $p^2(S \cap T, M) \subseteq p^2(S, M) \cap p^2(T, M)$. Let $0 \leqq x \in p^2(S, M) \cap p^2(T, M)$, let $0 \leqq y \in p(S \cap T, M)$, let $0 \leqq s \in S$, and let $0 \leqq t \in T$. Then $s \wedge t \in S \cap T$, therefore $y \wedge s \wedge t \in M$ and so $x \wedge y \wedge s \wedge t \in M$. It follows that $x \wedge y \wedge s \in p(T, M)$. $x \in p^2(T, M)$ implies that $x \wedge y \wedge s \in p^2(T, M)$. Therefore $x \wedge y \wedge s \in p(T, M) \cap p^2(T, M) = M$, hence $x \wedge y \in p(S, M)$. Now $x \wedge y \in p^2(S, M)$ as $x \in p^2(S, M)$. Thus $x \wedge y \in p(S, M) \cap p^2(S, M) = M$. Therefore $x \in p^2(S \cap T, M)$.

It is easy to construct examples to show that (3) of this lemma is not true for arbitrary intersections.

A *Boolean algebra* is a lattice with a smallest element 0 and a largest element 1 which is complemented and distributive. Let M be a fixed convex l-subgroup of G and let $\mathscr{B}$ denote the collection of all M-polars of G. By Lemma 3.3 $\mathscr{B} = \{p(C, M) \mid C$ is a convex l-subgroup of $G\} = \{p(D, M) \mid M \subseteq D$ and D is a convex l-subgroup of $G\}$. We define a partial order on $\mathscr{B}$ by set inclusion. For $\{p(S_\lambda, M) \mid \lambda \in A\} \subseteq \mathscr{B}$, define $\sqcup_\lambda p(S_\lambda, M) = p^2([\bigcup p(S_\lambda, M)], M)$ and $\sqcap_\lambda p(S_\lambda, M) = p([\bigcup p^2(S_\lambda, M)], M)$.

Theorem 3.10. *The collection $\mathscr{B} = \mathscr{B}(\sqcup, \sqcap, \subseteq)$ of all M-polars of G is a complete*

Boolean algebra where the 1 *is G and the* 0 *is M*. $p(A, M) \sqcup p(B, M) = p(A \cap B, M)$ *and* $p(A, M) \sqcap p(B, M) = p([A \cup B], M) = p(A, M) \cap p(B, M)$. *Moreover, if* $p(T, M)$, $p(S_\lambda, M) \in \mathscr{B}(\lambda \in \Lambda)$, *then* $\sqcap_\lambda p(S_\lambda, M) = \bigcap_\lambda p(S_\lambda, M)$ and $(T, M) \sqcap \sqcap (\sqcup_\lambda p(S_\lambda, M)) = \sqcup_\lambda (p(T, M) \sqcap p(S_\lambda, M))$.

Proof. Let $\{p(S_\lambda, M) \mid \lambda \in \Lambda\} \subseteq \mathscr{B}$. By Lemmas 3.9 and 3.4 it follows that $\sqcap p(S_\lambda, M) = p([\bigcup p^2(S_\lambda, M)], M) = \bigcap p^3(S_\lambda, M) = \bigcap p(S_\lambda, M) = p([\bigcup S_\lambda], M)$. Therefore $\bigcap p(S_\lambda, M)$ is an M-polar and is a lower bound for $\{p(S_\lambda, M) \mid \lambda \in \Lambda\}$. If $p(C, M)$ is any other lower bound for $\{p(S_\lambda, M) \mid \lambda \in \Lambda\}$, then $\bigcap p(S_\lambda, M) \supseteq p(C, M)$. Thus $\bigcap p(S_\lambda, M)$ is the greatest lower bound for $\{p(S_\lambda, M) \mid \lambda \in \Lambda\}$. For each $\alpha \in \Lambda$, $p(S_\alpha, M) \subseteq [\bigcup p(S_\lambda, M)]$, hence $p(S_\alpha, M) = p^3(S_\alpha, M) \subseteq p^2([\bigcup p(S_\lambda, M)], M)$. If $p(C, M)$ is any other upper bound for $\{p(S_\lambda, M) \mid \lambda \in \Lambda\}$, then $p(C, M) \supseteq [\bigcup p(S_\lambda, M)]$. Therefore $p(C, M) = p^3(C, M) \supseteq p^2([\bigcup p(S_\lambda, M)], M)$. Thus $p^2([\bigcup p(S_\lambda, M)], M)$ is the least upper bound for $\{p(S_\lambda, M) \mid \lambda \in \Lambda\}$. In particular, if Λ is finite, then it follows from Lemma 3.9 that $p^2([\bigcup p(S_\lambda, M)], M) = p(p([\bigcup p(S_\lambda, M)], M), M) = = p(\bigcap p^2(S_\lambda, M), M) = p^3(\bigcap S_\lambda, M) = p(\bigcap S_\lambda, M)$. Thus $\mathscr{B}$ is a complete lattice. Let $p(T, M) \in \mathscr{B}$. Then by Lemma 3.4, $M = p(T, M) \cap p^2(T, M)$ and from the above $G = p(M, M) = p(p(T, M) \cap p^2(T, M), M) = p(T, M) \sqcup p^2(T, M)$. Thus $\mathscr{B}$ has a 0 and a 1 and is complemented. To show that $\mathscr{B}$ is a distributive lattice, it suffices to show that $p(T, M) \sqcap (\sqcup_\lambda p(S_\lambda, M)) = \sqcup_\lambda (p(T, M) \sqcap p(S_\lambda, M))$. By an application of Lemmas 3.4 and 3.9, the definition of $\sqcup$, and Theorem 2 in [7], it follows that

