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VAN DER POL PERTURBATION OF THE EQUATION FOR A VIBRATING STRING

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0. By the Van der Pol Perturbation of the Equation for a vibrating string the following boundary value problem is meant

$$u_{tt} - u_{xx} = \varepsilon(1 - u^2)\, u_t\,, \tag{1,0}$$

$$0 \leqq x \leqq 1\,, \quad t \geqq 0\,, \quad u(0, t) = u(1, t) = 0\,, \tag{2,0}$$

ε being a small parameter. Instead of (1,0), (2,0) the more general problem (2,0),

$$u_{tt} - u_{xx} = \varepsilon\, h(u)\, u_t\,, \tag{3,0}$$

will be considered assuming that

(4,0) h is an even function defined on E_1, its derivative of the second order h'' is continuous, $h(0) > 0$, $h'(\xi) < 0$ for $\xi > 0$, $h''(\xi) < 0$ for $\xi > 0$.

The results are analogous to those proved in section 6 of the author's paper [1] for the problem

$$u_{tt} - u_{xx} = \varepsilon(1 - u_t^2)\, u_t \tag{5,0}$$

with boundary conditions (2,0).

In the paper quoted it was proved that for ε small enough there exist classical solutions of (5,0), (2,0) such that there exists the uniform limit $v(t, x) = \lim_{n=1,2,3,\ldots} u(t + 2n, x)$; of course, $v(t + 2, x) = v(t, x)$ and v is continuous, but the derivatives $\partial v/\partial t$, $\partial v/\partial x$ are continuous at (t, x) only, if $t + x \neq n + \vartheta$, and $t - x \neq n + \vartheta$, $n = \ldots -1, 0, 1, \ldots$, ϑ being a fixed real number.

Here it will be proved that there exist classical solutions u of (3,0), (2,0) such that there exists the limit $v(t, x) = \lim_{n=1,2,3,\ldots} u(t + 2n, x)$ if $t + x \neq n + \vartheta$, $t - x \neq$

$\neq n + \vartheta$, $n = \ldots -1, 0, 1, \ldots, \vartheta$ being a fixed real number; v is continuous at (t, x) only if $t + x \neq n + \vartheta$, and $t - x \neq n + \vartheta$, $n = \ldots -1, 0, 1, \ldots$

1. Let H be defined by $H(\xi) = \int_0^\xi h(\sigma)\,d\sigma$. It will be proved that there exists a unique $a > 0$, $H(a) = 0$. Put $q(\xi) = \frac{1}{2}a$ for $0 \leqq \xi < 1$, $q(\xi + 1) = -q(\xi)$ for $\xi \in E_1$. The main results of this paper are formulated in the following Theorem:

Theorem 1,1. *There exist functions φ and ψ which fulfil the conditions* (iii), (iv) *of section* 4, [1] *and to every $\delta > 0$ there exists an $\varepsilon_0 > 0$ that for $0 < \varepsilon \leqq \varepsilon_0$ the following assertions take place*:

(i) *the solution u of* (3,0), (2,0), $u(x, 0) = \varphi(x)$, $u_t(x, 0) = \psi(x)$ *exists for $t \geqq 0$ and is bounded,*

(ii) *there exists a number $\vartheta = \vartheta(\varepsilon)$ such that the limit*

$$\lim_{n\to\infty} u(x, t + 2n) = v(x, t),\ n = 1, 2, 3, \ldots \tag{1,1}$$

exists, if

$$x + t \neq \vartheta + 2n\,, \quad \text{and} \quad -x + t \neq \vartheta + 2n\,, \quad n = \ldots -1, 0, 1, \ldots \tag{2,1}$$

The limit (1,1) *exists uniformly for $(x, t) \in \tilde{D} \subset \langle 0, 1\rangle \times \langle 0, \infty)$, if $\tilde{D}$ is compact and* (2,1) *is fulfilled on $\tilde{D}$,*

(iii) *v is continuous at (x, t), if (x, t) fulfils* (2,1) *and*

$$|v(x, t) - q(x + t - \vartheta) + q(-x + t - \vartheta)| \leqq \delta\,, \tag{3,1}$$

if $x + t \neq \vartheta + n$, and $-x + t \neq \vartheta + n$, $n = \ldots -1, 0, 1, \ldots$.

Note 1,1. Let $(x_1, t_1) \in \langle 0, 1\rangle \times \langle 0, \infty)$, $x_1 + t_1 = \vartheta + n$, or $-x_1 + t_1 = \vartheta + n$, $n = \ldots -1, 0, 1, \ldots$. It follows from (3,1) that $\lim\limits_{(x,t)\to(x_1,t_1)} v(x, t)$ does not exist, if $\delta < a$.

Note 2,1. The same result is valid, if q is replaced by q^* which is defined as follows: there exist numbers $0 = \xi_0 < \xi_1 < \ldots < \xi_l = 1$, $q^*(\xi) = \frac{1}{2}a$ for $\xi_{2i} \leqq \xi < \xi_{2i+1}$, $i = 0, 1, \ldots, 2i + 1 \leqq l$, $q^*(\xi) = -\frac{1}{2}a$ for $\xi_{2i+1} \leqq \xi < \xi_{2i+2}$, $i = 0, 1, \ldots, 2i + 2 \leqq l$, $q^*(\xi + 1) = -q^*(\xi)$ for $\xi \in E_1$.

Note 3,1. Theorem 1,1 holds, if h'' is continuous and the assertion of Lemma 1,2 holds ((4,0) need not be fulfilled).

The problem (3,0), (2,0) will be treated by the method developed in section 4, [1]. Let L_1 be the Banach space, the elements of which are locally integrable functions $y = y(\xi)$ on E_1, $y(\xi + 2) = y(\xi)$ a.e. on E_1, $\int_0^2 y(\xi)\,d\xi = 0$, $\|y\| = \|y\|_{L_1} =$

$= \int_0^2 |y(\xi)| \, d\xi$. If $y \in L_1$, let $I_\xi y = Y$ be absolutely continuous, $dY/d\xi = y$ a.e., $\int_0^2 Y(\xi) \, d\xi = 0$. Obviously

$$(4,1) \quad Y(\xi) = \int_0^\xi v(\sigma) \, d\sigma + \frac{1}{2} \int_0^2 (\sigma - 2) \, v(\sigma) \, d\sigma = \frac{1}{2} \int_\xi^{\xi+2} (\sigma - \xi - 1) \, v(\sigma) \, d\sigma \, .$$

The transformed equation is (cf. [1], (18,4))

$$(5,1) \qquad \frac{dy}{d\tau} = f(y, \tau, \varepsilon) \, ,$$

f being defined for $y \in L_1$ by

$$(6,1) \qquad f(y, \tau, \varepsilon) = \tfrac{1}{2} h \left(Y(\xi) - Y\left(-\xi + 2 \frac{\tau}{\varepsilon}\right)\right) \left(y(\xi) - y\left(-\xi + 2 \frac{\tau}{\varepsilon}\right)\right)$$

and the averaged equation is (cf. [1], (20,4))

$$(7,1) \qquad \frac{dy}{d\tau} = f_0(y) \, ,$$

f_0 being defined for $y \in L_1$ by

$$(8,1) \qquad f_0(y) = \frac{1}{4} \int_0^2 h(Y(\xi) - Y(-\xi + 2\sigma)) \, (y(\xi) - y(-\xi + 2\sigma)) \, d\sigma \, .$$

Let C^1 be the Banach space the elements of which are functions $y \in L_1$ that have a continuous derivative $y' = dy/d\xi$, $\|y\| = \|y\|_{C^1} = \max_\xi |y'(\xi)|$. It may be proved in a similar way as in [1], Lemma 1,4, that $f(y, \tau, \varepsilon)$ and $f_0(y) \in L_1(C^1)$ if $y \in L_1(C^1)$ and obviously f and f_0 fulfil a Lipschitz condition with respect to y on every bounded subset of $L_1(C^1)$. Therefore equations (5,1), (7,1) may be examined on L_1 or on C^1. Let (5,1) be examined on C^1 and let $y = y(\tau)$ be a solution of (5,1) on $\langle \tau_1, \tau_2 \rangle$, $Y(\tau) = I_\xi \, y(\tau)$. Put $Y(\xi, \tau) = Y(\tau) \, (\xi)$ for $\xi \in E_1$, $\tau \in \langle \tau_1, \tau_2 \rangle$. It was shown in [1], section 4 that u defined by $u(x, t) = Y(x + t, \varepsilon t) - Y(-x + t, \varepsilon t)$ is a classical solution of (3,0), (2,0). Therefore Theorem 1,1 is a consequence of the following

Theorem 2,1. *There exists a function* $\tilde{y} \in C^1$ *(*$\tilde{y}$ *may be chosen analytic) and to every* $\delta > 0$ *there exists an* $\varepsilon_0 > 0$ *that for* $0 < \varepsilon \leqq \varepsilon_0$ *the following assertions take place:*

(i) *the solution* y *of* (5,1), $y(0) = \tilde{y}$ *exists for* $\tau \geqq 0$ *and* $Y(\tau) = I_\xi \, y(\tau)$ *is bounded for* $\tau \geqq 0$,

(ii) *put* $Y(\xi, \tau) = Y(\tau) \, (\xi)$; *there exists such a* $\vartheta = \vartheta(\varepsilon)$ *that the limit*

$$(9,1) \qquad \lim_{i \to \infty} Y(\xi, \tau + i\varepsilon) = Z(\xi, \tau) \, , \quad i = 1, 2, 3, \ldots$$

exists for $\xi \neq \vartheta + j$, $j = \dots -1, 0, 1, \dots$, $\tau \in E_1$. *The convergence in* (9,1) *is uniform for* $\xi \in A$, $\tau \in \langle 0, \infty)$, *A being a subset of* E_1, *whose distance from the set* $\{\vartheta + j\}$, $j = \dots -1, 0, 1, \dots$ *is positive,* ε *being fixed,*

(iii) *Z is continuous at every point* (ξ, τ), $\xi \neq \vartheta + j$, $j = \dots -1, 0, 1, \dots$ *and*

$$|Z(\xi, \tau) - q(\xi - \vartheta)| \leqq \delta \tag{10,1}$$

for $\xi \neq \vartheta + j$, $j = \dots -1, 0, 1, \dots$, $\tau \in E_1$.

The proof of Theorem 2,1 is contained at the end if section 4. Section 2 contains auxiliary results, in section 3 a set U is defined, which is invariant in the following sense: if $\tilde{y} \in U$, then $y(\varepsilon) \in U$. The proof of Theorem 2,1 is preceded by a series of Lemmas, the main result of which is that $Y(i\varepsilon)$, $i = 0, 1, 2, \dots$ is a Cauchy sequence in some sense.

2. Lemma 1,2. *Let h fulfils* (4,0). *Put* $H(\xi) = \int_0^\xi h(\sigma)\,\mathrm{d}\sigma$. *Then there exists a unique* $a > 0$ *that* $H(a) = 0$. $h(\xi - \frac{1}{2}a) + h(\xi + \frac{1}{2}a)$ *is decreasing on* $\langle 0, \frac{1}{2}a + 1\rangle$ *and*

$$h(-\tfrac{1}{2}a) + h(\tfrac{1}{2}a) > 0\,, \quad h(0) + h(a) < 0\,, \tag{1,2}$$

$$H(\xi - \tfrac{1}{2}a) + H(\xi + \tfrac{1}{2}a) > 0 \quad \textit{for} \quad \xi \in (0, \tfrac{1}{2}a)\,. \tag{2,2}$$

Proof. It is obvious that there exists an a_1 such that $h(\xi) > 0$ for $\xi \in \langle 0, a_1)$ and $h(\xi) < 0$ for $\xi \in (a_1, \infty)$; as $h'(\xi) < 0$ for $\xi > 0$, $\lim_{\xi \to \infty} h(\xi) < 0$. Therefore there exists a unique a such that $H(a) = 0$. Obviously $H(\xi - \frac{1}{2}a) + H(\xi + \frac{1}{2}a) = 0$ for $\xi = 0$ (H is odd) and $\xi = \frac{1}{2}a$, $(\mathrm{d}/\mathrm{d}\xi)\left(H(\xi - \frac{1}{2}a) + H(\xi + \frac{1}{2}a)\right) = h(\xi - \frac{1}{2}a) + h(\xi + \frac{1}{2}a) = h(\frac{1}{2}a - \xi) + h(\frac{1}{2}a + \xi)$. $h(\frac{1}{2}a - \xi) + h(\frac{1}{2}a + \xi)$ is decreasing for $\xi \geqq 0$ as the derivative $h'(\frac{1}{2}a + \xi) - h'(\frac{1}{2}a - \xi) = \int_{\frac{1}{2}a-\xi}^{\frac{1}{2}a+\xi} h''(\eta)\,\mathrm{d}\eta$ is negative for $\xi > 0$. Hence $h(-\frac{1}{2}a) + h(\frac{1}{2}a) > 0$, $h(0) + h(a) < 0$ and $H(\xi - \frac{1}{2}a) + H(\xi + \frac{1}{2}a) > 0$ on $(0, \frac{1}{2}a)$.

In the sequel the following Banach spaces will be used: L_1, C^1, M, L_2, L_1^1. L_1 and C^1 were introduced in section 1, the elements of M are bounded measurable functions Y defined on E_1, $Y(\xi + 2) = Y(\xi)$ on E_1, $\int_0^2 Y(\xi)\,\mathrm{d}\xi = 0$, $\|Y\|_M = \sup_\xi |Y(\xi)|$, the elements of L_2 are locally square integrable functions Y on E_1, $Y(\xi) = Y(\xi + 2)$ a.e., $\int_0^2 Y(\xi)\,\mathrm{d}\xi = 0$, $\|Y\|_{L_2} = (\int_0^2 Y^2(\xi)\,\mathrm{d}\xi)^{1/2}$ and the elementsof L_1^1 are absolutely continuous functions Y on E_1, $Y(\xi) = Y(\xi + 2)$ on E_1, $\int_0^2 Y(\xi)\,\mathrm{d}\xi = 0$, $\|Y\|_{L_1^1} = \int_0^2 |y(\xi)|\,\mathrm{d}\xi$, $y = (\mathrm{d}/\mathrm{d}\xi)\,Y$. $\hat{L}_1, \hat{C}^1, \hat{M}, \hat{L}_2, \hat{L}_1^1$ are such subspaces of the above spaces that in addition $Y(\xi + 1) = -Y(\xi)$ holds for $\xi \in E_1$.

For every fixed τ and $\varepsilon > 0$, $f(., \tau, \varepsilon)$ maps any of the above spaces into itself and a Lipschitz condition is fulfilled on any bounded set. As f is continuous in all variables, (5,1) may be examined on any of the above spaces. If X is one of the above spaces, the fact that (5,1) is examined in X and y is a solution defined on $\langle \tau_1, \tau_2\rangle$, will be described by y is a solution of (5,1) in X on $\langle \tau_1, \tau_2\rangle$ (and in an analogous way solutions of (7,1) and other equations will be described).

Let $y \in L_1$, $Y = I_\xi y$; then

$$(3,2) \quad \frac{\mathrm{d}}{\mathrm{d}\sigma} H(Y(\xi) - Y(-\xi + 2\sigma)) = 2 \, . \, h(Y(\xi) - Y(-\xi + 2\sigma)) \, y(-\xi + 2\sigma) \cdot$$

Hence

$$(4,2) \qquad f_0(y)(\xi) = y(\xi) \frac{1}{4} \int_0^2 h(Y(\xi) - Y(-\xi + 2\sigma)) \, \mathrm{d}\sigma =$$

$$= \frac{1}{4} \int_0^2 h(Y(\xi) - Y(-\xi + 2\sigma)) \, (y(\xi) + y(-\xi + 2\sigma)) \, \mathrm{d}\sigma =$$

$$= \frac{\mathrm{d}}{\mathrm{d}\xi} \frac{1}{4} \int_0^2 H(Y(\xi) - Y(-\xi + 2\sigma)) \, \mathrm{d}\sigma \, .$$

Define for $Y \in M$

$$(5,2) \qquad f_0^*(Y)(\xi) = \frac{1}{4} \int_0^2 H(Y(\xi) - Y(-\xi + 2\sigma)) \, \mathrm{d}\sigma \, .$$

Again $f_0^*(Y) \in M$, if $Y \in M$ (cf. [1], Lemma 1,4).

