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GENERALIZATION OF A THEOREM OF W. SIERPIŃSKI

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The following theorem was published by W. SIERPIŃSKI [1. p. 99] in the special case $\aleph_\vartheta = \aleph_0$, although in a little different form:

Theorem. *If $\aleph_\vartheta$ is an arbitrary infinite cardinal number, let E be a set of power $\aleph_\vartheta$. Then there exists a sequence $\{Q_\xi\}_{\xi<\omega_{\vartheta+1}}$ of subsets of E of type $\omega_{\vartheta+1}$ such that for each $\alpha < \beta < \omega_{\vartheta+1}$ the set $Q_\beta - Q_\alpha$ is of power $\aleph_\vartheta$ while $Q_\alpha - Q_\beta$ is of power less than $\aleph_\vartheta$.*

Loosely speaking, $\{Q_\xi\}_{\xi<\omega_{\vartheta+1}}$ is a sequence of sets such that each set in it almost contains its predecessors.

In this note we are going to prove this Theorem. We use the usual notations of set theory, therefore we mention only the following ones: ω_λ denotes the initial number of $\aleph_\lambda$ and $\overline{\overline{H}}$ the cardinality of the set H.

For the proof, we need, two well known lemmas of HAUSDORFF [2]:

Lemma 1. *Each partially ordered set has an ordered subset which is maximal with respect to the inclusion.*

Lemma 2. *Each ordered set has a well ordered subset which is confinal to it.*

Proof of the Theorem. We consider two cases:

a) $\aleph_\vartheta$ is regular,

b) $\aleph_\vartheta$ is singular.

a. Let $E = \bigcup_{\eta<\omega_\vartheta} E_\eta$ be a decomposition of E in the mutually disjoint sets E_η, each of which is of power $\aleph_\vartheta$, and let

$$\mathbf{P} = \{X : X \subset E \text{ and } \overline{\overline{X \cap E_\eta}} < \aleph_\vartheta \quad \text{for every} \quad \eta < \omega_\vartheta\}.$$

We define the partial ordering $(\mathbf{P}, \prec)$ as follows: if X and Y are any two elements of $\mathbf{P}$, $X \prec Y$ denotes that $\overline{\overline{Y - X}} = \aleph_\vartheta$ and $\overline{\overline{X - Y}} < \aleph_\vartheta$.

Lemma A. *If* $\{Q_\xi\}_{\xi<\lambda}$ *is a well ordered sequence of elements of* **P** *in the partial ordering* $(\mathbf{P}, \prec)$ *of type* $\lambda \leqq \omega_\vartheta$, *then there exists an element* M *of* **P** *such that* $Q_\xi \prec M$ *for every* $\xi < \lambda$.

Proof. Let

$$M_1 = \bigcup_{\eta<\omega_\vartheta} \bigcup_{\xi<\eta,\lambda} (Q_\xi \cap E_\eta).$$

Then it easy to see that

$$(1) \qquad M_1 \in \mathbf{P} \quad \text{and} \quad \overline{\overline{Q_\xi - M_1}} < \aleph_\vartheta \quad \text{for every} \quad \xi < \lambda .$$

Since, according to the definition of M_1 and **P**, the sets $E_\eta - M_1$ are of power $\aleph_\vartheta$ fo evevery $\eta < \omega_\vartheta$, we obtain by the axiom of choice that there exists a set $M_2 \subset E - M$ such that $\overline{\overline{M_2 \cap E_\eta}} = 1$ for every $\eta < \omega_\vartheta$. Put $M = M_1 \cup M_2$. Then by (1) it is easy to see that M satisfies the requirements of the lemma.