$$
\begin{aligned}
p(T, M) \sqcap (\sqcup_\lambda p(S_\lambda, M)) &= p^3(T, M) \cap (p^2([\textstyle\bigcup_\lambda p(S_\lambda, M)], M)) = \\
= p^2(p(T, M) \cap [\textstyle\bigcup_\lambda p(S_\lambda, M)], M) &= p^2([\textstyle\bigcup_\lambda (p(T, M) \cap p(S_\lambda, M))], M) = \\
= p^2([\textstyle\bigcup_\lambda p([T \cup S_\lambda]), M)], M) &= \sqcup_\lambda p([T \cup S_\lambda], M) = \\
= \sqcup_\lambda (p(T, M) \cap p(S_\lambda, M)) &= \sqcup_\lambda (p(T, M) \sqcap p(S_\lambda, M)) \,.
\end{aligned}
$$

This completes the proof of the theorem.

Let L and L' be lattices. If π is a mapping of L into L' with the property that $(x \vee y)\pi = x\pi \vee y\pi$ and $(x \wedge y)\pi = x\pi \wedge y\pi$ for all $x, y \in L$, then π is called a *lattice homomorphism*. A one to one lattice homomorphism is called a *lattice isomorphism*. If L' has a least element 0, then the set $K(\pi) = \{x \in L \mid x\pi = 0\}$ is called the *kernel* of π. If π_1 and π_2 are two lattice homomorphisms of a lattice L, then π_1 is said to be *greater than* π_2 (see [8]) if for all $x, y \in L$, $x\pi_2 = y\pi_2$ implies that $x\pi_1 = y\pi_1$.

Let $\mathscr{C}$ denote the lattice of all convex l-subgroups of G and let M be a fixed element of $\mathscr{C}$. For A in $\mathscr{C}$ define $A\pi = p^2(A, M)$. The next theorem is a generalization of a result of K. Lorenz ([7], Theorem 4).

Theorem 3.11. *π is a lattice homomorphism of $\mathscr{C}$ onto $\mathscr{B}$. If $\{C_\lambda \mid \lambda \in \Lambda\} \subseteq \mathscr{C}$, then $[\bigcup C_\lambda]\pi = \sqcup(C_\lambda \pi)$. M is the largest element in $K(\pi)$ and π may be characterised as a maximal lattice homomorphism of $\mathscr{C}$ such that M is the largest element in $K(\pi)$.*

Proof. Clearly π is a function. It follows from (3) of Lemma 3.4 that π restricted to $\mathscr{B}$ is the identity, hence π is onto. If $A, B \in \mathscr{C}$, then by Lemma 3.9, $A\pi \cap B\pi = p^2(A, M) \cap p^2(B, M) = p^2(A \cap B, M) = (A \cap B)\,\pi$, and by Lemma 3.9 and Theorem 3.10, $A\pi \sqcup B\pi = p^2(A, M) \sqcup p^2(B, M) = p(p(A, M), M) \sqcup \sqcup p(p(B, M), M) = p(p(A, M) \cap p(B, M), M) = p^2([A \cup B], M) = ([A \cup B])\,\pi$. Thus π is a lattice homomorphism. If $\{C_\lambda \mid \lambda \in \Lambda\} \subseteq \mathscr{C}$, then by successive use of Lemmas 3.9, 3.4, and 3.9 and the definition of $\sqcup$, it follows that $[\bigcup C_\lambda]\,\pi = = p^2([\bigcup C_\lambda], M) = p(p([\bigcup C_\lambda], M), M) = p(\bigcap p(C_\lambda, M), M) = p(\bigcap p^3(C_\lambda, M), M) = = p^2([\bigcup p^2(C_\lambda, M)], M) = \sqcup p^2(C_\lambda, M) = \sqcup(C_\lambda \pi)$.