The following lemma is a consequence of the above considerations (as I_ξ may be interpreted as a bounded operator from C^1 to M).

Lemma 2,2. *If* (7,1) *is examined on* C^1 *if* y *is a solution of* (7,1) *on* $\langle \tau_1, \tau_2 \rangle$ *and if* $Y(\tau) = I_\xi \, y(\tau)$, *then* Y *is a solution of*

$$(6,2) \qquad \frac{\mathrm{d}Y}{\mathrm{d}\tau} = f_0^*(Y)$$

(*considered on* M) *on* $\langle \tau_1, \tau_2 \rangle$.

Observe that

$$(7,2) \qquad f_0^*(q) = 0 \quad (\text{in } M) \, ,$$

q being defined at the beginning of section 1.

If $y \in \hat{L}_1$, then $f(y, \tau, \varepsilon) \in \hat{L}_1$. The following Lemma is a consequence of the uniqueness of solutions of (5,1).

Lemma 3,2. *Let* y *be a solution of* (5,1) (*examined on* L_1, C^1) *on* $\langle \tau_1, \tau_2 \rangle$, $y(\tau_1) \in \hat{L}_1, \hat{C}_1$. *Then* $y(\tau) \in \hat{L}_1, \hat{C}_1$ *for* $\tau \in \langle \tau_1, \tau_2 \rangle$.

Note 1,2. Lemma 3,2 may be modified as to hold for solutions of any of equations (5,1), (7,1) or (6,2) on any of the spaces M, C^1, L_1, L_2, L_1^1, if the right hand side of the equation is defined on the respective space.

In the sequel a series of estimates will be established. It is supposed that h is a fixed function; hence the number a and the function q are fixed. These estimates hold for $0 < \varepsilon \leqq \varepsilon_0$ and by ε_0 a sufficiently small positive constant will be denoted, which may be dimished, if necessary, without mention of it specially. At the beginning of section 4 a $\eta > 0$ (small) is introduced and a $D > 0$ (large) is introduced in Lemma 12,4. Both η and D are to be locked as free parametres, while $K_1, K_2, \ldots$ $\ldots, \varkappa_1, \varkappa_2, \ldots, \nu, \varrho, \gamma, T, a, R$ are fixed. (Especially these constants are independent of $\varepsilon, \eta, D, \tilde{y}$.) $K_1, K_2, \ldots$ are to be sufficiently large; it would be possible to replace them by a single K sufficiently large, but as γ and ϱ have to fulfil several inequalities where some of the constants $K_1, K_2, \ldots$ appear (cf. Theorem 2,3), $K_1, K_2, \ldots$ are distinguished by indices in order to avoid any misunderstanding. N_1 depends on η only, N_2 depend on η and D only (i.e. they are independent of ε, $\tilde{y}$).

Lemma 4,2. *Put $R = 2a + 12$. There exist $\varepsilon_0 > 0$ and $K_1 > 0$ that the following assertion holds: If $\tilde{y} \in L_1$, $\|\tilde{y}\|_{L_1} \leqq R$, $\tilde{\tau} \in E_1$, $0 < \varepsilon \leqq \varepsilon_0$, then there exist solutions y of (5,1) and y_0 of (7,1) (considered on L_1), $y(\tilde{\tau}) = \tilde{y} = y_0(\tilde{\tau})$ on $\langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$ and*

$$(8,2) \qquad y(\tilde{\tau} + \varepsilon) = \tilde{y} + f_0(\tilde{y})\,\varepsilon + z\,, \quad \|z\|_{L_1} \leqq K_1\varepsilon^2\,,$$

$$(9,2) \qquad y_0(\tilde{\tau} + \varepsilon) = \tilde{y} + f_0(\tilde{y})\,\varepsilon + z_0\,, \quad \|z_0\|_{L_1} \leqq K_1\varepsilon^2\,,$$

$$(10,2) \qquad \|y(\tau)\|_{L_1} \leqq R + 1\,, \quad \|y_0(\tau)\|_{L_1} \leqq R + 1 \quad \text{for} \quad \tau \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle\,.$$

Proof. The proof that y and y_0 exist on $\langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$ is standard and that (10,2) holds for sufficiently small ε. (8,2) follows from the fact that

$$y(\tau) = \tilde{y} + \int_{\tilde{\tau}}^{\tau} f(\tilde{y}, \sigma)\,\mathrm{d}\sigma + z_1(\tau)\,, \quad \|z_1(\tau)\|_{L_1} \leqq K\varepsilon^2 \quad \text{for} \quad \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + \varepsilon$$

and that $\int_{\tilde{\tau}}^{\tilde{\tau}+\varepsilon} f(\tilde{y}, \sigma)\,\mathrm{d}\sigma = \varepsilon f_0(\tilde{y})$. (9,2) follows in a similar way.

Lemma 5,2. *Let y be a solution of (5,1), (7,1) in L_1 on $\langle\tau_1, \tau_2\rangle$, $y(\tau_1) \in C^1$. Then $y(\tau) \in C^1$ for $\tau \in \langle\tau_1, \tau_2\rangle$, y is a solution in C^1 and $y(\xi, \tau) = y(\tau)(\xi)$ is a continuous function on $E_1 \times \langle\tau_1, \tau_2\rangle$. Conversely every solution of (5,1) (or (7,1)) in C_1 is a solution of (5,1) (or (7,1)) in L_1.*

The proof is omitted as it is a slight modification of the proof of Lemma 6,4, [1]. (The only change being in that, that $Y_1 = Y$ on $(0, L_1)$ and therefore Y_1 is bounded on $(0, L_1)$ and y_1 is bounded on $(0, L_1)$, as in case of equations (5,1) and (7,1) equation (34,4), [1] is linear in y_1. The converse statement follows from that the natural map of C^1 into L_1 is linear and bounded.)

Lemma 6,2. *Let $\tilde{y} \in C^1$, $\|\tilde{y}\|_{L_1} = R$, $\tilde{\tau} \in E_1$, $0 < \varepsilon \leqq \varepsilon_0$. Then the solution y of (5,1) in C^1 on $\langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$, $y(\tilde{\tau}) = \tilde{y}$ fulfils*

$$(11,2) \qquad y(\xi, \tau) - \tilde{y}(\xi) = \int_{\tilde{\tau}}^{\tau} h\left(Y(\xi, \sigma) - Y\left(-\xi + 2\frac{\sigma}{\varepsilon}, \sigma\right)\right) \cdot$$

$$\cdot \left[y(\xi, \sigma) - \tilde{y}(\xi) - y\left(-\xi + 2\frac{\sigma}{\varepsilon}, \sigma\right) + \tilde{y}\left(-\xi + 2\frac{\sigma}{\varepsilon}\right)\right]\mathrm{d}\sigma + z_2(\xi, \tau)\,,$$

$$|z_2(\xi, \tau)| \leqq \varepsilon K_2[|\tilde{y}(\tfrac{1}{2})| + 1]\,, \quad y(\xi, \tau) = y(\tau)(\xi)\,, \quad \xi \in E_1\,, \quad \tau \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle\,.$$

Proof. The solution y exists according to Lemmas 4,2 and 5,2. As the value at ξ is a linear bounded functional on C^1, we obtain

$$(12,2) \qquad y(\xi, \tau) = \tilde{y}(\xi) + \int_{\tilde{\tau}}^{\tau} h\left(Y(\xi, \sigma) - Y\left(-\xi + 2\frac{\sigma}{\varepsilon}, \sigma\right)\right) \cdot \left(y(\xi, \sigma) - y\left(-\xi + 2\frac{\sigma}{\varepsilon}, \sigma\right)\right) d\sigma .$$

Hence $z_2(\xi, \tau) = \int_{\tilde{\tau}}^{\tau} h(Y(\xi, \sigma) - Y(-\xi + 2\sigma/\varepsilon, \sigma))(\tilde{y}(\xi) - \tilde{y}(-\xi + 2\sigma/\varepsilon) d\sigma$. The estimate of $|z_2|$ in (11,2) holds, as $\|y(\sigma)\|_{L_1} \leqq R + 1$ for $\sigma \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$ (cf. (10,2)) and therefore $|Y(\xi, \sigma)| \leqq R + 1$ for $\xi \in E_1$, $\sigma \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$ (cf. (4,1)).

Lemma 7,2. *Let $\tilde{y} \in C^1$, $\|\tilde{y}\|_{L_1} \leqq R$, $\tilde{\tau} \in E_1$, $0 < \varepsilon \leqq \varepsilon_0$. Then the solution y of (7,1) in C^1 on $\langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$, $y(\tilde{\tau}) = \tilde{y}$ fulfils*

$$(13,2) \qquad y(\xi, \tau) - \tilde{y}(\xi) = \frac{1}{4}\int_{\tilde{\tau}}^{\tau}\int_0^2 h\left(Y(\xi, \eta) - Y\left(-\xi + 2\frac{\sigma}{\varepsilon}, \eta\right)(y(\xi, \eta) - \tilde{y}(\xi)\right) d\sigma\, d\eta + z_2^*(\xi, \eta),$$

$$|z_2^*(\xi, \tau)| \leqq \varepsilon K_2 |\tilde{y}(\xi)|, \quad \xi \in E_1, \quad \tau \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle .$$

The proof is similar to the one of Lemma 6,2.

Proof. Starting from

$$(14,2) \qquad y(\xi, \tau) = \tilde{y}(\xi) + \frac{1}{4}\int_{\tilde{\tau}}^{\tau}\int_0^2 h\left(Y(\xi, \eta) - Y\left(-\xi + 2\frac{\sigma}{\varepsilon}, \eta\right)\right) y(\xi, \eta)\, d\sigma\, d\eta ,$$

we obtain that $z_2^*(\xi, \eta) = \frac{1}{4}\tilde{y}(\xi) \int_{\tilde{\tau}}^{\tau}\int_0^2 h(Y(\xi, \eta) - Y(-\xi + 2\sigma/\varepsilon, \eta)\, d\sigma\, d\eta$ and the estimate (13,2) for $z_2^*(\xi, \eta)$ holds.

Lemma 8,2. *Let y have the same meaning as in Lemma 7,2. Then*

$$(15,2) \qquad |y(\xi, \tau) - \tilde{y}(\xi)| \leqq \varepsilon K(|\tilde{y}(\xi)| + 1), \quad \xi \in E_1, \quad \tau \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle .$$

The proof of Lemma 8,2 is based on the Lemma 9,2.

Let X be a Banach space; X will be called an ordered Banach space, if for some $x_1, x_2 \in X$ the relation $x_1 \leqq x_2$ is defined and if the following conditions are fulfilled:

(i) the relation $\leqq$ is transitive;

(ii) if $x_1 \leqq x_2$, then $x_1 + x \leqq x_2 + x$ for $x \in X$;

(iii) the set of all $x \geqq 0$ is closed.

(Of course $x_1 \leqq x_2$ has the same meaning as $x_2 \geqq x_1$.)

Lemma 9,2. *Let X be an ordered Banach space, let A be a linear bounded operator from X to X such that $\|A\| < 1$ and $Ax \geqq 0$ if $x \geqq 0$. Let $w \in X$ and let for some $x_1, x_2 \in X$*

$$x_1 \leqq Ax_1 + w\,, \quad Ax_2 + w \leqq x_2\,. \tag{16,2}$$

Then $x_1 \leqq x_2$.

Proof. Put $x = w + Aw + A^2w + \ldots$ Obviously $x = Ax + w$. It follows by induction that

$$x_1 \leqq w + Ax_1 \leqq w + Aw + A^2x_1 \leqq \ldots,$$
$$x_2 \geqq w + Ax_2 \geqq w + Aw + A^2x_2 \geqq \ldots$$

Hence $x_1 \leqq x \leqq x_2$.

Proof of Lemma 8,2: Put $x_1(\xi, \tau) = |y(\xi, \tau) - \tilde{y}(\xi)|$. If y is a solution of (5,1), then it follows from (11,2) (cf. (10,2)) that

$$x_1(\xi, \tau) \leqq K \int_{\tilde{\tau}}^{\tau} \left(x_1(\xi, \sigma) + x_1\left(-\xi + 2\frac{\sigma}{\varepsilon}, \sigma\right)\right) \mathrm{d}\sigma + \varepsilon K(|\tilde{y}(\xi)| + 1)\,, \tag{17,2}$$

$$\xi \in E_1\,, \quad \tau \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle\,.$$

Let X be the space of bounded and continuous functions x on $E_1 \times \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$ with the usual norm and order relation. For $x \in X$ put $(Ax)(\xi, \tau) = K \int_{\tilde{\tau}}^{\tau} (x(\xi, \sigma) + x(-\xi + 2\sigma/\varepsilon, \sigma))\, \mathrm{d}\sigma$; obviously $\|A\| \leqq 2K\varepsilon$. Of course, we assume $2\varepsilon_0 K < 1$. (K being kept fixed during this proof.) Put $w(\xi, \tau) = \varepsilon K[|\tilde{y}(\xi)| + 1]$, $x_2 = 2w$. (17,2) may be written as $x_1 \leqq Ax_1 + w$. It remains to verify that $Ax_2 + w \leqq x_2$, i.e. that

$$2\varepsilon K(|\tilde{y}(\xi)| + 1) \geqq K \int_{\tilde{\tau}}^{\tau} 2\varepsilon K \left(|\tilde{y}(\xi)| + \left|\tilde{y}\left(-\xi + 2\frac{\sigma}{\varepsilon}\right)\right| + 2\right) \mathrm{d}\sigma + \varepsilon K(|\tilde{y}(\xi)| + 1)$$

which is equivalent to

$$|\tilde{y}(\xi)| + 1 \geqq 2\varepsilon K|\tilde{y}(\xi)| + \varepsilon K \int_0^2 |\tilde{y}(\sigma)|\, \mathrm{d}\sigma + 4\varepsilon K\,.$$

As $\int_0^2 |\tilde{y}(\sigma)|\, \mathrm{d}\sigma = \|\tilde{y}\|_{L_1} \leqq R$, the last inequality is fulfilled for $0 < \varepsilon \leqq \varepsilon_0$, ε_0 being sufficiently small. (15,2) is a consequence of (16,2) (K in (15,2) being equal to $2K$ from this proof). If y is a solution of (7,1), then it follows from (12,2) that

$$x_1(\xi, \tau) \leqq \tfrac{1}{2}K \int_{\tilde{\tau}}^{\tau} x_1(\xi, \eta)\, \mathrm{d}\eta + \varepsilon K|\tilde{y}(\xi)|\,.$$

Gronwell inequality implies that $x_1(\xi, \tau) \leqq \varepsilon K|\tilde{y}(\xi)| \exp\left(\tfrac{1}{2}K\varepsilon\right)$ and (13,2) holds in this case. Lemma 8,2 is proved.

