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A CLASS OF CLOSE-TO-CONVEX FUNCTIONS

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1. Introduction. Let $f(z)$, $F(z)$ be regular in the unit disk $D(|z| < 1)$ and satisfy the conditions $f(0 = F(0) = 0, f'(0) = 1, F'(0) = e^{i\beta}$, where β is real. If

$$(1.1)\qquad \operatorname{Re}\left\{\frac{z f'(z)}{F(z)}\right\} \geqq \lambda \quad \text{and} \quad \operatorname{Re}\left\{\frac{z F'(z)}{F(z)}\right\} \geqq \sigma$$

for z in D and $0 \leqq \lambda, \sigma \leqq 1$, then $f(z)$ is close-to-convex of order λ and type σ with respect to $F(z)$. This class of functions is discussed by R. J. Libra in [2].

Let

$$(1.2)\qquad f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

and

$$(1.3)\qquad F(z) = e^{i\beta} z + b_2 z^2 + \ldots + b_n z^n + \ldots$$

Since

$$(1.4)\qquad \left.\frac{z f'(z)}{F(z)}\right|_{z=0} = e^{-i\beta} = \operatorname{Cos}\beta - i \operatorname{Sin}\beta\,, \quad \operatorname{Cos}\beta \geqq \lambda\,,$$

we have

$$(1.5)\qquad \operatorname{Re}\left\{\operatorname{Sec}\beta \frac{z f'(z)}{F(z)} + i \tan\beta\right\} \geqq \lambda \operatorname{Sec}\beta\,.$$

Denote by $C^*(\lambda, \sigma)$, $0 \leqq \lambda, \sigma \leqq 1$, the family of all functions $f(z)$ and $F(z)$ which satisfy (1.1) and

$$(1.6)\qquad \left|\frac{\operatorname{Sec}\beta \dfrac{z f'(z)}{F(z)} + i \tan\beta - \lambda \operatorname{Sec}\beta}{1 - \lambda \operatorname{Sec}\beta} - \alpha\right| < \alpha\,, \quad (\alpha \geqq 1) \quad \text{for} \quad |z| < 1\,.$$

The results obtained by R. J. Libra in [2] shall follow as special cases from our results by taking $\alpha = \infty$.

2. We first prove the following lemmas which will be used in the subsequent work.

Lemma (2.1). *Let*

$$P(z) = 1 + p_1 z + p_2 z^2 + \dots + p_n z^n + \dots \tag{2.1}$$

be regular in D and satisfy the condition

$$|P(z) - \alpha| < \alpha, (\alpha \geqq 1) \quad \textit{for} \quad |z| < 1 ,$$

then

$$P(z) = \frac{1 + \varphi(z)}{1 - \left(1 - \frac{1}{\alpha}\right)\varphi(z)} , \tag{2.2}$$

where $\varphi(0) = 0$, $\varphi(z)$ *is regular in* $|z| < 1$, *satisfying* $|\varphi(z)| \leqq |z|$ *in* $|z| < 1$.

$$\frac{1 - |z|}{1 + \left(1 - \frac{1}{\alpha}\right)|z|} \leqq |P(z)| \leqq \frac{1 + |z|}{1 - \left(1 - \frac{1}{\alpha}\right)|z|} . \tag{2.3}$$

$$\left|\frac{z P'(z)}{P(z)}\right| \leqq \frac{\left(2 - \frac{1}{\alpha}\right)|z|}{1 - \frac{1}{\alpha}|z| - \left(1 - \frac{1}{\alpha}\right)|z|^2} . \tag{2.4}$$

All these inequalities are the best possible.

Proof.

Let

$$\psi(z) = \frac{P(z)}{\alpha} - 1 , \tag{2.5}$$

then

$$\psi(0) = \frac{1}{\alpha} - 1 . \tag{2.6}$$

Let

$$\varphi(z) = \frac{\psi(z) - \psi(0)}{1 - \psi(0)\,\psi(z)} , \tag{2.7}$$

then $\varphi(0) = 0$ and $|\varphi(z)| < 1$, therefore by Schwarz's lemma

$$|\varphi(z)| \leqq |z| . \tag{2.8}$$

From (2.5), (2.6) and (2.7) we obtain

$$P(z) = \frac{1 + \varphi(z)}{1 - \left(1 - \frac{1}{\alpha}\right)\varphi(z)},$$

which proves (2.2).