$M\pi = p^2(M, M) = M$, hence $M \in K(\pi)$. If $A \in \mathscr{C}$ such that $A\pi = M$, then $G = p(M, M) = = p^3(A, M) = p(A, M)$ and so by Lemma 3.3, $A \subseteq M$. Let τ be any lattice homomorphism of $\mathscr{C}$ such that M is the largest element in $K(\tau)$. For each A in $\mathscr{C}$ let $\triangle(A) = = \{C \in \mathscr{C} \mid A\tau \wedge C\tau = M\tau\}$. Now suppose that there exists $A, B \in \mathscr{C}$ such that $A\tau = = B\tau$. Then $\triangle(A) = \triangle(B)$. If $C \in \triangle(A)$, then $(A \cap C)\,\tau = A\tau \wedge C\tau = M\tau$ and so $A \cap C \subseteq M$. By Lemma 3.4, $C \subseteq p(A, M)$. In particular, $p(A, M) \in \triangle(A)$ and is the largest member of $\triangle(A)$. Similarly $p(B, M)$ is the largest member in $\triangle(B)$ and since $\triangle(A) = \triangle(B)$, it follows that $p(A, M) = p(B, M)$. Therefore $A\pi = p^2(A, M) = = p^2(B, M) = B\pi$.

It is easy to show that the mapping $p(C, M) \to p^2(C, M)$ is an anti-lattice isomorphism of $\mathscr{B}$ onto $\mathscr{B}$. Now let $\mathscr{D} = \{p^2(a, M) \mid a \in G^+\}$. We shall call the elements of $\mathscr{D}$ *principal M-bipolars*. The next theorem uses the result by K. Lorenz ([7], Lemma 1) that for $a, b \in G^+$, $G(a \wedge b) = G(a) \cap G(b)$ and that $G(a \vee b) = = [G(a) \cup G(b)]$. With this we extend Theorem 3 in [7].

Theorem 3.12. *The set $\mathscr{D}$ is a sublattice of $\mathscr{B}$, where $p^2(a, M) \cap p^2(b, M) = = p^2(a \wedge b, M)$ and $p^2(a, M) \sqcup p^2(b, M) = p^2(a \vee b, M)$, $a, b \in G^+$. Thus the mapping ϱ of G^+ into $\mathscr{D}$ defined by $a\varrho = p^2(a, M)$ is a lattice homomorphism of G^+ onto $\mathscr{D}$ with kernel M^+. Moreover, if $\{g_\alpha \mid \alpha \in \Lambda\} \subseteq G^+$ such that $\bigvee g_\alpha$ exists and if M is closed, then $(\bigvee g_\alpha)\,\varrho = \sqcup(g_\alpha \varrho)$.*

Proof. By Corollary 3.2, $p^2(a, M) = p^2(G(a), M)$. Let $p^2(a, M), p^2(b, M) \in \mathscr{D}$. Then by Lemma 3.9, $a\varrho \cap b\varrho = p^2(a, M) \cap p^2(b, M) = p^2(G(a) \cap G(b), M) = = p^2(G(a \wedge b), M) = (a \wedge b)\,\varrho$ and by Theorem 3.11, $a\varrho \sqcup b\varrho = p^2(a, M) \sqcup \sqcup p^2(b, M) = p^2([G(a) \cup G(b)], M) = p^2(G(a \vee b), M) = (a \vee b)\,\varrho$. Therefore $\mathscr{D}$ is a sublattice of $\mathscr{B}$ and ϱ is a lattice homomorphism of G^+ onto $\mathscr{D}$. If $a\varrho = M$, then $p(a, M) = p^3(a, M) = p(M, M) = G$, hence by Lemma 3.3, $a \in M^+$. Conversely if $a \in M^+$, then $a\varrho = p^2(a, M) \subseteq p^2(M, M) = M$ and so $a \in K(\varrho)$.