Let us examine the equation of variations of (5,2) for $Y = q$

$$\frac{dW}{d\tau} = F'_{0Y}(q)\, W, \tag{18,2}$$

the right hand side of (18,2) being defined by

$$(F'_{0Y}(q)\, W)(\xi) = \tag{19,2}$$
$$= \frac{1}{2}\int_0^2 h(q(\xi) - q(-\xi + 2\sigma))\,[W(\xi) - W(-\xi + 2\sigma)]\,d\sigma\,.$$

Lemma 10,2. *Let W be a solution of* (18,2) *in M on $\langle\tau_1, \tau_2\rangle$. Then*

$$\|W(\tau)\|_{L_2} \leqq \|W(\tau_1)\|_{L_2} \exp\left(\tfrac{1}{2}(h(0) + h(a))(\tau - \tau_1)\right), \quad \tau \in \langle\tau_1, \tau_2\rangle\,. \tag{20,2}$$

Proof. It may be verified that $(d/d\tau)\,\|W(\tau)\|^2_{L_2} = Q_1(W(\tau)) - Q_2(W(\tau))$, Q_1, Q_2 being defined by

$$Q_1(U)(\xi) = \int_0^2\int_0^2 h(q(\xi) - q(-\xi + 2\sigma))\, U^2(\xi)\,d\sigma\,d\xi\,,$$

$$Q_2(U)(\xi) = \int_0^2\int_0^2 h(q(\xi) - q(-\xi + 2\sigma))\, U(-\xi + 2\sigma)\, U(\xi)\,d\sigma\,d\xi$$

for $U \in M$. The definition of q (cf. the beginning of section 1) implies that $\int_0^2 h(q(\xi) - q(-\xi + 2\sigma))\,d\sigma = h(0) + h(a)$ so that

$$Q_1(U) = (h(0) + h(a))\,\|U\|^2_{L_2}\,. \tag{21,2}$$

Let us substitute $\eta = -\xi + 2\sigma$ for σ in the expression for Q_2:

$$Q_2(U)(\xi) = \int_0^2\int_0^2 h(q(\xi) - q(\eta))\, U(\xi)\, U(\eta)\,d\xi\,d\eta\,.$$

It follows from the definition of q that $h(q(\xi) - q(\eta)) = h(0)$ for $0 \leqq \xi < 1$, $0 \leqq \eta < 1$ and for $1 \leqq \xi < 2$, $1 \leqq \eta < 2$, $h(q(\xi) - q(\eta)) = h(a)$ for $0 \leqq \xi < 1$, $1 \leqq \eta < 2$ and for $1 \leqq \xi < 2$, $0 \leqq \eta < 1$. Put $\omega(\xi) = 1$ for $0 \leqq \xi < 1$, $\omega(\xi) = 0$ for $1 \leqq \xi < 2$. Then $h(q(\xi) - q(\eta)) = h(0)\,[\omega(\xi)\,\omega(\eta) + (1 - \omega(\xi))\,(1 - \omega(\eta))] + h(a)\,[\omega(\xi)\,(1 - \omega(\eta)) + (1 - \omega(\xi))\,\omega(\eta)]$ for $\xi, \eta \in \langle 0, 2)$. Hence (as $\int_0^2 W\,d\xi = 0$) $Q_2(W) = 2(h(0) - h(a))\,.\,(\int_0^1 W(\xi)\,d\xi)^2$. As $h(0) > 0$, $h(0) + h(a) < 0$ (cf. (4,0), (1,2)) it follows that $h(0) - h(a) > 0$; therefore

$$Q_2(W) \geqq 0\,, \quad \frac{d}{d\tau}\,\|W\|^2_{L_2} \leqq (h(0) + h(a))\,\|W\|^2_{L_2} \tag{22,2}$$

and (20,2) follows.

3. The purpose of this section is to prove that if $\tilde{y} \in C^1$ and $\tilde{Y} = I_\xi \tilde{y}$ fulfil certain conditions, $\tilde{\tau} \in E_1$, $0 < \varepsilon \leqq \varepsilon_0$, then there exists the solution of (5,1) ((7,1)) in C^1 on $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$, $y(\tilde{\tau}) = \tilde{y}$ and that $y(\tilde{\tau} + \varepsilon)$, $Y(\tilde{\tau} + \varepsilon)$ fulfil the same conditions as $\tilde{y}$, $\tilde{Y}$.

Let us describe the conditions which will be imposed on $\tilde{y}$, $\tilde{Y}$. Let us choose numbers $\mu_0, \mu_1, \mu_2, \mu_3$, $0 < \mu_1 < \mu_2 < \frac{1}{2}a < \mu_3 < \frac{1}{2}a + 1$, $\mu_2 + \mu_3 = a$, $\mu_3 - \frac{1}{2}a \leqq \frac{1}{2}\delta$ (cf. (10,1)) in such a way that $|H(\eta) - h(0)\,\eta| + |H(a + \eta) - h(a)\,\eta| \leqq \frac{1}{8}|h(0) + h(a)|\,|\eta|$ for $|\eta| \leqq \mu_3 - \mu_2$,

$$(1,3)\qquad |H(\eta_2) - H(\eta_1) - h(0)\,(\eta_2 - \eta_1)| \leqq \tfrac{1}{16}|h(0) + h(a)|\,|\eta_2 - \eta_1|\,,$$

$$|H(a + \eta_2) - H(a + \eta_1) - h(a)\,(\eta_2 - \eta_1)| \leqq \tfrac{1}{16}|h(0) + h(a)|\,|\eta_2 - \eta_1|$$

for $|\eta_2|, |\eta_1| \leqq \mu_3 - \mu_2$,

$$(2,3)\qquad |h(\eta) - h(0)| \leqq \tfrac{1}{8}|h(0) + h(a)| \quad \text{for} \quad 2\mu_2 - a \leqq \eta \leqq 2\mu_3 - a\,,$$

$$|h(\eta) - h(a)| \leqq \tfrac{1}{8}|h(0) + h(a)| \quad \text{for} \quad 2\mu_2 \leqq \eta \leqq 2\mu_3\,,$$

$$(3,3)\qquad h(\mu_0 - \tfrac{1}{2}a) + h(\mu_0 + \tfrac{1}{2}a) > 0\,,$$

$$(4,3)\qquad h(\mu_1 - \tfrac{1}{2}a) + h(\mu_1 + \tfrac{1}{2}a) < 0\,,$$

$$(5,3)\qquad \inf_{\mu_0 \leqq \xi \leqq \mu_2} [H(\xi - \tfrac{1}{2}a) + H(\xi + \tfrac{1}{2}a)] > 12(\mu_1 - \mu_0)\,|h\,(\mu_1 - \tfrac{1}{2}a) + h\,(\mu_1 + \tfrac{1}{2}a)|\,.$$

It follows from Lemma 1,2 that it is possible to fulfil conditions (1,3)–(5,3).

Let ν, ϱ and γ be positive numbers, $0 < \gamma < 1$; $\nu \geqq 1$ may be arbitrarily large and ϱ and $1 - \gamma$ are assumed to be small. Let $|A|$ denote the Lebesgue measure of A, A being a Lebesgue measurable subset of the real line. The conditions which are to be satisfied by $\tilde{y}$, $\tilde{Y}$ are the following ones:

$$(6,3)\qquad \tilde{y} \in \hat{C}^1\,,$$

(7,3) there exist numbers $\alpha', \alpha, \beta, \beta'$, $-\frac{1}{2} < \alpha' < \alpha < \beta < \beta' < \frac{1}{2}$ such that

$$\begin{aligned} -\mu_3 \leqq \tilde{Y}(\xi) &\leqq -\mu_1 \quad \text{for} \quad -\tfrac{1}{2} \leqq \xi \leqq \alpha'\,, \\ -\mu_1 \leqq \tilde{Y}(\xi) &\leqq -\mu_0 \quad \text{for} \quad \alpha' \leqq \xi \leqq \alpha\,, \\ -\mu_0 \leqq \tilde{Y}(\xi) &\leqq \mu_0 \quad \text{for} \quad \alpha \leqq \xi \leqq \beta\,, \\ \mu_0 \leqq \tilde{Y}(\xi) &\leqq \mu_1 \quad \text{for} \quad \beta \leqq \xi \leqq \beta'\,, \\ \mu_1 \leqq \tilde{Y}(\xi) &\leqq \mu_3 \quad \text{for} \quad \beta' \leqq \xi \leqq \tfrac{1}{2}\,, \end{aligned}$$

(8,3) $|A| \geqq \gamma$, A being the set of these ξ from $\langle -\frac{1}{2}, \frac{1}{2} \rangle$ that $\mu_2 \leqq |\tilde{Y}(\xi)| \leqq \mu_3$,

$$(9,3)\qquad \tilde{y}(\xi) \geqq \nu \geqq 1 \quad \text{for} \quad \xi \in \langle \alpha, \beta \rangle\,,$$

(10,3) $$\tilde{y}(\xi) \geqq \nu[1 - (-\tilde{Y}(\xi) - \mu_0)/2(\mu_1 - \mu_0)] \quad \text{for} \quad \xi \in \langle \alpha', \alpha \rangle\,,$$
$$\tilde{y}(\xi) \geqq \nu[1 - (\tilde{Y}(\xi) - \mu_0)/2(\mu_1 - \mu_0)] \quad \text{for} \quad \xi \in \langle \beta, \beta' \rangle\,,$$

(11,3) $$\tilde{y}(\xi) \geqq -2 \quad \text{for} \quad \xi \in \langle -\tfrac{1}{2}, \tfrac{1}{2} \rangle\,,$$

(12,3) $$\textstyle\int_0^2 (\tilde{Y}(\xi) - q(\xi - \vartheta_0))^2 \, d\xi \leqq \varrho^2,$$

$\vartheta_0 \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$ being defined by $\tilde{Y}(\vartheta_0) = 0$(of course, ϑ_0 is unique and $\alpha < \vartheta_0 < \beta$),

(13,3) $$\|\tilde{y}\|_{L_1} \leqq R\,.$$

Denote by $U = U(\nu, \varrho, \gamma)$ the set of such $\tilde{y}$, that conditions (4,3)–(13,3) are satisfied by $\tilde{y}$, $\tilde{Y} = I_\xi \tilde{y}$.

Observe that (6,3), (9,3) imply that $\nu(\beta - \alpha) \leqq 2\mu_0$, $\beta' < \alpha' \leqq 1 - \gamma$. Let $\tilde{y} \in U$, $\tilde{\tau} \in E_1$. (13,3) and Lemmas 4,2, 5,2 and 3,2 imply that the solution y of (5,1) (or of (7,1)) in C^1, $y(\tilde{\tau}) = \tilde{y}$ exists on $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$ and fulfils (8,2), (9,2), (10,2) and $y(\tau) \in \hat{C}^1$ for $\tau \in \langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$. Hence

Lemma 1,3. *$y(\tilde{\tau} + \varepsilon)$ fulfils (6,3) (for $0 < \varepsilon \leqq \varepsilon_0$).*

The proof that $Y(\tilde{\tau} + \varepsilon)$ fulfils (7,3) and (8,3) is based on the formula

(14,3) $$Y(\xi, \tilde{\tau} + \varepsilon) = \tilde{Y}(\xi) + \frac{1}{4}\, \varepsilon \,.\int_0^2 H(\tilde{Y}(\xi) - \tilde{Y}(-\xi + \sigma))\, d\sigma + Z_3(\xi)\,,$$
$$\xi \in E_1\,, \quad |Z_3(\xi)| \leqq K_3 \varepsilon^2\,, \quad 0 < \varepsilon \leqq \varepsilon_0\,.$$

In order to prove (14,3) suppose that y is a solution of (5,1) and start from

$$y(\xi, \tilde{\tau} + \varepsilon) = \tilde{y}(\xi) +$$
$$+ \frac{1}{2} \int_{\tilde{\tau}}^{\tilde{\tau}+\varepsilon} h\left(Y(\xi, \tau) - Y\left(-\xi + 2\frac{\tau}{\varepsilon}, \tau\right)\right)\left[y(\xi, \tau) - y\left(-\xi + 2\frac{\tau}{\varepsilon}, \tau\right)\right] d\tau\,.$$

Applying the operation I_ξ one finds that

$$Y(\xi, \tilde{\tau} + \varepsilon) = \tilde{Y}(\xi) +$$
$$+ \frac{1}{2} \int_{\tilde{\tau}}^{\tilde{\tau}+\varepsilon} I_\xi \left\{h\left(Y(\xi, \tau) - Y\left(-\xi + 2\frac{\tau}{\varepsilon}, \tau\right)\right)\left[y(\xi, \tau) + y\left(-\xi + 2\frac{\tau}{\varepsilon}, \tau\right)\right]\right\} d\tau$$
$$- \int_{\tilde{\tau}}^{\tilde{\tau}+\varepsilon} I_\xi \left\{h\left(Y(\xi, \tau) - Y\left(-\xi + 2\frac{\tau}{\varepsilon}, \tau\right)\right) y\left(-\xi + 2\frac{\tau}{\varepsilon}, \tau\right)\right\} d\tau\,.$$

As $I_\xi\{h(Y(\xi,\tau) - Y(-\xi + 2\tau/\varepsilon, \tau))\,[y(\xi,\tau) + y(-\xi + 2\tau/\varepsilon,\tau)]\} = H(Y(\xi,\tau) - Y(-\xi + 2(\tau/\varepsilon),\tau))$, we obtain that

$$Z_3(\xi) = \frac{1}{2}\int_{\tilde\tau}^{\tilde\tau+\varepsilon}\left[H\left(Y(\xi,\tau) - Y\left(-\xi + 2\frac{\tau}{\varepsilon},\tau\right)\right) - H\left(\tilde Y(\xi) - \tilde Y\left(-\xi + 2\frac{\tau}{\varepsilon}\right)\right)\right]\mathrm{d}\tau - I_\xi\left\{\int_{\tilde\tau}^{\tilde\tau+\varepsilon} h\left(Y(\xi,\tau) - Y\left(-\xi + 2\frac{\tau}{\varepsilon},\tau\right)\right) y\left(-\xi + 2\frac{\tau}{\varepsilon},\tau\right)\mathrm{d}\tau\right\}.$$

Observe that $\int_{\tilde\tau}^{\tilde\tau+1} h(\tilde Y(\xi) - \tilde Y(-\xi + 2\sigma))\,\tilde y(-\xi + 2\sigma)\,\mathrm{d}\sigma \equiv 0$ (cf. (3.2)). Therefore it follows from Lemma 8,2 that there exists such a K_4 (depending on R, K_1, K_2) that

$$\left|\int_{\tilde\tau}^{\tilde\tau+\varepsilon} h\left(Y(\xi,\tau) - Y\left(-\xi + 2\frac{\tau}{\varepsilon},\tau\right)\right) y\left(-\xi + 2\frac{\tau}{\varepsilon},\tau\right)\mathrm{d}\tau\right| < \tfrac{1}{2}K_4\varepsilon^2$$

and that

$$\left|\frac{1}{2}\int_{\tilde\tau}^{\tilde\tau+\varepsilon}\left[H\left(Y(\xi,\tau) - Y\left(-\xi + 2\frac{\tau}{\varepsilon},\tau\right)\right) - H\left(\tilde Y(\xi) - \tilde Y\left(-\xi + 2\frac{\tau}{\varepsilon}\right)\right)\right]\mathrm{d}\tau\right| \leqq \tfrac{1}{2}K_4\varepsilon^2 .$$

(14,3) holds, y being a solution of (5,1), $K_3 \geqq K_4$. If y is a solution of (7,1) then Y is a solution of (6,2) (cf. Lemma 2,2) and we may start from

$$Y(\xi, \tilde\tau + \varepsilon) = \tilde Y(\xi) + \frac{1}{4}\int_{\tilde\tau}^{\tilde\tau+\varepsilon}\int_0^2 H(Y(\xi,\tau) - Y(-\xi + \sigma,\tau))\,\mathrm{d}\sigma\,\mathrm{d}\tau =$$

$$= \tilde Y(\xi) + \frac{1}{4}\,\varepsilon\,.\int_0^2 H(\tilde Y(\xi) - \tilde Y(-\xi + \sigma))\,\mathrm{d}\sigma + Z_3^*(\xi),$$

$$Z_3^*(\xi) = \frac{1}{4}\int_{\tilde\tau}^{\tilde\tau+\varepsilon}\int_0^2 [H(Y(\xi,\tau) - Y(-\xi + \sigma,\tau)) - H(\tilde Y(\xi) - \tilde Y(-\xi + \sigma))]\,\mathrm{d}\sigma .$$

As $\|y(\tau)\|_{L_1} \leqq R + 1$ (cf. (10,2)), it follows that $\|Y(\tau)\|_M \leqq R + 1$ and (6,2) implies that $\|Y(\tau) - \tilde Y\|_M \leqq K_5|\tau - \tilde\tau|$ for $\tau \in \langle\tilde\tau, \tilde\tau + \varepsilon\rangle$. Hence there exists K_6 such that $|Z_3^*(\xi)| \leqq K_6\varepsilon^2$. (14,3) holds for $K_3 = \max(K_4, K_6)$.