(2.2) gives in connection with (2.8)

$$\frac{1 - |z|}{1 + \left(1 - \frac{1}{\alpha}\right)|z|} \leqq |P(z)| \leqq \frac{1 + |z|}{1 - \left(1 - \frac{1}{\alpha}\right)|z|}.$$

Equality holds for

$$P(z) = \frac{1 + \varepsilon z}{1 - \varepsilon\left(1 - \frac{1}{\alpha}\right)z}, \quad |\varepsilon| = 1.$$

From (2.2)

$$P(z) = \frac{1 + \varphi(z)}{1 - \left(1 - \frac{1}{\alpha}\right)\varphi(z)}.$$

Differentiation and simplification give

$$\frac{z\,P'(z)}{P(z)} = \frac{\left(2 - \frac{1}{\alpha}\right)z\,\varphi'(z)}{1 + \frac{1}{\alpha}\varphi(z) - \left(1 - \frac{1}{\alpha}\right)\varphi^2(z)}.$$

Therefore

$$\left|\frac{z\,P'(z)}{P(z)}\right| \leqq \frac{\left(2 - \frac{1}{\alpha}\right)|z|}{1 - |z|^2} \cdot \frac{1 - |\varphi(z)|^2}{1 - \frac{1}{\alpha}|\varphi(z)| - \left(1 - \frac{1}{\alpha}\right)|\varphi(z)|^2}, \tag{2.9}$$

Where we used the estimate $|\varphi'(z)| \leqq (1 - |\varphi(z)|^2)/(1 - |z|^2)$ (see [1], p. 18). It can be easily shown that

$$\frac{1 - |\varphi(z)|^2}{1 - \frac{1}{\alpha}|\varphi(z)| - \left(1 - \frac{1}{\alpha}\right)|\varphi(z)|^2} \leqq \frac{1 - |z|^2}{1 - \frac{1}{\alpha}|z| - \left(1 - \frac{1}{\alpha}\right)|z|^2}. \tag{2.10}$$

(2.9) gives in connection with (2.10)

$$\left|\frac{z\,P'(z)}{P(z)}\right| \leqq \frac{\left(2-\frac{1}{\alpha}\right)|z|}{1-\frac{1}{\alpha}|z|-\left(1-\frac{1}{\alpha}\right)|z|^2},$$

with equality holding for

$$P(z)=\frac{1+\varepsilon z}{1-\varepsilon\left(1-\frac{1}{\alpha}\right)z}, \quad |\varepsilon|=1.$$

Lemma (2.2). *If $P(z)$ satisfies the conditions of Lemma (2.1), then*

(2.11) $$|p_n| \leqq \left(2-\frac{1}{\alpha}\right) \quad \textit{for all } n.$$

The bounds are sharp.

Proof. From (2.2)

$$P(z)=\frac{1+\varphi(z)}{1-\left(1-\frac{1}{\alpha}\right)\varphi(z)},$$

where $\varphi(z)$ is regular in D and satisfies the conditions $\varphi(0)=0$ and $|\varphi(z)| \leqq 1$ for z in D. Therefore

$$\left[\left(2-\frac{1}{\alpha}\right)+\left(1-\frac{1}{\alpha}\right)\sum_{k=1}^{n-1} p_k z^k\right]\varphi(z)=\sum_{k=1}^{n} p_k z^k+\sum_{k=n+1}^{\infty} c_k z^k,$$

where $\sum_{k=n+1}^{\infty} c_k z^k$ converges in D. Then, since $|\varphi(z)| \leqq 1$,

$$\left|\left(2-\frac{1}{\alpha}\right)+\left(1-\frac{1}{\alpha}\right)\sum_{k=1}^{n-1} p_k z^k\right| \geqq \left|\sum_{k=1}^{n} p_k z^k+\sum_{k=n+1}^{\infty} c_k z^k\right|, \quad n \geqq 1.$$