Next suppose that $\{g_\alpha \mid \alpha \in \Lambda\} \subseteq G^+$ such that $g = \bigvee g_\alpha$ exists and suppose that M is closed. $g \geqq g_\alpha$ implies that $p(g, M) \subseteq p(g_\alpha, M)$ for all α in Λ. Therefore $p(g, M) \subseteq \subseteq \bigcap p(g_\alpha, M)$. Let $0 \leqq x \in \bigcap p(g_\alpha, M)$. Then $x \wedge g_\alpha \in M$ for all α and so $x \wedge g = = x \wedge (\bigvee g_\alpha) = \bigvee(x \wedge g_\alpha) \in M$. Thus $p(g, M) = \bigcap p(g_\alpha, M)$. Therefore by Lemmas 3.4 and 3.9 and the definition of $\sqcup$, it follows that $g\varrho = p^2(g, M) = p(p(g, M), M) = = p(\bigcap p^3(g_\alpha, M), M) = p^2([\bigcup p^2(g_\alpha, M)], M) = \sqcup p^2(g_\alpha, M) = \sqcup(g_\alpha \varrho)$.

In the case $M = \{0\}$ there is a natural lattice isomorphism of the lattice of all carriers of G (see [6], p. 72) onto the collection of all principal bipolars of G.

A convex l-subgroup A of G is called an *M-summand* of G if there exists a convex l-subgroup B of G such that $G = [A \cup B]$ and $A \cap B = M$. If this is the case, then it will be denoted by $G = A \mid + \mid B$.

Lemma 3.13. (1) *If* $G = A \mid + \mid B$ *and if* C *is a convex l-subgroup of* G *that contains* M, *then* $C = (C \cap A) \mid + \mid (C \cap B)$.

(2) If $G = A \mid + \mid B$, *then* $A = p(B, M)$ *and* $M = p(A, M) \cap p(B, M)$.

Proof. (1) $(C \cap A) \cap (C \cap B) = C \cap A \cap B = C \cap M = M$ and $[(C \cap A) \cup \cup (C \cap B)] = C \cap [A \cup B] = C \cap G = C$.

(2) By Lemma 3.4, $A \cap B = M$ implies $A \subseteq p(B, M)$. If $0 \leqq x \in p(B, M)$, then $x = a_1 + b_1 + \ldots + a_n + b_n$, where $a_i \in A$ and $b_i \in B$ and without loss of generality it may be assumed that a_i and b_i are greater than or equal to 0. Thus for each $i(1 \leqq \leqq i \leqq n)$, $0 \leqq b_i = b_i \wedge b_i \leqq (a_1 + b_1 + \ldots + a_n + b_n) \wedge b_i = x \wedge b_i \in M$ as $x \in p(B, M)$. Therefore $b_i \in M \subseteq A$ and so $x \in A$. $M = p(G, M) = p([A \cup B], M) = = p(A, M) \cap p(B, M)$ by Lemma 3.9.

For a fixed convex l-subgroup M of G, let $\mathscr{M}$ be the collection of all M-summands of G. In particular, $G, M \in \mathscr{M}$. If $M = \{0\}$, then this is precisely the collection of all cardinal summands of G.

Theorem 3.14. $\mathscr{M}$ *is a subalgebra of* $\mathscr{B}$. *Moreover, for* $A, C \in \mathscr{M}$, $A \sqcup C = = [A \cup C]$. *Thus* $\mathscr{M}$ *is a sublattice of* $\mathscr{C}$.

Proof. By Lemma 3.13 $\mathscr{M}$ is a subset of $\mathscr{B}$. If $A, C \in \mathscr{M}$, then $G = A \mid + \mid B = = C \mid + \mid D$ for some $B, D \in \mathscr{M}$. By Lemma 3.13 it follows that $G = (A \cap C) \mid + \mid (B \cap \cap C) \mid + \mid D = A \mid + \mid (B \cap C) \mid + \mid (B \cap D)$. $A \cap C$, $A \mid + \mid (B \cap C) \in \mathscr{M}$ and clearly $[A \cup C] = A \mid + \mid (B \cap C)$. Thus $A \sqcup C = p^2([A \cup C], M) = p^2(A \mid + \mid (B \cap \cap C), M) = A \mid + \mid (B \cap C) = [A \cup C]$. It follows from Lemma 3.13 that if $A \in \mathscr{M}$, then $p(A, M) \in \mathscr{M}$. Hence $\mathscr{M}$ is a subalgebra of $\mathscr{B}$. Since $\sqcap$ in $\mathscr{B}$ agrees with $\bigcap$ in $\mathscr{C}$, it follows that $\mathscr{M}$ is a sublattice of $\mathscr{C}$.

In general $\mathscr{M}$ is not a complete subalgebra of $\mathscr{B}$. It is not difficult to construct examples to show that the hypothesis that $\mathscr{M}$ is a complete subalgebra of $\mathscr{B}$ is not sufficient to insure that $\mathscr{M}$ will be a complete sublattice of $\mathscr{C}$.

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