Lemma 2,3.

$$(15,3)\qquad Y(\xi, \tilde\tau + \varepsilon) = q(\xi - \vartheta_0) + W(\xi)\,[1 + \tfrac{1}{4}\varepsilon(h(0) + h(a))] + Z_4(\xi),$$
$$|Z_4(\xi)| \leqq \tfrac{1}{4}\varepsilon\,.\,[\tfrac{1}{2}|h(0) + h(a)|\,(\mu_3 - \tfrac{1}{2}a) + 4K_3\varepsilon + K_7(1 - \nu + \varrho)],$$

if $\mu_2 \leqq |\tilde Y(\xi)| \leqq \mu_3$.

In order to prove (15,3) choose such a ξ that $\mu_2 \leqq \tilde{Y}(\xi) \leqq \mu_3$ and start from (14,3). Put $W(\xi) = \tilde{Y}(\xi) - q(\xi - \vartheta_0)$, $\vartheta_0 \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$ being defined by (12,3). It follows that

$$Y(\xi, \tilde{\tau} + \varepsilon) = q(\xi - \vartheta_0) + W(\xi) + W(\xi) \frac{1}{4} \varepsilon \, . \int_0^2 h(q(\xi - \vartheta_0) - \\ - q(-\xi + \sigma - \vartheta_0) \, d\sigma + Z_4(\xi) ,$$

$$Z_4(\xi) = Z_3(\xi) - \frac{1}{4} \varepsilon \, . \int_0^2 h(q(\xi - \vartheta_0) - q(-\xi + \sigma - \vartheta_0)) \, W(-\xi + \sigma) \, d\sigma + \\ + \varepsilon \, . \frac{1}{4} \int_0^2 [H(q(\xi - \vartheta_0) - q(-\xi + \sigma - \vartheta_0) + W(\xi) - W(-\xi + \sigma)) - \\ - h(q(\xi - \vartheta_0) - q(-\xi + \sigma - \vartheta_0)) (W(\xi) - W(-\xi + \sigma))] \, d\sigma \, .$$

Let A' be the set of such $\sigma \in \langle -1, 1 \rangle$ that $|W(-\xi + \sigma)| \leqq \mu_3 - \frac{1}{2}a$; as $|A'| \geqq 2\gamma$ (cf. (8,3)), the last integral may be estimated by (cf. (1,3)) $2 \frac{1}{8} |h(0) + h(a)| \, 2(\mu_3 - - \frac{1}{2}a) + 2K_4(1 - \gamma)$ and

$$(16,3) \qquad \left| \int_0^2 h(q(\xi - \vartheta_0) - q(-\xi + \sigma - \vartheta_0)) \, W(-\xi + \sigma) \, d\sigma \right| \leqq K_7 \varrho \, ,$$

K_7 being a positive constant (cf. (12,3)). Therefore

$$|Z_4(\xi)| \leqq K_3 \varepsilon^2 + \varepsilon \, . \tfrac{1}{4} K_7 \varrho + \varepsilon \, . \tfrac{1}{8} |h(0) + h(a)| \, (\mu_3 - \tfrac{1}{2}a) + 2K_4(1 - \gamma) \, ;$$

as

$$\int_0^2 h(q(\xi - \vartheta_0) - q(-\xi + \sigma - \vartheta_0)) \, d\sigma = h(0) + h(a) \, ,$$

(15,3) holds.

Let $\xi \in A'$ (cf. (8,3)). Then $|W(\xi)| \leqq \mu_3 - \frac{1}{2}a$ and it follows from (15,3) that

$$(17,3) \qquad |Y(\xi, \tilde{\tau} + \varepsilon) - q(\xi - \vartheta_0)| \leqq (\mu_3 - \tfrac{1}{2}a) \, [1 + \tfrac{1}{4}\varepsilon(h(0) + h(a))] + \\ + \tfrac{1}{4}\varepsilon[\tfrac{1}{2}|h(0) + h(a)| \, (\mu_3 - \tfrac{1}{2}a) + 4K_3\varepsilon + K_7(1 - \gamma + \varrho)] \leqq \mu_3 - \tfrac{1}{2}a \, , \\ 0 < \varepsilon \leqq \varepsilon_0$$

provided that $\frac{1}{4}\varepsilon_0 |h(0) + h(a)| < 1$ and $4K_3\varepsilon_0 + K_7(1 - \gamma + \varrho) \leqq \frac{1}{2}|h(0) + h(a)| \, . \, . \, (\mu_3 - \frac{1}{2}a)$. The last inequality holds for ε_0 sufficiently small if

$$(18,3) \qquad K_7(1 - \gamma + \varrho) < \tfrac{1}{2}|h(0) + h(a)| \, (\mu_3 - \tfrac{1}{2}a)$$

which is one of the conditions for γ and ϱ.

Let A_1 be the set of such $\xi \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$ that $\mu_3 \geqq |Y(\xi, \tilde{\tau} + \varepsilon)| \geqq \mu_2$. (17,3) implies that there $A \subset A_1$,i.e.

Lemma 3,3. $Y(\tilde{\tau} + \varepsilon)$ *fulfils* (8,3).

Put $\varkappa_1 = \inf_{\mu_1 \leqq \xi \leqq \mu_2} [H(\xi - \frac{1}{2}a) + H(\xi + \frac{1}{2}a)]$; $\varkappa_1 > 0$ according to Lemma 1,2.

Lemma 4,3. *If* $\mu_0 \leqq \tilde{Y}(\xi_1) \leqq \mu_2$ *for some* $\xi_1 \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$, *then* $Y(\xi_1, \tilde{\tau} + \varepsilon) \geqq \geqq \tilde{Y}(\xi_1) + \frac{1}{8}\varepsilon\varkappa_1$, *if* $-\mu_2 \leqq \tilde{Y}(\xi_2) \leqq -\mu_0$ *for some* $\xi_2 \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$, *then* $Y(\xi_2, \tilde{\tau} + \varepsilon) \leqq \leqq \tilde{Y}(\xi_2) - \frac{1}{8}\varepsilon\varkappa_1$ (*y being a solution of* (5,1) *or* (7,1), $0 < \varepsilon \leqq \varepsilon_0$).

Proof. Let $\mu_0 \leqq \tilde{Y}(\xi_1) \leqq \mu_2$; it follows from (14,3) that

$$Y(\xi_1, \tilde{\tau} + \varepsilon) = \tilde{Y}(\xi_1) + \frac{1}{4}\varepsilon \cdot \int_0^2 H(\tilde{Y}(\xi_1) - q(-\xi_1 + \sigma - \vartheta_0))\, d\sigma + Z_5(\xi_1)\,, \quad \xi_1 \in E_1\,,$$

$$Z_5(\xi_1) = Z_3(\xi_1) + \tfrac{1}{4}\varepsilon \int_0^2 [H(\tilde{Y}(\xi_1) - \tilde{Y}(-\xi_1 + \sigma)) - $$
$$- H(\tilde{Y}(\xi_1) - q(-\xi_1 + \sigma - \vartheta_0))]\, d\sigma\,.$$

As $\tilde{Y}(-\xi_1 + \sigma) = q(-\xi_1 + \sigma - \vartheta_0) + W(-\xi_1 + \sigma - \vartheta_0)$, we obtain that $|Z_5(\xi_1)| \leqq \leqq K_3\varepsilon^2 + \frac{1}{4}\varepsilon \,.\, K_8\varrho$ (cf. (12,3));

$$\int_0^2 H(\tilde{Y}(\xi_1) - q(-\xi_1 + \sigma - \vartheta_0))\, d\sigma = H(\tilde{Y}(\xi_1) - \tfrac{1}{2}a) + H(\tilde{Y}(\xi_1) + \tfrac{1}{2}a) \geqq \varkappa_1\,.$$

As the case $-\mu_2 \leqq \tilde{Y}(\xi_2) \leqq -\mu_1$ is analogous, Lemma 4,3 holds, provided that $K_3\varepsilon_0^2 + \frac{1}{4}K_8\varrho\varepsilon_0 \leqq \frac{1}{8}\varkappa_1\varepsilon_0$. The last inequality holds for ε_0 sufficiently small, if

$$2K_8\varrho < \varkappa\,, \tag{19,3}$$

this being another condition on ϱ.

Put $\varkappa_2 = h(\mu_0 - \frac{1}{2}a) + h(\mu_0 + \frac{1}{2}a)$, $-\varkappa_3 = h(\mu_1 - \frac{1}{2}a) + h(\mu_1 + \frac{1}{2}a)$. According to (3,3), (4,3) and Lemma 1,2 one obtains that $\varkappa_2 > 0$, $\varkappa_3 > 0$, $h(\xi - \frac{1}{2}a) + h(\xi + \frac{1}{2}a) \geqq \varkappa_2$ for $0 \leqq \xi \leqq \mu_0$, $h(\xi - \frac{1}{2}a) + h(\xi + \frac{1}{2}a) \leqq -\varkappa_3$ for $\mu_1 \leqq \xi \leqq \mu_3$.

Lemma 5,3. (i) *If* $|\tilde{Y}(\xi_1)| \leqq \mu_0$, *for some* $\xi_1 \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$, *then* $y(\xi_1, \tilde{\tau} + \varepsilon) \geqq \geqq \tilde{y}(\xi_1)(1 + \frac{1}{8}\varepsilon \,.\, \varkappa_2)$.

(ii) *If* $|\tilde{y}(\xi_2)| \geqq 1$, $\mu_1 \leqq |\tilde{Y}(\xi_2)| \leqq \mu_3$ *for some* $\xi_2 \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$, *then* $|y(\xi_2, \tilde{\tau} + \varepsilon)| \leqq \leqq |\tilde{y}(\xi_2)|(1 - \frac{1}{8}\varepsilon \,.\, \varkappa_2)$.

Proof. Let y be a solution of (5,1). Starting from (12,2) one obtains that

$$y(\xi, \tilde{\tau} + \varepsilon) = \tilde{y}(\xi) + \tfrac{1}{4}\varepsilon\, \tilde{y}(\xi) \int_0^2 h(\tilde{Y}(\xi) - \tilde{Y}(-\xi + \sigma))\, d\sigma + z_5(\xi)\,, \tag{20,3}$$

$$z_5(\xi) = \tfrac{1}{4}\varepsilon \int_0^2 [h(Y(\xi, \tfrac{1}{2}\sigma) - Y(-\xi + \sigma, \tfrac{1}{2}\sigma))(y(\xi, \tfrac{1}{2}\sigma) - y(-\xi + \sigma, \tfrac{1}{2}\sigma)) - $$

$$- h(\tilde{Y}(\xi) - \tilde{Y}(-\xi + \sigma))(\tilde{y}(\xi) - \tilde{y}(-\xi + \sigma))]\, d\sigma\,, \quad \xi \in E_1\,, \quad 0 < \varepsilon \leqq \varepsilon_0$$

(cf. (3,2)). By means of (15,2) and (10,2) one may estimate

$$(21,3) \qquad |z_5(\xi)| \leqq \varepsilon^2 K_9(|\tilde{y}(\xi)| + 1), \quad \xi \in E_1 .$$

If y is a solution of (7,1), then starting from (14,2) one obtains

$$(22,3) \quad y(\xi, \tilde{\tau} + \varepsilon) = \tilde{y}(\xi) + \tfrac{1}{4}\,\varepsilon\,\tilde{y}(\xi)\int_0^2 h(\tilde{Y}(\xi) - \tilde{Y}(-\xi + 2\sigma))\,\mathrm{d}\sigma + z_5^*(\xi),$$

$$z_5^*(\xi) = \tfrac{1}{4}\int_{\tilde{\tau}}^{\tilde{\tau}+\varepsilon}\int_0^2 h(Y(\xi, \eta) - Y(-\xi + 2\sigma, \eta))\,y(\xi, \eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta -$$

$$- \frac{1}{4}\varepsilon\,\tilde{y}(\xi)\int_0^2 h(\tilde{Y}(\xi) - \tilde{Y}(\xi + 2\sigma))\,\mathrm{d}\sigma, \quad \xi \in E_1 .$$

Again (15,2) implies that the estimate (21,3) holds for $z_5^*(\xi)$, $\xi \in E_1$ (K_9 being enlarged, if necessary). As

$$\int_0^2 h(\tilde{Y}(\xi) - \tilde{Y}(-\xi + \sigma))\,\mathrm{d}\sigma = \int_0^2 h(\tilde{Y}(\xi) - q(-\xi + \sigma - \vartheta_0))\,\mathrm{d}\sigma +$$

$$+ \int_0^2 [h(\tilde{Y}(\xi) - q(-\xi + \sigma - \vartheta_0) - W(-\xi + \sigma)) - h(\tilde{Y}(\xi) - q(-\xi + \sigma - \vartheta_0))]\,\mathrm{d}\sigma =$$

$$= [h(\tilde{Y}(\xi) - \tfrac{1}{2}a) + h(\tilde{Y}(\xi) + \tfrac{1}{2}a)] + z_6(\xi), \quad |z_6(\xi)| \leqq K_{10}\varrho, \quad \xi \in E_1$$

one obtains that

$$(23,3) \quad y(\xi, \tilde{\tau} + \varepsilon) = \tilde{y}(\xi)\,[1 + \tfrac{1}{4}\varepsilon(h(\tilde{Y}(\xi) - \tfrac{1}{2}a) + h(\tilde{Y}(\xi) + \tfrac{1}{2}a))] + z_7(\xi),$$

$$|z_7(\xi)| \leqq \varepsilon^2 K_9 + |\tilde{y}(\xi)|\,\varepsilon[\varepsilon K_9 + \tfrac{1}{4}\varrho K_{10}], \quad \xi \in \langle -\tfrac{1}{2}, \tfrac{1}{2}\rangle, \quad 0 < \varepsilon \leqq \varepsilon_0,$$

y being a solution of (5,1) or (7,1).

As $|\tilde{Y}(\xi_1)| \leqq \mu_0$, it follows that $h(\tilde{Y}(\xi_1) - \tfrac{1}{2}a) + h(\tilde{Y}(\xi_1) + \tfrac{1}{2}a) \geqq \varkappa_2$ and (i) of Lemma 3,3 holds, provided that $\tfrac{1}{8}\varkappa_2 \geqq 2\varepsilon_0 K_9 + \tfrac{1}{4}\varrho K_{10}$. This inequality is fulfilled for ε_0 sufficiently small, if

$$(24,3) \qquad 2K_{10}\varrho < \varkappa_2,$$

which is another condition on ϱ.