Squaring both sides and integrating around $|z|=r<1$

$$\left(2-\frac{1}{\alpha}\right)^2+\left(1-\frac{1}{\alpha}\right)^2\sum_{k=1}^{n-1}|p_k|^2 r^{2k} \geqq \sum_{k=1}^{n}|p_k|^2 r^{2k}+\sum_{k=n+1}^{\alpha}|c_k|^2 r^{2k} \geqq$$

$$\geqq \sum_{k=1}^{n}|p_k|^2 r^{2k}, \quad n \geqq 1.$$

Let $r \to 1$, then

$$\left(2-\frac{1}{\alpha}\right)^2+\left(1-\frac{1}{\alpha}\right)^2\sum_{k=1}^{n-1}|p_k|^2 \geqq \sum_{k=1}^{n}|p_k|^2,$$

or

$$|p_n|^2 \leqq \left(2 - \frac{1}{\alpha}\right)^2 - \left[1 - \left(1 - \frac{1}{\alpha}\right)^2\right] \sum_{k=1}^{n-1} |p_k|^2 .$$

Therefore,

$$|p_n| \leqq \left(2 - \frac{1}{\alpha}\right), \quad n \geqq 1 .$$

Considering now the function

$$P(z) = \frac{1 + z^n}{1 - \left(1 - \frac{1}{\alpha}\right) z^n}, \quad |z| < 1$$

for which

$$\left|\frac{1}{\alpha} P(z) - 1\right| = \left|\frac{\left(\frac{1}{\alpha} - 1\right) + z^n}{1 + \left(\frac{1}{\alpha} - 1\right) z^n}\right| \leqq 1$$

for $|z| < 1$, $(1/\alpha - 1) < 1$.

Also $P(z)$ has the expansion

$$P(z) = 1 + \left(2 - \frac{1}{\alpha}\right) z^n + \dots$$

showing that the estimate is sharp.

3. Some properties of $c^*(\lambda, \sigma)$. Theorem (3.1) *If $f(z)$ belongs to $c^*(\lambda, \sigma)$, then the radius of convexity of $f(z)$ is greater than or equal to the smallest positive root of*

(3.1) $$(1 - 2\sigma)\left[\left(1 - \frac{1}{\alpha}\right)(1 - \lambda b) - \lambda b\right] r^3 - \\ - \left[(2\sigma - 1) + 2(1 - \sigma)\left\{\left(1 - \frac{1}{\alpha}\right)(1 - \lambda b) - \lambda b\right\} + \left(2 - \frac{1}{\alpha}\right)(1 - \lambda b)\right] r^2 + \\ + (2\sigma - 3) r + 1 = 0 .$$

where $b = \text{Sec}\, \beta$.

Proof. Let

(3.2) $$P(z) = \frac{\text{Sec}\, \beta \dfrac{z f'(z)}{F(z)} + i \tan \beta - \lambda \,\text{Sec}\, \beta}{1 - \lambda \,\text{Sec}\, \beta},$$

then $P(z)$ satisfies the conditions of Lemma (2.1). Differentiating (3.2) and simplifying we get

$$(3.2') \qquad 1 + \frac{z f''(z)}{f'(z)} = \frac{z F'(z)}{F(z)} + \frac{z P'(z)}{P(z) + \eta}, \; \eta = \frac{\lambda \operatorname{Sec} \beta - i \tan \beta}{1 - \lambda \operatorname{Sec} \beta},$$

here $\operatorname{Re} \{\eta\} \geqq 0$, then

$$(3.3) \qquad \operatorname{Re} \left\{1 + \frac{z f''(z)}{f'(z)}\right\} \geqq \min \operatorname{Re} \left\{\frac{z F'(z)}{F(z)}\right\} - \max \left|\frac{z P'(z)}{P(z) + \eta}\right|.$$

Since $F(z)$ is starlike of order σ, therefore [3].

$$(3.4) \qquad \operatorname{Re} \left\{\frac{z F'(z)}{F(z)}\right\} \geqq \frac{1 - (1 - 2\sigma) r}{1 + r}, \quad |z| = r, \quad 0 \leqq r < 1 .$$

$$\left|\frac{z P'(z)}{P(z) + \eta}\right| = \frac{\left|\frac{z P'(z)}{P(z)}\right|}{1 + \frac{\eta}{P(z)}} \leqq \frac{\left|\frac{z P'(z)}{P(z)}\right|}{\left[1 + \operatorname{Re}\left\{\frac{\eta}{P(z)}\right\}\right]}$$

using (2.3) and (2.4)

$$(3.5) \qquad \frac{\left(2 - \frac{1}{\alpha}\right)|z|}{1 - \frac{1}{\alpha}|z| - \left(1 - \frac{1}{\alpha}\right)|z|^2} \; \frac{1}{1 + \operatorname{Re}\{\eta\} \dfrac{1 - |z|}{1 + \left(1 - \frac{1}{\alpha}\right)|z|}} =$$

$$= \frac{\left(2 - \frac{1}{\alpha}\right) r(1 - \lambda b)}{(1 - r)\left[1 + r\left\{\left(1 - \frac{1}{\alpha}\right)(1 - \lambda b) - \lambda b\right\}\right]},$$

here $b = \operatorname{Sec} \beta$, $|z| = r$.