Now the Theorem in case a) can be proved as follows. By Lemma 1, **P** has a maximal ordered subset **P**′ in the partial ordering $(\mathbf{P}, \prec)$. Then Lemma 2 provides the existence of a well ordered subset **Q** of **P**′ which is confinal to **P**. If the ordinal type of **Q** is $<\omega_{\vartheta+1}$, then **Q** has a subset $\{Q_\xi\}_{\xi<\lambda}$ with $\lambda \leqq \omega_\vartheta$ which is confinal to **Q**. However, by Lemma *A* this contradicts the maximality of **P**′; i.e. **Q** must have a type $\geqq \omega_{\vartheta+1}$. This proves the Theorem in case a.

b. Let $E = \bigcup_{\eta>\mathfrak{C}} E_\eta$ be a decomposition of E into the mutually disjoint sets E_η, such that $\overline{\overline{E_\eta}} = \aleph_{\eta+1}$ for each $\eta < \vartheta$, and let

$$\mathbf{P} = \{X : X \subset E \text{ and } \overline{\overline{X \cap E_\eta}} \leqq \aleph_\eta \text{ for every } \eta < \vartheta\}.$$

We define the partial ordering $(\mathbf{P}, \prec)$ as follows: if X and Y are any two elements of **P**, $X \prec Y$ denotes that $\overline{\overline{Y - X}} = \aleph_\vartheta$ and $\overline{\overline{X - Y}} < \aleph_\vartheta$.

Lemma B. *If* $\{Q_\xi\}_{\xi<\lambda}$ *is a well ordered sequence of elements of* **P** *in the partial ordering* $(\mathbf{P}, \prec)$ *of the type* $\lambda < \omega_\vartheta$, *then there exists an element* M *of* **P** *such that* $Q_\alpha \prec M$ *for every* $\alpha < \lambda$.

Proof. Let $\aleph_\mu$ be the cardinality of $\{Q_\xi\}_{\xi<\lambda}$. Since $\lambda < \omega_\vartheta$, we obtain $\aleph_\mu < \aleph_\vartheta$, and so the singularity of $\aleph_\vartheta$ implies that $\aleph_{\mu+1} < \aleph_\vartheta$. Let

$$M_1 = \bigcup_{\omega_{\mu+1}<\eta<\omega_\vartheta} \bigcup_{\xi<\lambda} (Q_\xi \cap E_\eta).$$

Then it is easy to see that

$$(2) \qquad M_1 \in \mathbf{P} \quad \text{and} \quad \overline{\overline{Q_\xi - M_1}} < \aleph_\vartheta \quad \text{for every} \quad \xi < \lambda .$$

Since, according to the definition of M_1 and **P**, for each $\eta < \vartheta$ the set $E_\eta - M_1$ is of power $\aleph_{\eta+1}$, it can be easily seen (by transfinite induction or by a simple direct application of the choice axiom) that there exists a set $M_2 \subset E - M_1$ such that $\overline{\overline{M_2 \cap E_\eta}} = \aleph_\eta$ for every $\eta < \vartheta$. Put $M = M_1 \cup M_2$. Then by (2) it is clear that M satisfies the requirements of the lemma.

Now the Theorem in case b) can be verified in a similar way as in case a) as follows. By Lemma 1, $\mathbf{P}$ has a maximal ordered subset $\mathbf{P}'$ in the partial ordering $(\mathbf{P}, \prec)$. Then Lemma 2 provides the existence of a well ordered subset $\mathbf{Q}$ of $\mathbf{P}'$ which is confinal to $\mathbf{P}'$. If the ordinal type of $\mathbf{Q}$ is $<\omega_{\vartheta+1}$, then, with the aid of singularity of $\aleph_\vartheta$, we obtain that $\mathbf{Q}$ has a subset $\{Q_\xi\}_{\xi<\lambda}$ with $\lambda < \omega_\vartheta$, which is confinal to $\mathbf{Q}$. However, by Lemma B this contradicts the maximality of $\mathbf{P}'$; i.e. $\mathbf{Q}$ must have a type $\geqq \omega_{\vartheta+1}$. This proves the Theorem in case b.

References

[1] *W. Sierpiński:* Sur un problème de M. J. Novák, Czechosl. Math. Journ. Vol. *1* (*76*), (1951), 97–101.

[2] *F. Hausdorff:* Grundzüge einer Theorie der geordneten Mengen. Math. Annalen, *65* (1908), 435–505.

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