Let $|\tilde{y}(\xi_2)| \geqq 1$, $\mu_1 \leqq |\tilde{Y}(\xi_2)| \leqq \mu_3$; as $h(\tilde{Y}(\xi_2) - \tfrac{1}{2}a) + h(\tilde{Y}(\xi_2) + \tfrac{1}{2}a) \leqq -\varkappa_3$, (ii) of Lemma 3,3 follows from (23,3) provided that $\tfrac{1}{4}\varepsilon_0|h(\mu_3 - \tfrac{1}{2}a) + h(\mu_3 + \tfrac{1}{2}a)| < 1$, $\tfrac{1}{8}\varkappa_3 \geqq 2\varepsilon_0 K_9 + \tfrac{1}{4}\varrho K_{10}$. The last inequality is fulfilled for ε_0 sufficiently small, if

$$(25,3) \qquad 2K_{10}\varrho < \varkappa_3 .$$

Lemma 5,3 is proved.

Put $\varkappa_4 = \min(\tfrac{1}{8}\varkappa_1, \tfrac{1}{8}\varkappa_2, \tfrac{1}{48}\varkappa_1/(\mu_1 - \mu_0))$.

Lemma 6,3. *Let y be the same as above.*

(i) *If* $\mu_0 \leqq Y(\xi_1, \tilde{\tau} + \varepsilon) \leqq \mu_1$, *for some* $\xi_1 \in \langle -\frac{1}{2}, \frac{1}{2}\rangle$ *then*

$$y(\xi_1, \tilde{\tau} + \varepsilon) \geqq \nu(1 + \varepsilon\varkappa_4)\left[1 - (Y(\xi_1, \tilde{\tau} + \varepsilon) - \mu_0)/2(\mu_1 - \mu_0)\right];$$

(ii) *If* $-\mu_1 \leqq Y(\xi_2, \tilde{\tau} + \varepsilon) \leqq -\mu_0$, *for some* $\xi_2 \in \langle -\frac{1}{2}, \frac{1}{2}\rangle$, *then*

$$y(\xi, \tilde{\tau} + \varepsilon) \geqq \nu(1 + \nu\, \varepsilon\varkappa_4)\left[1 + (Y(\xi, \tilde{\tau} + \varepsilon) + \mu_0)/2(\mu_1 - \mu_0)\right].$$

Proof. It follows from Lemma 2,3 that $0 \leqq \tilde{Y}(\xi_1) \leqq \mu_1$ (of course, $\tilde{Y}(\xi_1) < 0$ is not possible, as $|Y(\xi_1, \tilde{\tau} + \varepsilon) - \tilde{Y}(\xi_1)| \leqq \|Y(\tilde{\tau} + \varepsilon) - \tilde{Y}\|_M \leqq \|y(\tilde{\tau} + \varepsilon) - \tilde{y}\|_{L_1} \leqq \leqq \varepsilon\|f_0(\tilde{y})\|_{L_1} + K_1\varepsilon^2 \leqq \mu_0$, $0 < \varepsilon \leqq \varepsilon_0$, ε_0 being sufficiently small (cf. Lemma 4,2). If $0 \leqq \tilde{Y}(\xi_1) \leqq \mu_0$, then (i) of Lemma 6,3 holds according to Lemma 5,3, (9,3) and (7,3). Let therefore $\mu_0 < \tilde{Y}(\xi_1) \leqq \mu_1$. It follows from (20,3), (22,3) and (21,3) that

$$y(\xi_1, \tilde{\tau} + \varepsilon) \geqq \tilde{y}(\xi_1)\left[1 + \tfrac{1}{4}\varepsilon\int_0^2 h(\tilde{Y}(\xi_1) - \tilde{Y}(-\xi_1 + \sigma))\, d\sigma\right].$$

$$.(1 - \tfrac{1}{48}\varepsilon\varkappa_1/(\mu_1 - \mu_0)) + . \tfrac{1}{96}\varepsilon\varkappa_1\, \tilde{y}(\xi_1)/(\mu_1 - \mu_0) + z_5(\xi_1),$$

$$|z_5(\xi_1)| \leqq \varepsilon^2 K_9(\tilde{y}(\xi_1) + 1) \leqq 2\varepsilon^2 K_9\, \tilde{y}(\xi_1),$$

ε_0 being so small that $\frac{1}{4}\varepsilon_0 \int_0^2 h(\tilde{Y}(\xi_1) - \tilde{Y}(-\xi_1 + \sigma))\, d\sigma \geqq -\frac{1}{2}$. If in addition $\varkappa_1 \geqq \geqq 96\varepsilon_0(\mu_1 - \mu_0) K_9$; then

$$y(\xi_1, \tilde{\tau} + \varepsilon) \geqq \tilde{y}(\xi_1)\left[1 + \tfrac{1}{4}\varepsilon\int_0^2 h(\tilde{Y}(\xi_1) - \tilde{Y}(-\xi_1 + \sigma))\, d\sigma\right](1 - \tfrac{1}{48}\varepsilon\varkappa_1/(\mu_1 - \mu_0)).$$

As $\tilde{y}(\xi_1) \geqq \nu[1 - (\tilde{Y}(\xi_1) - \mu_0)/2(\mu_1 - \mu_0)]$ (cf. 10,3) and

$$\int_0^2 h(\tilde{Y}(\xi_1) - \tilde{Y}(-\xi_1 + \sigma))\, d\sigma = \int_0^2 h(\tilde{Y}(\xi_1) - q(-\xi_1 + \sigma - \vartheta_0))\, d\sigma +$$

$$+ \int_0^2 [h(\tilde{Y}(\xi_1) - q(-\xi_1 + \sigma - \vartheta_0) - W(-\xi_1 + \sigma)) - h(\tilde{Y}(\xi_1) - q(-\xi + \sigma))]\, d\sigma \geqq$$

$$\geqq h(\tilde{Y}(\xi_1) - \tfrac{1}{2}a) + h(\tilde{Y}(\xi_1) + \tfrac{1}{2}a) - K_{11}\varrho \geqq -\varkappa_3 - K_{11}\varrho$$

(cf. (12,3), Lemma 1,2), it follows that

$$y(\xi_1, \tilde{\tau} + \varepsilon) \geqq \nu[1 - (\tilde{Y}(\xi_1) - \mu_0)/2(\mu_1 - \mu_0)]\left[1 - \tfrac{1}{4}\varepsilon\,.\,(\varkappa_3 + K_{11}\varrho)\right].$$

$$.\left[1 - \tfrac{1}{48}\varepsilon\varkappa_1/(\mu_1 - \mu_0)\right]$$

Taking into account that $Y(\xi_1, \tilde{\tau} + \varepsilon) \geqq \tilde{Y}(\xi_1) + \frac{1}{8}\varepsilon\varkappa_1$ we obtain that

$$y(\xi_1, \tilde{\tau} + \varepsilon) \geqq \{v[1 - (Y(\xi_1, \tilde{\tau} + \varepsilon) - \mu_0)/2(\mu_1 - \mu_0) + \tfrac{1}{16}\varepsilon\varkappa_1/2(\mu_1 - \mu_0)] \, .$$
$$. \, [1 - \tfrac{1}{4}\varepsilon \, . \, (\varkappa_3 + K_{11}\varrho)] \, [1 - \tfrac{1}{48}\varepsilon\varkappa_1/(\mu_1 - \mu_0)] \, [1 - \tfrac{1}{48}\varepsilon\varkappa_1/(\mu_1 - \mu_0)] \, .$$
$$. \, [1 + \tfrac{1}{48}\varepsilon\varkappa_2/(\mu_1 - \mu_0] \geqq v(1 + \tfrac{1}{48}\varepsilon\varkappa_1/(\mu_1 - \mu_0)) \, .$$
$$. \, [1 - (Y(\xi, \tilde{\tau} + \varepsilon) - \mu_0)/2(\mu_1 - \mu_0)] \, . \, [1 + \tfrac{1}{16}\varepsilon\varkappa_1/(\mu_1 - \mu_0)] \, .$$
$$. \, [1 - \tfrac{1}{4}\varepsilon(\varkappa_3 + K_{11}\varrho) - \tfrac{2}{48}\varepsilon\varkappa_1/(\mu_1 - \mu_0)] \, .$$

If

$$(26,3) \qquad [1 + \tfrac{1}{16}\varepsilon\varkappa_1/(\mu_1 - \mu_0)] \, [1 - \tfrac{1}{4}\varepsilon(\varkappa_3 + K_{11}\varrho) - \tfrac{1}{24}\varepsilon\varkappa_1(\mu_1 - \mu_0)] \geqq 1 \, ,$$

then

$$(27,3) \qquad y(\xi_1, \tilde{\tau} + \varepsilon) \geqq v(1 + \tfrac{1}{48}\varepsilon\varkappa_1/(\mu_1 - \mu_0)) \, [1 - (Y(\xi_1, \tilde{\tau} + \varepsilon) - \mu_0)/2(\mu_1 - \mu_0)] \, .$$

(26,3) is fulfilled, if $\varkappa_1/(\mu_1 - \mu_0) > 12\varkappa_3 + K_{11}\varrho$, $0 < \varepsilon \leqq \varepsilon_0$, ε_0 being sufficiently small. Therefore (27,3) holds, if

$$(28,3) \qquad \varrho < (K_{11}(\mu_1 - \mu_0))^{-1} \, . \, [\varkappa_1 - 12\varkappa_3(\mu_1 - \mu_0)] \, .$$

The right hand side of (28,3) is positive (cf. (5,3) and the definitions of $\varkappa_1$, $\varkappa_3$ before Lemma 5,3) and (28,3) is another condition on ϱ. (27,3) makes the proof of (i) of Lemma 4,3 complete.

As (ii) is analogous, Lemma 6,3 is proved.

Some of the above results make possible to prove the following theorem.

Theorem 1,3. *Let* $v \geqq 1$, $\varrho > 0$, $0 < \gamma < 1$ *and let* ϱ *and* γ *fulfil* (18,3), (19,3), (24,3), (25,3) *and* (28,3), $\tilde{y} \in U(v, \varrho, \gamma)$. *Then* $y(\tilde{\tau} + \varepsilon)$, $Y(\tilde{\tau} + \varepsilon)$ *fulfil* (6,3) *and*

(29,3) *there exist such numbers* $\alpha_1', \alpha_1, \beta_1, \beta_1'$, $-\frac{1}{2} < \alpha_1' < \alpha_1 < \beta_1 < \beta_1' < \frac{1}{2}$, $\alpha < \alpha_1$, $\beta_1 < \beta$, $\alpha' < \alpha_1'$, $\beta_1' < \beta'$ such that

$$\begin{aligned}
-\mu_3 \leqq Y(\xi, \tilde{\tau} + \varepsilon) &\leqq -\mu_1 \quad \textit{for} \quad -\tfrac{1}{2} \leqq \xi \leqq \alpha_1' \, , \\
-\mu_1 \leqq Y(\xi, \tilde{\tau} + \varepsilon) &\leqq -\mu_0 \quad \textit{for} \quad \alpha_1' \leqq \xi \leqq \alpha_1 \, , \\
-\mu_0 \leqq Y(\xi, \tilde{\tau} + \varepsilon) &\leqq \mu_0 \quad \textit{for} \quad \alpha_1 \leqq \xi \leqq \beta_1 \, , \\
\mu_0 \leqq Y(\xi, \tilde{\tau} + \varepsilon) &\leqq \mu_1 \quad \textit{for} \quad \beta_1 \leqq \xi \leqq \beta_1' \, , \\
\mu_1 \leqq Y(\xi, \tilde{\tau} + \varepsilon) &\leqq \mu_3 \quad \textit{for} \quad \beta_1' \leqq \xi \leqq \tfrac{1}{2} \, ,
\end{aligned}$$

$$(30,3) \qquad y(\xi, \tilde{\tau} + \varepsilon) \geqq v(1 + \varepsilon\varkappa_4) \quad \textit{for} \quad \xi \in \langle\alpha_1, \beta_1\rangle \, ,$$

$$(31,3) \qquad y(\xi, \tilde{\tau} + \varepsilon) \geqq v(1 + \varepsilon\varkappa_4) \, .$$
$$. \, [1 - (-Y(\xi, \tilde{\tau} + \varepsilon) - \mu_0)/2(\mu_1 - \mu_0)] \quad \textit{for} \quad \xi \in \langle\alpha_1', \alpha_1\rangle \, ,$$

$$y(\xi, \tilde{\tau} + \varepsilon) \geqq \nu(1 + \varepsilon\varkappa_4) \,.$$
$$. \left[1 - (Y(\xi, \tilde{\tau} + \varepsilon) - \mu_0)/2(\mu_1 - \mu_0)\right] \quad \textit{for} \quad \xi \in \langle \beta_1, \beta'_1 \rangle \,,$$

(32,3) $$y(\xi, \tilde{\tau} + \varepsilon) \geqq -2 \quad \textit{for} \quad \xi \in \langle -\tfrac{1}{2}, \tfrac{1}{2} \rangle \,.$$

As usually y is a solution of (5,1) *or of* (7,1) *in* C^1 *on* $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$, $y(\tilde{\tau}) = \tilde{y}$.

Note 1,3. Observe that (29,3)–(32,3) with the exception of $\alpha < \alpha_1$, $\beta_1 < \beta$, $\alpha' < \alpha'_1$, $\beta'_1 < \beta'$ may be formulated shortly: $Y(\tilde{\tau} + \varepsilon)$, $y(\tilde{\tau} + \varepsilon)$ fulfil (7,3)–(11,3).