From (3.3), (3.4) and (3.5) we get

$$(3.6) \qquad \operatorname{Re} \left\{1 + z \frac{f''(z)}{f'(z)}\right\} \geqq \frac{1 - (1 - 2\sigma) r}{1 + r} -$$

$$- \frac{\left(2 - \frac{1}{\alpha}\right)(1 - \lambda b) r}{(1 - r)\left[1 + r\left\{\left(1 - \frac{1}{\alpha}\right)(1 - \lambda b) - \lambda b\right\}\right]} =$$

$$= \left\{(1-2\sigma)\left[\left(1-\frac{1}{\alpha}\right)(1-\lambda b)-\lambda b\right]r^3 - \right.$$

$$-\left[(2\sigma-1)+2(1-\sigma)\left\{\left(1-\frac{1}{\alpha}\right)(1-\lambda b)-\lambda b\right\}+\left(2-\frac{1}{\alpha}\right)(1-\lambda b)\right]r^2 +$$

$$\left.+ 2(\sigma-3)\,r+1\right\}.\left\{(1-r^2)\left[1+r\left\{\left(1-\frac{1}{\alpha}\right)(1-\lambda b)-\lambda b\right\}\right]\right\}^{-1}.$$

$f(z)$ is Convex whenever the last expression is positive. Denoting by $P(r)$, the numerator on the right hand side of the ineuqality (3,6), observe that $P(0) = 1 > 0$ and $P(1) = -2(2 - 1/\alpha)(1 - \lambda b) < 0$.

Therefore the smallest positive root of the equation $P(r) = 0$ lies between 0 and 1. If we denote this root by r_0, it follows that the inequality (3.6) holds for $r = |z| < r_0$. Hence the radius of convexity of $f(z)$ is greater than or equal to the smallest positive root of

$$(1-2\sigma)\left[\left(1-\frac{1}{\alpha}\right)(1-\lambda b)-\lambda b\right]r^3 -$$

$$-\left[(2\sigma-1)+2(1-\sigma)\left\{\left(1-\frac{1}{\alpha}\right)(1-\lambda b)-\lambda b\right\}+\left(2-\frac{1}{\alpha}\right)(1-\lambda b)\right]r^2 +$$

$$+(2\sigma-3)\,r+1=0\,.$$

Theorem (3.2) *If $f(z) \in c^*(\lambda, \sigma)$, then for $|z| = r$, $0 \leqq r < 1$,*

(3.7)
$$\frac{1-r}{(1+r)^{2(1-\sigma)}\left[1+r\left\{\left(1-\frac{1}{\alpha}\right)(1-\lambda)-\lambda\xi\right\}\right]} \leqq |f'(z)| \leqq$$

$$\leqq \frac{1+r\left[\left(1-\frac{1}{\alpha}\right)(1-\lambda)-\lambda\right]}{(1-r)^{3-2\sigma}}$$

and

(3.8)
$$\int_0^r \frac{(1-r)\,\mathrm{d}r}{(1+r)^{2(1-\sigma)}\left[1+r\left\{\left(1-\frac{1}{\alpha}\right)(1-\lambda)-\lambda\right\}\right]} \leqq |f(z)| \leqq$$

$$\leqq \left\{2r(1-\sigma)\left[\left(1-\frac{1}{\alpha}\right)(1-\lambda)-\lambda\right]+\left[2(\lambda-\sigma)+\frac{1}{\alpha}(1-\lambda)\right].\right.$$

$$\left.\cdot\left[1-(1-r)^{2(1-\sigma)}\right]\right\}\left[2(1-\sigma)(1-2\sigma)(1-r)^{2(1-\sigma)}\right]^{-1}, \quad \sigma \neq \tfrac{1}{2}, 1\,,$$

$$\left|\begin{array}{ll} \leqq \left(2-\frac{1}{\alpha}\right)(1-\lambda)\frac{r}{1-r}+\left[\left(1-\frac{1}{\alpha}\right)(1-\lambda)-\lambda\right]\log(1-r)\,, & \sigma=\frac{1}{2} \\ \leqq \left[\lambda-\left(1-\frac{1}{\alpha}\right)(1-\lambda)\right]r-\left(2-\frac{1}{\alpha}\right)(1-\lambda)\log(1-r)\,, & \sigma=1\,. \end{array}\right.$$

The above estimates are all sharp.

Proof. Let $f(z) \in c^*(\lambda, \sigma)$ with respect to $F(z)$, $F'(0) = e^{i\beta}$ and $\operatorname{Sec}\beta = b$. Then, since

$$\text{(3.9)} \qquad \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} = r\frac{\partial}{\partial r}\log|f'(z)|\,, \quad |z| = r\,,$$

(3.9) gives in connection with (3.6)

$$\text{(3.10)} \quad \frac{\partial}{\partial r}\log|f'(z)| \geqq \frac{-2(1-\sigma)}{1+r} - \frac{1}{1-r} - \frac{\left(1-\frac{1}{\alpha}\right)(1-\lambda b)-\lambda b}{1+r\left[\left(1-\frac{1}{\alpha}\right)(1-\lambda b)-\lambda b\right]}\,.$$

Integrating both sides of (3.10) from 0 to r, we obtain

$$\text{(3.11)} \qquad |f''(z)| \geqq \frac{1-r}{(1+r)^{2(1-\sigma)}\left[1+r\left\{\left(1-\frac{1}{\alpha}\right)(1-\lambda b)-\lambda b\right\}\right]} \geqq$$

$$\geqq \frac{1-r}{(1+r)^{2(1-\sigma)}\left[1+r\left\{\left(1-\frac{1}{\alpha}\right)(1-\lambda)-\lambda\right\}\right]}\,,$$

since $\lambda b \geqq \lambda$.

Again from (3.2), (3.5) and Knovn bounds on $\operatorname{Re}\{P(z)\}$, [3] we have

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \leqq \frac{1+(1-2\sigma)r}{1-r} + \frac{\left(2-\frac{1}{\alpha}\right)(1-\lambda b)r}{(1-r)\left[1+r\left\{\left(1-\frac{1}{\alpha}\right)(1-\lambda b)-\lambda b\right\}\right]}\,,$$

for $|z| = r$; from which we get

$$\frac{\partial}{\partial r}\log|f'(z)| \leqq \frac{3-2\sigma}{1-r} + \frac{\left(1-\frac{1}{\alpha}\right)(1-\lambda b)-\lambda b}{1+r\left\{\left(1-\frac{1}{\alpha}\right)(1-\lambda b)-\lambda b\right\}}\,.$$

Integrating from 0 to r, we get

$$(3.12) \qquad |f'(z)| \leqq \frac{1 + r\left[\left(1 - \frac{1}{\alpha}\right)(1 - \lambda b) - \lambda b\right]}{(1 - r)^{3-2\sigma}} \leqq$$

$$\leqq \frac{r + 1\left[\left(1 - \frac{1}{\alpha}\right)(1 - \lambda) - \lambda\right]}{(1 - r)^{3-2\sigma}}, \quad \text{since} \quad \lambda b \geqq \lambda .$$

From (3.12) we have

$$|f(z)| = \left|\int_0^z f'(z)\,\mathrm{d}z\right| \leqq \int_0^r |f'(z)|\,\mathrm{d}r \leqq$$

$$\leqq \int_0^r \frac{1 + r\left[\left(1 - \frac{1}{\alpha}\right)(1 - \lambda) - \lambda\right]}{(1 - r)^{3-2\sigma}}\,\mathrm{d}r .$$

On carrying out this integration, we obtain the upper bounds of (3.8).

If L is the arc in $|z| < 1$ which is mapped by $f(z)$ on the line segment $[0, f(z)]$, then

$$|f(z)| = \int_L |f'(z)|\,|\mathrm{d}z| \geqq \int_L |f'(z)|\,\mathrm{d}r \geqq$$

$$\geqq \int_0^r \frac{(1 - r)\,\mathrm{d}r}{(1 + r)^{2(1-\sigma)}\left[1 + r\left\{\left(1 - \frac{1}{\alpha}\right)(1 - \lambda) - \lambda\right\}\right]} .$$

The estimates are all sharp.