Proof. $y(\tilde{\tau} + \varepsilon)$ fulfils (6,3) according to Lemma 1,3. It follows from Lemmas 4,3, 5,3 that $Y(\alpha, \tilde{\tau} + \varepsilon) \leqq \mu_0 - \frac{1}{8}\varkappa_1\varepsilon$, $Y(\beta, \tilde{\tau} + \varepsilon) \geqq \mu_0 + \frac{1}{8}\varkappa_1\varepsilon$, $y(\xi, \tilde{\tau} + \varepsilon) \geqq \nu(1 + \frac{1}{8}\varepsilon\varkappa_2) \geqq \nu(1 + \varepsilon\varkappa_4)$ for $\xi \in \langle \alpha, \beta \rangle$. Therefore there exist $\alpha_1, \beta_1, \alpha < \alpha_1 < \beta_1 < \beta$, $Y(\alpha_1, \tilde{\tau} + \varepsilon) = -\mu_0$, $Y(\beta_1, \tilde{\tau} + \varepsilon) = \mu_0$ and (30,3) is fulfilled. It follows from Lemma 4,3 that the image of $\langle -\frac{1}{2}, \frac{1}{2} \rangle$ by $Y(., \tilde{\tau} + \varepsilon)$ contains the interval $\langle -\mu_2 - \varepsilon\varkappa_4, \mu_2 + \varepsilon\varkappa_4 \rangle$. Therefore there exist $\alpha'_1, \beta'_1 \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$, $Y(\alpha'_1, \tilde{\tau} + \varepsilon) = -\mu_1$, $Y(\beta'_1, \tilde{\tau} + \varepsilon) = \mu_1$. According to Lemma 6,3 $y(\xi_3, \tilde{\tau} + \varepsilon) \geqq \frac{1}{2}\nu$ if $\mu_0 \leqq |Y(\xi_3, \tilde{\tau} + \varepsilon)| \leqq \mu_1$, for some $\xi_3 \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$. Therefore the numbers $\alpha'_1, \alpha_1, \beta_1, \beta'_1$ are unique, $\alpha'_1 < \alpha_1$, $\beta_1 < \beta'_1$ and $Y(\xi, \tilde{\tau} + \varepsilon) \leqq -\mu_1$ on $\langle \frac{1}{2}, \alpha'_1 \rangle$ and $Y(\xi, \tilde{\tau} + \varepsilon) \geqq \mu_1$ on $\langle \beta'_1, \frac{1}{2} \rangle$. As $Y(\alpha', \tilde{\tau} + \varepsilon) \leqq -\mu_1 - \varepsilon\varkappa_4$, $Y(\beta', \tilde{\tau} + \varepsilon) \geqq \mu_1 + \varepsilon\varkappa_4$ (cf. Lemma 4,3), we obtain that $\alpha' < \alpha'_1$, $\beta'_1 < \beta'$. Let us prove that $|Y(\xi, \tilde{\tau} + \varepsilon)| \leqq \mu_3$ for $\xi \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$. If $|Y(\xi_4, \tilde{\tau} + \varepsilon)| > \mu_3$ for some $\xi_4 \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$, then (14,3) implies that $|\tilde{Y}(\xi_4)| \geqq \mu_2$, ε_0 being sufficiently small (ε_0 is independent on $\tilde{Y}$). $|\tilde{Y}(\xi_4)| \leqq \mu_3$, therefore necessarily $\xi_4 \in A$ and (17,3) implies that $|Y(\xi_4, \tilde{\tau} + \varepsilon)| \leqq \mu_3$. Therefore (29,3) and (30,3) are fulfilled. (31,3) holds according to Lemma 6,3. Let us prove (32,3). It follows from (30,3), (31,3) and Lemma 5,3, (ii) that $y(\xi) \geqq -2$, if $\xi \in \langle \alpha'_1, \beta'_1 \rangle \cup \langle -\frac{1}{2}, \alpha' \rangle \cup \langle \beta', \frac{1}{2} \rangle$. Let $\xi_1 \in (\beta'_1, \beta')$; it follows that $\tilde{Y}(\xi_1) < \mu_1$, $Y(\xi_1, \tilde{\tau} + \varepsilon) > \mu_1$. Obviously $\alpha' < \alpha'_1 < \beta'_1 < \beta'$ (cf. (29,3)); therefore $\tilde{y}(\xi_1) \geqq \frac{1}{2}\nu > 0$. It follows from (15,2) that $y(\xi_1, \tilde{\tau} + \varepsilon) \geqq \tilde{y}(\xi_1)\left[1 - 2\varepsilon \,.\, K_2\right] - 2\varepsilon \,.\, K_2$; hence $y(\xi_1, \tilde{\tau} + \varepsilon) \geqq -1$ if $2\varepsilon_0 \,.\, K_2 \leqq 1$. As the case that $\xi_1 \in (\alpha', \alpha'_1)$ is analogous, (32,3) holds.

Lemma 7,3. $\|y(\tilde{\tau} + \varepsilon)\|_{L_1} \leqq R$.

Proof. Put $y_+(\xi) = \max(y(\xi, \tilde{\tau} + \varepsilon), 0)$, $y_-(\xi) = \max(-y(\xi, \tilde{\tau} + \varepsilon), 0)$.

$$\|y(\tilde{\tau} + \varepsilon)\|_{L_1} = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} y_+(\xi)\,d\xi + 2\int_{-\frac{1}{2}}^{\frac{1}{2}} y_-(\xi)\,d\xi \,.$$

As $y_-(\xi) \leqq 2$ (cf. (32,3)), $y(\xi, \tilde{\tau} + \varepsilon) = y_+(\xi) - y_-(\xi)$, $|Y(\xi, \tilde{\tau} + \varepsilon)| \leqq \mu_3$, it follows that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} y_+(\xi)\,d\xi = Y(\tfrac{1}{2}, \tilde{\tau} + \varepsilon) - Y(-\tfrac{1}{2}, \tilde{\tau} + \varepsilon) + \int_{-\frac{1}{2}}^{\frac{1}{2}} y_-(\xi)\,d\xi \leqq 2\mu_3 + 2 \leqq a + 4 \,.$$

Hence $\|y(\tilde{\tau} + \varepsilon)\|_{L_1} \leqq 2a + 8 + 4 = R$.

Let $q_0 \in M$ be defined by $q_0(\xi) = q(\xi - \vartheta_0)$ for $\xi \in E_1$ (ϑ_0 being defined in (12,3)) and put $\varkappa_5 = \frac{1}{8}|h(0) + h(a)|$.

Lemma 8,3. *Let S_1, S_2 be solutions of (6,2) in M on $\langle\tau_1, \tau_2\rangle$, $\|S_j(\tau) - q_0\|_M \leqq \leqq \mu_3 - \frac{1}{2}a$, $\tau \in \langle\tau_1, \tau_2\rangle$, $j = 1, 2$. Then $\|S_2(\tau) - S_1(\tau)\|_{L_2} \leqq \exp\{-2\varkappa_5(\tau - \tau_1)\} \cdot \cdot \|S_1(\tau_1) - S_2(\tau_1)\|_{L_2}$, $\tau \in \langle\tau_1, \tau_2\rangle$.*

Proof. It may be verified that

$$\frac{d}{d\tau}\int_0^2 (S_2(\xi, \tau) - S_1(\xi, \tau))^2 \, d\xi =$$

$$= \int_0^2 \int_0^2 (S_2(\xi, \tau) - S_1(\xi, \tau)) \left[H(S_2(\xi, \tau) - S_2(-\xi + 2\sigma, \tau)) - \right.$$

$$\left. - H(S_1(\xi, \tau) - S_1(-\xi + 2\sigma, \tau))\right] d\sigma \, d\tau =$$

$$= \int_0^2 \int_0^2 (S_2(\xi, \tau) - S_1(\xi, \tau)) \, h(\lambda(S_2(\xi, \tau) - S_2(-\xi + 2\sigma, \tau)) +$$

$$+ (1 - \lambda)(S_1(\xi, \tau) - S_1(-\xi + 2\sigma, \tau)))(S_2(\xi, \tau) - S_2(-\xi + 2\sigma, \tau) -$$

$$- S_1(\xi, \tau) + S_1(-\xi + 2\sigma, \tau)) \, d\sigma \, d\tau \,, \quad 0 < \lambda < 1 \,, \quad \lambda = \lambda(\xi, \tau) \,.$$

From the above assumptions it follows that

$$|\lambda(S_2(\xi, \tau) - S_2(-\xi + 2\sigma, \tau)) + (1 - \lambda)(S_1(\xi, \tau) - S_1(-\xi + 2\sigma, \tau)) - q_0(\xi)| \leqq$$
$$\leqq \mu_3 - \tfrac{1}{2}a$$

and according to (2,3) $h(\lambda(S_2(\xi, \tau) - S_2(-\xi + 2\sigma, \tau)) + (1 - \lambda)(S_1(\xi, \tau) - - S_1(-\xi + 2\sigma, \tau))) = h(q_0(\xi) - q_0(-\xi + 2\sigma)) + z_8(\xi, \tau, \sigma)$, $|z_8(\xi, \tau, \sigma)| \leqq \frac{1}{8}|h(0) + + h(a)|$. Hence (cf. proof of Lemma 10,2)

$$\frac{d}{d\tau}\int_0^2 (S_2(\xi, \tau) - S_1(\xi, \tau))^2 \, d\xi = Q_1(S_1(\tau) - S_2(\tau)) - Q_2(S_1(\tau) - S_2(\tau)) +$$

$$+ \int_0^2 \int_0^2 (S_1(\xi, \tau) - S_2(\xi, \tau)) \, z_8(\xi, \tau, \sigma)(S_1(\xi, \tau) - S_2(\xi, \tau) -$$

$$- S_1(-\xi + 2\sigma, \tau) + S_2(-\xi + 2\sigma, \tau)) \, d\sigma \, d\xi \,.$$

Taking (21,2), (22,2), (1,2), the definition of $\varkappa_5$ (before Lemma 8,3) and the estimate for z_8 into account, we obtain that

$$\frac{d}{d\tau}\int_0^2 (S_2(\xi, \tau) - S_1(\xi, \tau))^2 \, d\xi \leqq [-8\varkappa_5 + \varkappa_5 + 2\varkappa_5] \int_0^2 (S_2(\xi, \tau) - S_1(\xi, \tau))^2 \, d\xi \,.$$

Therefore $\|S_2(\tau) - S_1(\tau)\|_{L_2} \leqq \|S_2(\tau_1) - S_1(\tau_1)\|_{L_2} \exp(-2\varkappa_5(\tau - \tau_1))$ and Lemma 8,3 holds.

Lemma 9,3. *Let* $\tilde{S} \in M$, $\|\tilde{S} - q_0\|_M \leqq \mu_3 - \frac{1}{2}a$, $\|\tilde{S} - q_0\|_{L_2} \leqq \varrho$, $\tilde{\tau} \in E_1$, $\varrho > 0$ *being sufficiently small* (cf. 34,3)). *Then there exists the solution* S *of* (6,2) *in* M *on* $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$, $S(\tilde{\tau}) = \tilde{S}$ *and* $\|S(\tau) - q_0\|_M \leqq \mu_3 - \frac{1}{2}a$ *for* $\tau \in \langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$, $0 < \varepsilon \leqq \varepsilon_0$ (ε_0 *being small enough*).

Proof. A standard argument proves that S exists on $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$ and that $\|S(\tau)\| \leqq \leqq \mu_3 + 1$,

$$S(\tau) = \tilde{S} + (\tau - \tilde{\tau}) f_0^*(\tilde{S}) + Z_1(\tau), \quad \|Z_1(\tau)\| \leqq K_{12}(\tau - \tilde{\tau})^2 \tag{33,3}$$

for $\tau \in \langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$. Put $V = \tilde{S} - q_0, f_0^*(\tilde{S})(\xi) = \frac{1}{4} \int_0^2 h(q_0(\xi) - q_0(-\xi + \sigma))(V(\xi) - - V(-\xi + \sigma))\, d\sigma + Z_2(\xi)$. Taking into account that 0 and a are the only values of $q_0(\xi) - q_0(-\xi + \sigma)$, we obtain from (1,3) that $|Z_2(\xi)| \leqq \frac{1}{32}|h(0) + h(a)|(\mu_3 - \mu_2)$,

$$\int_0^2 h(q_0(\xi) - q_0(-\xi + \sigma))\, d\sigma = h(0) + h(a) = -|h(0) + h(a)|.$$

As

$$\left| \int_0^2 h(q_0(\xi) - q_0(-\xi + \sigma))\, V(-\xi + \sigma)\, d\sigma \right| \leqq K_{13}\varrho,$$

one obtains from (33,3) that

$$S(\xi, \tau) - q_0(\xi) = V(\xi) - (\tau - \tilde{\tau}) \,.\, \tfrac{1}{4}|h(0) + h(a)|\, V(\xi) + Z_5(\xi, \tau),$$
$$|Z_5(\xi, \tau)| \leqq (\tau - \tilde{\tau}) K_{13}\varrho + (\tau - \tilde{\tau}) \,.\, \tfrac{1}{32}|h(0) + h(a)|(\mu_3 - \mu_2) + K_{12}(\tau - \tilde{\tau})^2.$$

As $|V(\xi)| \leqq \mu_3 - \frac{1}{2}a$, $\mu_3 - \mu_2 = 2(\mu_3 - \frac{1}{2}a) \leqq 2$, it follows that $|S(\xi, \tau) - q_0(\xi)| \leqq \leqq (\mu_3 - \frac{1}{2}a)\,[1 - (\tau - \tilde{\tau}) \,.\, \frac{1}{4}|h(0) + h(a)|] + (\tau - \tilde{\tau}) K_{13}\varrho + (\tau - \tilde{\tau}) \,.\, \frac{1}{8}|h(0) + + h(a)|(\mu_3 - \frac{1}{2}a) + K_{12}(\tau - \tilde{\tau})^2$. Hence $|S(\xi, \tau) - q_0(\xi)| \leqq \mu_3 - \frac{1}{2}a$ for $\tilde{\tau} \leqq \tau \leqq \tilde{\tau} + + \varepsilon$, $(0 < \varepsilon \leqq \varepsilon_0)$ if $\frac{1}{8}|h(0) + h(a)|(\mu_3 - \frac{1}{2}a) \geqq K_{13}\varrho + K_{12}\varepsilon_0$. The last inequality holds if ε_0 is sufficiently small and

$$\tfrac{1}{8}|h(0) + h(a)|(\mu_3 - \tfrac{1}{2}a) > K_{13}\varrho. \tag{34,3}$$

Lemma 9,3 is proved.

Lemma 10,3. *Let* $\tilde{S} \in M$ *be the same as in Lemma* 9,3. *Then there exists a solution* S *of* (6,2) *in* M *on* $\langle \tilde{\tau}, \infty)$, $S(\tilde{\tau}) = \tilde{S}$ *and* $\|S(\tau) - q_0\|_M \leqq \mu_3 - \frac{1}{2}a$, $\|S(\tau) - - q_0\|_{L_2} \leqq \exp(-2\varkappa_5(\tau - \tilde{\tau}))\, \|\tilde{S} - q_0\|_{L_2}$.

Proof. As q is on time independent solution of (6,2), it follows from Lemmas 8,3 and 9,3 that S exists on $\langle \tilde{\tau}, \tilde{\tau} + \varepsilon \rangle$ and fulfils $\|S(\tau) - q_0\|_M \leqq \mu_3 - \frac{1}{2}a$, $\|S(\tau) - - q_0\|_{L_2} \leqq \exp(-2\varkappa_5(\tau - \tilde{\tau}))\, \|\tilde{S} - q_0\|_{L_2}$ there. Then S and both inequalities are extended to $\langle \tilde{\tau}, \tilde{\tau} + 2\varepsilon \rangle$ and Lemma 10,3 follows by induction.

It follows from (29,3) and (30,3) that there exists a unique $\vartheta_1 \in \langle \frac{1}{2}, \frac{1}{2} \rangle$ such that $Y(\vartheta_1, \tilde{\tau} + \varepsilon) = 0$; obviously $\alpha_1 < \vartheta_1 < \beta_1$. Put $q_1(\xi) = q(\xi - \vartheta_1)$.

Lemma 11,3. $\|Y(\tilde{\tau} + \varepsilon) - q_1\|_{L_2} \leqq \varrho$, *y being as usually the solution of* (5,1) *or of* (7,1) *in* C^1 *on* $\langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$, $y(\tilde{\tau}) = \tilde{y} \in U(v, \varrho, \gamma)$, ϱ, γ *fulfilling* (18,3), (19,3), (24,3), (25,3), (28,3) *and* (34,3).