Equality holds on the left side of (3.7) and (3.8) for the function

$$f(z) = \int_0^z \frac{(1 - z)\,\mathrm{d}z}{(1 + z)^{2(1-\sigma)}\left[1 + z\left\{\left(1 - \frac{1}{\alpha}\right)(1 - \lambda) - \lambda\right\}\right]} .$$

Replacing r by z in the right side of (3.8) we obtain a function in $c^*(\lambda, \sigma)$ for which equality holds in both the upper bounds of Theorem (3.2).

Theorem (3.3). *If* $f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$ *is in* $c^*(\lambda, \sigma)$, *then*

$$(3.13) \qquad |a_n| \leqq \frac{2(3 - 2\sigma)\ldots(n - 2\sigma)}{n!}\left[(1 - \sigma) + \frac{n - 1}{2}\left(2 - \frac{1}{\alpha}\right)(1 - \lambda)\right] .$$

for $0 \leqq \lambda, \sigma \leqq 1$ *and all* n.

For $n = 2$, the inequality is best possible for every $\alpha \geqq 1$.

Proof. Substituting the power series for $f(z)$, $F(z)$ and $P(z)$ from (1.2), (1.3) and (2.1) respectively in (3.2) we obtain after some simplification

$$z + 2a_2z^2 + \ldots + na_nz^n + \ldots = z + [b_2\bar{b}_1 + b_1(\cos\beta - \lambda)\, p_1]\, z^2 + \ldots + \\ + [b_n\bar{b}_1 + b_{n-1}) \cos\beta - \lambda)\, p_1 + \ldots + b_1(\cos\beta - \lambda)\, p_{n-1}]\, z^n + \ldots$$

Equating Coefficients it gives

$$n|a_n| \leqq |b_n| + |b_{n-1}|\, |\cos\beta - \lambda|\, |p_1| + \ldots + |b_1|\, |\cos\beta - \lambda|\, |p_{n-1}|$$

and hence on using (2.11) we have

$$|a_n| \leqq \frac{|b_n|}{n} + \frac{\left(2 - \frac{1}{\alpha}\right)(1 - \lambda)}{n}\, [|b_{n-1}| + |b_{n-2}| + \ldots + |b_1|]\,. \tag{3.14}$$

If $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is starlike of order 6, then

$$|b_n| \leqq \frac{(2 - 2\sigma)\ldots(n - 2\sigma)}{(n - 1)!}\,, \quad n = 2, 3, \ldots, [4]\,. \tag{3.15}$$

(3.14) gives in connection with (3.15)

$$|a_n| \leqq \frac{2(3 - 2\sigma)\ldots(n - 2\sigma)}{n!}\left[(1 - \sigma) + \frac{n - 1}{2}\left(2 - \frac{1}{\alpha}\right)(1 - \lambda)\right].$$

For general value of α the sharp upper bound is achieved by the function

$$F(z) = \int_0^z \frac{1 + \left[1 - \lambda + \lambda\left(\frac{1}{\alpha} - 1\right)\right] z}{\left[1 + \left(\frac{1}{\alpha} - 1\right) z\right](1 - z)^{2(1-\sigma)}}$$

for $n = 2$ only.

If $\alpha \to \infty$, then the extremal function is

$$f(z) = \int_0^z \frac{1 + (1 - 2\lambda)\, z}{(1 - z)\, 3 - 2\sigma}\, dz =$$

$$= \frac{z(1 - \sigma)(1 - 2\lambda) + (\lambda - \sigma)\,[1 - (1 - z)^{2(1-\sigma)}]}{(1 - \sigma)(1 - 2\sigma)(1 - z)^{2(1-\sigma)}}$$

for $0 \leqq \lambda \leqq 1$ and $0 \leqq \sigma < 1$, $\sigma \neq \frac{1}{2}$;

$$f(z) = (1 - 2\lambda) \log(1 - z) + 2\,\frac{(1 - \lambda)\, z}{1 - z}$$

for $0 \leqq \lambda \leqq 1$ and $\sigma = \frac{1}{2}$; and

$$f(z) = 2(\lambda - 1)\log(1 - z) + (2\lambda - 1)z$$

for $0 \leqq \lambda \leqq 1$ and $\sigma = 1$. In this case the upper bounds are sharp for all n.