Proof. Define $\tilde{S}$ as follows: $\tilde{S}(\xi) = \tilde{Y}(\xi)$ if $\mu_2 \leqq |\tilde{Y}(\xi)| \leqq \mu_3$, $\tilde{S}(\xi) = \frac{1}{2}a$, if $0 \leqq \tilde{Y}(\xi) < \mu_2$, $\tilde{S}(\xi) = -\frac{1}{2}a$, if $-\mu_2 < \tilde{Y}(\xi) < 0$. It follows from Lemma 10,3 that there exists the solution S of (6,2) in M on $\langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$, $S(\tilde{\tau}) = \tilde{S}$ and that

$$(35,3) \qquad \|S(\tilde{\tau} + \varepsilon) - q_0\|_{L_2} \leqq \exp(-2\varkappa_5\varepsilon)\|\tilde{S} - q_0\|_{L_2}\,.$$

Let Y_0 be the solution of (6,2) in M on $\langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$, $Y_0(\tilde{\tau}) = \tilde{Y}$, $\tilde{Y} = I_\xi\tilde{y}$. Y_0 exists and $|Y_0(\tau)| \leqq R$, $\tau \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$, as in the case that y is the solution of (7,1), $Y_0(\tau) = Y(\tau)$ for $\tau \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$ (cf. Lemma 2,2). One verifies easily that $\|(\mathrm{d}/\mathrm{d}\tau)(Y_0(\tau) - S(\tau))\|_{L_2} \leqq K_{14}\|Y_0(\tau) - S(\tau)\|_{L_2}$. Hence

$$(36,3) \qquad \|Y_0(\tau) - S(\tau)\|_{L_2} \leqq (1 + K_{15}\varepsilon)\|\tilde{Y} - \tilde{S}\|_{L_2}\,, \quad \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + \varepsilon\,, \quad 0 < \varepsilon \leqq \varepsilon_0\,,$$

ε_0 being small enough (cf. (8,3)). If y is a solution of (5,1), then $\|y(\tilde{\tau} + \varepsilon) - y_0(\tilde{\tau} + \varepsilon)\|_{L_1} \leqq 2K_1\varepsilon^2$ and

$$(37,3) \qquad \|Y(\tilde{\tau} + \varepsilon) - Y_0(\tilde{\tau} + \varepsilon)\|_M \leqq 2K_1\varepsilon^2\,,$$

I_ξ being a bounded linear operation from L_1 to M, $\|I_\xi\| = 1$. Of course (36,3) is fulfilled, if y is a solution of (7,1) ($Y = Y_0$, as stated above). It follows from the definition of $\tilde{S}$ and from (12,3) that $\|\tilde{Y} - \tilde{S}\|_{L_2} + \|\tilde{S} - q_0\|_{L_2} = \|\tilde{Y} - q_0\|_{L_2} \leqq \varrho$. Hence and from (37,3), (36,3) and (35,3) one obtains that $\|Y(\tilde{\tau} + \varepsilon) - q_0\|_{L_2} \leqq 2K_1\varepsilon^2 + (1 + K_{15}\varepsilon)\|\tilde{Y} - \tilde{S}\|_{L_2} + \exp(-2\varkappa_5\varepsilon)(\varrho - \|\tilde{Y} - \tilde{S}\|_{L_2})$. Using $\|\tilde{Y} - \tilde{S}\|_{L_2} \leqq 4\mu_3(1 - \gamma)^{1/2}$ we obtain that

$$(38,3) \qquad \|Y(\tilde{\tau} + \varepsilon) - q_0\|_{L_2} \leqq$$
$$\leqq \varrho\exp(-2\varkappa_5\varepsilon) + (K_{15} + 2\varkappa_5)\,\varepsilon\,.\,4\mu_3(1 - \gamma)^{1/2} + 2K_1\varepsilon^2\,.$$

Assume that $\varrho(1 - \exp(2\varkappa_5\varepsilon_0)) \geqq 2K_1\varepsilon_0^2 + (K_{15} + 2\varkappa_5)(1 - \gamma)^{1/2}\,\varepsilon_0$ which is fulfilled for ε_0 sufficiently small, if

$$(39,3) \qquad 2\varkappa_5\varrho > (K_{15} + 2\varkappa_5)(1 - \gamma)^{1/2}\,.$$

Then it follows from (38,3) that $\|Y(\tilde{\tau} + \varepsilon) - q_0\|_{L_2} \leqq \varrho$ and Lemma 11,3 holds, as the definition of q_1 implies that $\|Y(\tilde{\tau} + \varepsilon) - q_1\|_{L_2} \leqq \|Y(\tilde{\tau} + \varepsilon) - q_0\|_{L_2}$.

Theorem 2,3. *Let* $v \geqq 2$, *let* ϱ *and* γ, $\varrho > 0$, $0 < \gamma < 1$ *fulfil* (18,3), (19,3), (24,3), (25,3), (28,3), (34,3), *and* (39,3), *let* $\tilde{y} \in U(v,\varrho,\gamma)$, $\tilde{\tau} \in E_1$, $0 < \varepsilon \leqq \varepsilon_0$, ε_0 *being sufficiently small. Then there exists the solution y of* (5,1) (*or of* (7,1)) *in* C^1 *on* $\langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$, $y(\tilde{\tau}) = y$ *and* $\|y(\tau)\|_{L_1} \leqq R + 1$ *for* $\tau \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$, *and* $y(\tilde{\tau} + \varepsilon) \in U(v(1 + \varepsilon\varkappa_4), \varrho, \gamma)$.

Proof. The existence of y and $\|y(\tau)\|_{L_1} \leqq R + 1$ *for* $\tau \in \langle\tilde{\tau}, \tilde{\tau} + \varepsilon\rangle$ follows from Lemmas 4,2 and 5,2. It was proved in Theorem 1,3 that $y(\tilde{\tau} + \varepsilon)$, $Y(\tilde{\tau} + \varepsilon)$ fulfil (6,3), (7,3), (8,3), (9,3), (10,3), (11,3). Lemma 3,2 reads that $Y(\tilde{\tau} + \varepsilon)$ fulfils (8,3) and $y(\tilde{\tau} + \varepsilon)$ fulfils (13,3) and (12,3) according to Lemmas 7,3 and 11,3. Theorem 2,3 is proved.

4. In this section Theorem 2,1 will be proved. Let $\tilde{y} \in U(v, \gamma, \varrho)$. According to Theorem 2,3 there exists the solution y of (5,1) or of (7,1) in C^1 on $\langle 0, \varepsilon\rangle$, $y(0) = \tilde{y}$ and $y(\varepsilon) \in U(v(1 + \varepsilon\varkappa_4), \gamma, \varrho)$. It follows by induction that the solution y of (5,1) or of (7,1) in C^1, $y(0) = \tilde{y}$ exists on $\langle 0, \infty)$ and $y(i\varepsilon) \in U(v(1 + \varepsilon\varkappa_4)^i, \gamma, \varrho)$, $i = 0, 1, 2, \ldots$. Put $\tilde{y}_i = y(i\varepsilon)$, $\tilde{Y}_i = Y(i\varepsilon)$. As $y_i \in U(v(1 + \varepsilon\varkappa_4)^i, \gamma, \varrho)$, there exist numbers α'_i, α_i, β_i β'_i, $-\frac{1}{2} < \alpha'_i < \alpha_i < \beta_i < \beta'_i < \frac{1}{2}$ such that (cf. (7,3), (29,3))

(1,4)
$$\begin{aligned}
-\mu_3 \leqq \tilde{Y}_i(\xi) &\leqq -\mu_1 \quad \text{for} \quad -\tfrac{1}{2} \leqq \xi \leqq \alpha'_i\,,\\
-\mu_1 \leqq \tilde{Y}_i(\xi) &\leqq -\mu_0 \quad \text{for} \quad \alpha'_i \leqq \xi \leqq \alpha_i\,,\\
-\mu_0 \leqq \tilde{Y}_i(\xi) &\leqq \mu_0 \quad \text{for} \quad \alpha_i \leqq \xi \leqq \beta_i\,,\\
\mu_0 \leqq \tilde{Y}_i(\xi) &\leqq \mu_1 \quad \text{for} \quad \beta_i \leqq \xi \leqq \beta'_i\,,\\
\mu_1 \leqq \tilde{Y}_i(\xi) &\leqq \mu_3 \quad \text{for} \quad \beta'_i \leqq \xi \leqq \tfrac{1}{2}\,, \quad i = 0, 1, 2, \ldots
\end{aligned}$$

and (cf. (9,3), (30,3))

(2,4)
$$\tilde{y}_i(\xi) \geqq v(1 + \varepsilon\varkappa_4)^i \quad \text{for} \quad \xi \in \langle\alpha_i, \beta_i\rangle\,, \quad i = 0, 1, 2, \ldots\,.$$

Hence

(3,4)
$$\beta_i - \alpha_i \leqq 2\mu_0 v^{-1}(1 + \varepsilon\varkappa_4)^{-i}\,, \quad i = 0, 1, 2, \ldots$$

and (29,3) implies that

(4,4)
$$\alpha_i < \alpha_{i+1} < \beta_{i+1} < \beta_i\,, \quad \alpha'_i < \alpha'_{i+1} < \beta'_{i+1} < \beta'_i\,, \quad i = 0, 1, 2, \ldots\,.$$

Therefore then exists a number ϑ,

(5,4)
$$\vartheta = \lim_{i\to\infty} \alpha_i = \lim_{i\to\infty} \beta_i\,, \quad \alpha_i < \vartheta < \beta_i\,, \quad i = 0, 1, 2, \ldots$$

Put $\varkappa_6 = 8(\mu_2 - \mu_0)\,\varkappa_1^{-1}$. It follows from Lemma 4,3 that

(6,4)
$$\begin{aligned}
\tilde{Y}_i(\xi) &\leqq -\mu_2 \quad \text{for} \quad \xi \in \langle -\tfrac{1}{2}, \alpha_{i-k}\rangle\,, \quad i \geqq k \geqq \varkappa_6/\varepsilon\,,\\
\tilde{Y}_i(\xi) &\geqq \mu_2 \quad \text{for} \quad \xi \in \langle\beta_{i-k}, \tfrac{1}{2}\rangle\,, \quad i \geqq k \geqq \varkappa_6/\varepsilon\,.
\end{aligned}$$

The decisive step for the proof of Theorem 2,1 is in proving that Y_i is a Cauchy--sequence (in a sense which will be precised later). For this purpose choose an $\eta > 0$ (arbitrarily small $\eta < \frac{1}{2} - |\vartheta|$) and put $J(\eta) = \bigcup_{l=-\infty}^{\infty} \langle\vartheta + l + \eta,\ \vartheta + l + 1 - \eta\rangle$,

$J_c(\eta) = E_1 - J(\eta)$ and define $\tilde{S}_i \in \hat{M}$ by

$$\tilde{S}_i(\xi) = \tilde{Y}_i(\xi) \quad \text{for} \quad \xi \in J(\eta),$$
$$\tilde{S}_i(\xi) = -\tfrac{1}{2}a \quad \text{for} \quad \xi \in (\vartheta - \eta, \vartheta),$$
$$\tilde{S}_i(\xi) = \tfrac{1}{2}a \quad \text{for} \quad \xi \in \langle \vartheta, \vartheta + \eta),$$

and $q' \in M$ by $q'(\xi) = q(\xi - \vartheta)$ for $\xi \in E_1$. Let $\| \ \|_\eta$ be the quasinorm in M defined by $\|Y\|_\eta = \sup_\xi |Y(\xi)|$, ξ being restricted to $\langle \vartheta - 1 + \eta, \vartheta - \eta \rangle \cup \langle \vartheta + \eta, \vartheta + 1 - \eta \rangle$, $Y \in M$. From $|\tilde{Y}_i(\xi)| \leqq \mu_3$ for $\xi \in E_1$ it follows that

$$\|\tilde{Y}_i - \tilde{S}_i\|_{L_2} \leqq 4\mu_3 \eta^{1/2} . \tag{7,4}$$

Let S_i be the solution of (6,2) in M, $S_i(i\varepsilon) = \tilde{S}_i$. Put $N_1 = N_1(\eta) = 2\mu_0 \nu^{-1} \varkappa_4^{-1} \eta^{-1} + \varkappa_6 + 2$.

Lemma 1,4. *Let $i < N_1/\varepsilon$, $0 < \varepsilon \leqq \varepsilon_0 \leqq 1$. Then*

(i) $\tilde{Y}_i(\xi) \leqq -\mu_2$ for $\xi \in \langle -\frac{1}{2}, \vartheta - \eta \rangle$, $\tilde{Y}_i(\xi) \geqq \mu_2$ for $\xi \in \langle \vartheta + \eta, \frac{1}{2} \rangle$,

(ii) *S_i exists on $\langle i\varepsilon, \infty \rangle$ and $\|S_i(\tau) - q'\|_M \leqq \mu_3 - \frac{1}{2}a$ for $\tau \in \langle i\varepsilon, \infty)$.*

Proof. Let $\alpha, \beta \geqq 0$; $(1 + \alpha)^{\beta/\alpha}$ is nonincreasing in α, hence

$$(1 + \gamma)^\delta \geqq 1 + \gamma\delta \quad \text{for} \quad \gamma \geqq 0, \quad \delta \geqq 1 . \tag{8,4}$$

Let $k \geqq \varkappa_6/\varepsilon$, k being the least integer,

$$2\mu_0 \nu^{-1}(1 + \varepsilon\varkappa_4)^{-i+k} \leqq 2\mu_0 \nu^{-1}\left((1 + \varepsilon\varkappa_4)^{(2\mu_0\nu^{-1}\varkappa_4^{-1}\eta^{-1}+1)\varepsilon^{-1}}\right)^{-1} \leqq \eta .$$

Hence according to (6,4), (3,4) and (5,4) $\tilde{Y}_i(\xi) \leqq -\mu_2$ for $\xi \in \langle -\frac{1}{2}, \vartheta - \eta \rangle$, $\tilde{Y}_i(\xi) \geqq \mu_2$ for $\xi \in \langle \vartheta + \eta, \frac{1}{2} \rangle$. (i) is proved. As $Y_i \in \hat{M}$, $\|\tilde{Y}_i\|_M \leqq \mu_3$, $\mu_3 + \mu_2 = a$, it follows that $\|\tilde{S}_i - q'\|_M \leqq \mu_3 - \frac{1}{2}a$ and Lemma 1,4 follows by induction from Lemma 9,3.

Let v_i be the solution of (7,1) in M, $v_i(i\varepsilon) = \tilde{y}_i$, $i = 0, 1, 2, \ldots$, $V_i(\tau) = I_\xi\, v_i(\tau)$.

Lemma 2,4. (i) *v_i exists on $(i\varepsilon, \infty)$, $v_i(j\varepsilon) \in U(\nu(1 + \varepsilon\varkappa_4)^j, \gamma, \varrho)$,*

(ii) *if $0 \leqq i \leqq j$, $j \geqq N_1/\varepsilon$, (i, j being integers), then*

$$V_i(\xi, j\varepsilon) \leqq -\mu_2 \quad \textit{for} \quad \xi \in \langle -\tfrac{1}{2}, \vartheta - \eta \rangle ,$$
$$V_i(\xi, j\varepsilon) \geqq \mu_2 \quad \textit{for} \quad \xi \in \langle \vartheta + \eta, \tfrac{1}{2} \rangle .$$

Proof. It was stated at the beginning of this section that $\tilde{y}_i \in U(\nu(1 + \varepsilon\varkappa_4)^i, \gamma, \varrho)$ and (i) follows from Theorem 2,3 by induction. (ii) may be proved in an analogous manner as (i) of Lemma 1,4, as Theorem 2,3 holds for solutions (5,1) and (7,1) as well.

Lemma 3,4.

$$(9,4) \qquad \|S_i(i\varepsilon + \sigma) - S_j(j\varepsilon + \sigma)\|_{L_2} \leqq \exp(-2\varkappa_5\sigma) \|\tilde{S}_i - \tilde{S}_j\|_{L_2}, \quad \sigma \geqq 0, \quad i, j \geqq N_1/\varepsilon.$$

Proof. As (6,2) is autonomous, Lemma 3,4 follows from Lemmas 8,3, 1.4.