Remarks 1. $\lambda = \sigma = 0$ gives the class of close-to-convex functions.

2. For $\sigma = 1$, $F(z) = z$ and $\text{Re}\{f'(z)\} \geqq 0$, therefore $\lambda = 0$, $\sigma = 1$ gives the class of functions whose derivatives have positive real part in the unit circle.

3. If $\lambda = 1$, $\sigma = 0$, then $\text{Re}\{zf'(z)/F(z)\} \geqq 1$ and $zf'(z) = F(z)$; thus we get the class of convex functions.

4. When $\lambda = \sigma$, $F(z) = f(z)$, we obtain the class of starlike functions of order σ.

4. A class of function which are real on the real axis and convex in the direction of imaginary axis.

Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is real for real z and maps D onto a domain convex in the direction of imaginary axis, then $\text{Re}\{f(z)/z\} > \frac{1}{2}$, [4].

Let F denote the class of all such functions which satisfy the condition

$$\left|\left(2\frac{f(z)}{z} - 1\right) - \alpha\right| < \alpha, \quad (\alpha \geqq 1), \quad \text{for} \quad |z| < 1. \tag{4.1}$$

Putting

$$P(z) = \frac{2f(z)}{z} - 1, \tag{4.2}$$

we see that $P(0) = 1$, $\text{Re}\{P(z)\} > 0$. Therefore, form (2.2) and (4.2) we get

$$f(z) = \frac{z}{2}\,\frac{2 + \frac{1}{\alpha}\varphi(z)}{1 - \left(1 - \frac{1}{\alpha}\right)\varphi(z)},$$

which gives rise to the following theorem.

Theorem (4.1) *If $f(z) \in F$, then*

$$\frac{|z|}{2}\,\frac{2 - \frac{|z|}{\alpha}}{1 + \left(1 - \frac{1}{\alpha}\right)|z|} \leqq |f(z)| \leqq \frac{|z|}{2}\,\frac{2 + \frac{|z|}{\alpha}}{1 - \left(1 - \frac{1}{\alpha}\right)|z|}.$$

The equality is obtained for

$$f(z) = \frac{z}{2} \frac{2 + \frac{\varepsilon}{\alpha} . z}{1 - \varepsilon\left(1 - \frac{1}{\alpha}\right) z}, \quad \varepsilon = \pm 1 .$$

By following a method similar to that of Lemma (2.2) we can prove the following theorem.

Theorem (4.2) *If $f(z) \in F$, then*

$$|a_n| \leqq 1 - \frac{1}{2\alpha} \quad \textit{for all } n .$$

The equality is achieved by the functions

$$f(z) = \frac{z}{2} \frac{2 + \frac{z^{n-1}}{\alpha}}{1 - \left(1 - \frac{1}{\alpha}\right) z^{n-1}}, \quad n \geqq 2 .$$

Theorem (4.3) *If $f_1(z)$ and $f_2(z)$ belong to F, then $\lambda f_1(z) + (1 - \lambda) f_2(z)$ also belongs to F, $(0 \leqq \lambda \leqq 1)$.*

Proof: Since f_1, (z) and f_2 (z) belongs to F, therefore,

$$\left|\frac{1}{\alpha}\left(\frac{2f_1(z)}{z} - 1\right) - 1\right| < 1 \quad \text{and} \quad \left|\frac{1}{\alpha}\left(\frac{2\, f_2(z)}{z} - 1\right) - 1\right| < 1 .$$

New

$$\left|\frac{1}{\alpha}\left(2 \frac{\lambda f_1(z) + (1 - \lambda) f_2(z)}{z} - 1\right) - 1\right| \leqq$$

$$\leqq \lambda \left|\frac{1}{\alpha}\left(\frac{2f_1(z)}{z} - 1\right) - 1\right| + (1 - \lambda) \left|\frac{1}{\alpha}\left(\frac{2f_2(z)}{z} - 1\right) - 1\right| \leqq \lambda + (1 - \lambda) = 1 ,$$

which proves the theorem.

Remarks. (1) Throughout the paper we have taken $\alpha \geqq 1$ for the sake of simplicity otherwise all the above theorems remain valid with slight modification when $\frac{1}{2} <$ $< \alpha < 1$.

(2) Similar theorems can also be proved for functions which are typically real and functions which are starlike in one direction.

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