Lemma 4,4.

$$(10,4) \qquad \|S_i(i\varepsilon + \tau) - S_j(j\varepsilon + \tau)\|_M \leqq \leqq K_{19} \exp(-\varkappa_5\tau) \|\tilde{S}_i - \tilde{S}_j\|_M \quad \text{for} \quad \tau \geqq 0, \quad i, j \geqq N_1/\varepsilon.$$

Proof. S_i, S_j are solutions of (6,2) (in M), which is an autonomous equation and the value at a fixed ξ is a continuous functional on M; hence

$$(11,4) \qquad \frac{\partial S_j}{\partial \tau}(\xi, j\varepsilon + \tau) - \frac{\partial S_i}{\partial \tau}(\xi, i\varepsilon + \tau) =$$

$$= \frac{1}{4}\int_0^2 [H(S_j(\xi, j\varepsilon + \tau) - S_j(-\xi + 2\sigma, j\varepsilon + \tau)) - H(S_i(\xi, i\varepsilon + \tau) - S_i(-\xi + 2\sigma, i\varepsilon + \tau))] \,d\sigma = \tfrac{1}{4}(h(0) + h(a)) [S_j(\xi, j\varepsilon + \tau) - S_i(\xi, i\varepsilon + \tau)] + Z(\xi, \tau),$$

$$Z(\xi, \tau) = \frac{1}{4}\int_{\xi+\vartheta}^{\xi+\vartheta+2} \{H(S_j(\xi, j\varepsilon + \tau) - S_j(-\xi + \sigma, j\varepsilon + \tau) - H(S_i(\xi, i\varepsilon + \tau) - S_i(-\xi + \sigma, i\varepsilon + \tau)) - b(\sigma) [S_j(\xi, j\varepsilon + \tau) - S_j(-\xi + \sigma, j\varepsilon + \tau) - S_i(\xi, i\varepsilon + \tau) + S_i(-\xi + \sigma, i\varepsilon + \tau)]\} \,d\sigma -$$

$$- \frac{1}{4}\int_{\xi+\vartheta}^{\xi+\vartheta+2} b(\sigma) [S_j(-\xi + \sigma, j\varepsilon + \tau) - S_i(-\xi + \sigma, i\varepsilon + \tau)] \,d\sigma = B_1 + B_2.$$

$b(\sigma) = h(0)$ for $\xi + \vartheta \leqq \sigma < \xi + \vartheta + 1$, $b(\sigma) = h(a)$ for $\xi + \vartheta + 1 \leqq \sigma < \xi + \vartheta + 2$. As $|S_i(\xi, i\varepsilon + \tau) - q`(\xi)| \leqq \mu_3 - \frac{1}{2}a$, $|S_j(\xi, j\varepsilon + \tau) - q`(\xi)| \leqq \mu_3 - \frac{1}{2}a$ (cf. Lemma 9,3), it follows from (1,3) that (cf. Lemma 8,3 and $\|\tilde{S}_j - \tilde{S}_i\|_{L_2} \leqq \sqrt{(2)} \|\tilde{S}_j - \tilde{S}_i\|_M$)

$$|B_1| \leqq \tfrac{1}{4} \cdot \tfrac{1}{16} |h(0) + h(a)| \int_{\xi+\vartheta}^{\xi+\vartheta+2} [|S_j(\xi, j\varepsilon + \tau) - S_i(\xi, i\varepsilon + \tau)| + |S_j(-\xi + \sigma, j\varepsilon + \tau) - S_i(-\xi + \sigma, j\varepsilon + \tau)|] \,d\sigma \leqq \tfrac{1}{4} \cdot \tfrac{1}{8} |h(0) + h(a)| \,.$$

$$. \,|S_j(\xi, j\varepsilon + \tau) - S_i(\xi, i\varepsilon + \tau)| + K_{16} \|S_j(j\varepsilon + \tau) - S_i(i\varepsilon + \tau)\|_{L_2} \leqq$$

$$\leqq \tfrac{1}{32} |h(0) + h(a)| \,.\, |S_j(\xi, j\varepsilon + \tau) - S_i(\xi, i\varepsilon + \tau)| + \sqrt{(2)}\, K_{16} \exp(-2\varkappa_5\tau) \|\tilde{S}_j - \tilde{S}_i\|_M,$$

$|B_2| \leqq K_{17} \|S_j(j\varepsilon + \tau) - S_i(i\varepsilon + \tau)\|_{L_2} \leqq \sqrt{(2)} K_{17} \exp(-2\varkappa_5 \tau) \|\tilde{S}_i - \tilde{S}_j\|_M$; therefore $|Z(\xi, \tau)| \leqq \frac{1}{32} |h(0) + h(a)| \,.\, |S_j(\xi, j\varepsilon + \tau) - S_i(\xi, i\varepsilon + \tau)| + K_{18} \exp(-2\varkappa_5\tau) \,.\, \|\tilde{S}_i - \tilde{S}_j\|_M = A$. Z is continuous in τ (ξ being fixed). Put

$$d(\tau) = \tfrac{1}{4} \,.\, \tfrac{1}{8} |h(0) + h(a)| \, Z(\xi, \tau) \operatorname{sgn}(S_j(\xi, j\varepsilon + \tau) - S_i(\xi, i\varepsilon + \tau)) \, A^{-1}$$
$$c(\tau) = K_{18} \exp(-2\varkappa_5 \tau) \|\tilde{S}_j - \tilde{S}_i\|_M \, Z(\xi, \tau) \, A^{-1}.$$

Then $Z(\xi, \tau) = d(\tau)(S_j(\xi, j\varepsilon + \tau) - S_i(\xi, i\varepsilon + \tau)) + c(\tau)$,

$$(12,4) \qquad |d(\tau)| \leqq \tfrac{1}{8} \,.\, \tfrac{1}{4} |h(0) + h(a)|\,, \quad |c(\tau)| \leqq K_{18} \exp(-2\varkappa_5\tau) \|\tilde{S}_j - \tilde{S}_i\|_M.$$

Instead of (11,4) we have

$$\frac{\partial S_j}{\partial \tau}(\xi, j\varepsilon + \tau) - \frac{\partial S_i}{\partial \tau}(\xi, i\varepsilon + \tau) =$$
$$= [\tfrac{1}{4}(h(0) + h(a)) + d(\tau)] (S_j(\xi, j\varepsilon + \tau) - S_i(\xi, i\varepsilon + \tau)) + c(\tau).$$

The variation of constants formula gives

$$(13,4) \qquad S_j(\xi, j\varepsilon + \tau) - S_i(\xi, i\varepsilon + \tau)) =$$
$$= (\tilde{S}_j(\xi) - \tilde{S}_i(\xi)) \exp \int_0^\tau [\tfrac{1}{4}(h(0) + h(a)) + d(\sigma)] \, d\sigma +$$
$$+ \int_0^\tau \exp \left\{ \int_0^\tau [\tfrac{1}{4}(h(0) + h(a)) + d(\sigma_1)] \, d\sigma_1 \right\} c(\sigma) \, d\sigma \,.$$

(13,4) together with (12,4) imply

$$|S_j(\xi, j\varepsilon + \tau) - S_i(\xi, i\varepsilon + \tau)| \leqq \|\tilde{S}_j - \tilde{S}_i\|_M \exp\{\tfrac{1}{8}(h(0) + h(a))\tau\} +$$
$$+ \|\tilde{S}_j - \tilde{S}_i\|_M K_{18} \int_0^\tau \exp\{-2\varkappa_5\sigma + \tfrac{1}{8}(h(0) + h(a))(\tau - \sigma)\} \, d\sigma \leqq$$
$$\leqq \|\tilde{S}_j - \tilde{S}_i\|_M K_{19} \exp(-\varkappa_5\tau) \,.$$

Lemma 4,4 is proved.

Let v_i, V_i be the same as in Lemma 2.4.

Lemma 5,4. *V_i exists on $\langle i\varepsilon, \infty)$, $\|V_i(\tau)\|_M \leqq R + 1$ for $\tau \in \langle i\varepsilon, \infty)$ and*

$$(14,4) \qquad \|V_i(i\varepsilon + \sigma) - S_i(i\varepsilon + \sigma)\|_{L_2} \leqq \|\tilde{Y}_i - \tilde{S}_i\|_{L_2} \exp(K_{20}\sigma)\,,$$
$$\sigma \geqq 0\,, \quad i \geqq N_1/\varepsilon\,.$$

Proof. It follows from Theorems 1,3 and 2,3 that V_i exists on $\langle j\varepsilon, (j + 1)\varepsilon\rangle$ and that $\|v_i(j\varepsilon)\|_{L_1} \leqq R$, $j = i, i + 1, i + 2, \ldots$, according to (10,2) $\|v_i(\tau)\|_{L_1} \leqq R + 1$, $\tau \in \langle i\varepsilon, \infty)$; hence $\|V_i(\tau)\|_M \leqq R + 1$, $\tau \in \langle i\varepsilon, \infty)$. As V_i is a solution of (6,2) (in M), and $\|V_i(\tau)\|_M$ is bounded, it may be verified that $(d/d\sigma)\,\|V_i(i\,\varepsilon + \sigma) - S_i(i\varepsilon + \sigma)\|_{L_2}^2 \leqq$ $\leqq 2K_{20}\|V_i(i\varepsilon + \sigma) - S_i(i\varepsilon + \sigma)\|_{L_2}^2$ and (14,4) holds.

Lemma 6,4.

(15,4) $\|V_i(i\varepsilon + \tau) - S_i(i\varepsilon + \tau)\|_\eta \leqq \|\tilde{Y}_i - \tilde{S}_i\|_{L_2} K_{22} \exp(K_{22}\tau)\,, \quad i \geqq N_1/\varepsilon\,.$

Proof. As S_i and V_i are solutions of (6,2) in M (cf. Lemma 2,2), S_i, V_i, $\partial S_i/\partial\tau$, $\partial V_i/\partial\tau$ are continuous in τ (ξ being fixed) and

$$\frac{\partial V_i}{\partial\tau}(\xi, i\varepsilon + \tau) - \frac{\partial S_i}{\partial\tau}(\xi_i\, i\varepsilon + \tau) = \frac{1}{4}\int_0^2 [H(V_i(\xi, \tau) - V_i(-\xi + 2\sigma, \tau)) -$$
$$- H(S_i(\xi, \tau) - S_i(-\xi + 2\sigma, \tau))]\, d\sigma\,.$$

Hence

$$\left|\frac{\partial V_i}{\partial\tau}(\xi, i\varepsilon + \tau) - \frac{\partial S_i}{\partial\tau}(\xi, i\varepsilon + \tau)\right| \leqq$$
$$\leqq K_{21}|V_i(\xi, \tau) - S_i(\xi, \tau)| + K_{21}\int_0^2 |V_i(-\xi + 2\sigma, \tau) - S_i(-\xi + 2\sigma, \tau)|\, d\sigma \leqq$$
$$\leqq K_{21}|V_i(\xi, \tau) - S_i(\xi, \tau)| + \sqrt{(2)}\, K_{21} \exp(K_{20}\tau)\, \|\tilde{Y}_i - \tilde{S}_i\|_{L_2}\,.$$

Remember that $V_i(\xi_1, i\varepsilon) = S_i(\xi_1, i\varepsilon)$ if $\xi_1 \in \langle\vartheta - 1 + \eta, \vartheta - \eta\rangle \cup \langle\vartheta + \eta, \vartheta +$ $+ 1 - \eta\rangle$. Therefore

$$|V_i(\xi_1, i\varepsilon + \tau) - S_i(\xi_1, i\varepsilon + \tau)| \leqq \|\tilde{Y}_i - \tilde{S}_i\|_{L_2} K_{22} \exp(K_{22}\tau)$$

for $\tau \geqq 0$, and Lemma 6,4 holds.

It was proved in Lemma 2,2 that Y is a solution of (6,2) (in M), if y is a solution of (7,1) (in C^1) and $Y(\tau) = I_\xi\, y(\tau)$. A similar result holds for the solution of (5,1), if the concept of generalized differential equation (cf. [1], section 1) is used. For $y \in C^1$, $\tau \in E_1$, $\varepsilon > 0$ put $F(y, \tau, \varepsilon) = \int_0^\tau f(y, \sigma, \tau)\, d\sigma$, f being defined in (6,1), i.e. (cf. (3,2))

(16,4) $$F(y, \tau, \varepsilon)(\xi) =$$
$$= \frac{1}{2}\int_0^\tau h(Y(\xi) - Y(-\xi + 2\sigma/\varepsilon)(y(\xi) + y(-\xi + 2\sigma/\varepsilon) - 2y(-\xi + 2\sigma/\varepsilon))\, d\sigma =$$
$$= \frac{1}{2}\int_0^\tau h(Y(\xi) - Y(-\xi + 2\sigma/\varepsilon))(y(\xi) + y(-\xi + 2\sigma/\varepsilon))\, d\sigma -$$
$$- \tfrac{1}{2}\varepsilon H(Y(\xi) - Y(-\xi + 2\tau/\varepsilon))\,,$$

(17,4) $$F_0(y, \tau) = f_0(y)\,\tau\,, \quad f_0 \text{ being defined by (8,1).}$$

It was proved in [1], section 1 that every solution of (3,1), [1] is simultaneously a solution of (52,1), [1]. Hence

Lemma 7,4. *If y is a solution of (5,1) ((7,1)) (in C^1) on $\langle \tau_1, \tau_2 \rangle$, then y is simultaneously a solution of*

$$\frac{dy}{d\tau} = D_\tau F(y, \tau, \varepsilon), \tag{18,4}$$

$$\frac{dy}{d\tau} = D_\tau F_0(y, \tau) \tag{19,4}$$

in C^1 on $\langle \tau_1, \tau_2 \rangle$.

For $y \in C^1$, $\tau \in E_1$ and $\varepsilon > 0$ put $F^*(y, \tau, \varepsilon) = I_\xi F(y, \tau, \varepsilon)$. It follows from (16,4) that

$$F^*(y, \tau, \varepsilon)(\xi) = \frac{1}{2}\int_0^\tau H(Y(\xi) - Y(-\xi + 2\sigma/\varepsilon))\, d\sigma - \tfrac{1}{2}\varepsilon I_\xi H(Y(\xi) - Y(-\xi + 2\tau/\varepsilon)).$$

The right hand side of the last formula contains Y and not y; define for $Y \in M$

$$F_1^*(Y, \tau, \varepsilon)(\xi) = \frac{1}{2}\int_0^\tau H(Y(\xi) - Y(-\xi + 2\sigma/\varepsilon))\, d\sigma, \tag{20,4}$$

$$F_2^*(Y, \tau, \varepsilon)(\xi) = \tfrac{1}{2}\varepsilon I_\xi H(Y(\xi) - Y(-\xi + 2\tau/\varepsilon)),$$

$$F^*(Y, \tau, \varepsilon)(\xi) = F_1^*(Y, \tau, \varepsilon)(\xi) - F_2^*(Y, \tau, \varepsilon)(\xi).$$

As H is odd and $Y(\xi') - Y(-\xi' + 2\sigma/\varepsilon) = -Y(\xi'') + Y(-\xi'' + 2\sigma/\varepsilon)$ if $\xi'' = -\xi' + 2\sigma/\varepsilon$, it follows that $F_1^*(., \tau, \varepsilon)$, $F_2^*(., \tau, \varepsilon)$, $F^*(., \tau, \varepsilon)$ are maps from M into itself.

Remember that a map Y from $\langle \tau_1, \tau_2 \rangle$ to M is said to be a solution of the generalized differential equation

$$\frac{dY}{d\tau} = D_\tau F^*(Y, \tau, \varepsilon) \tag{21,4}$$

in M, if $Y(\tau) \in M$ for $\tau \in \langle \tau_1, \tau_2 \rangle$, if $\int_{\tau_3}^{\tau_4} D_\sigma F^*(Y(\tau), \sigma, \varepsilon)$ exists (with respect to the norm $\| \ \|_M$) and if $Y(\tau_4) = Y(\tau_3) + \int_{\tau_3}^{\tau_4} D_\sigma F^*(Y(\tau), \sigma, \varepsilon)$ for $\tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle$. Put

$$F_0^*(Y, \tau) = \tau f_0^*(Y), \tag{22,4}$$

f_0^* being defined in (5,2), i.e.

$$F_0^*(Y, \tau)(\xi) = \frac{1}{4}\tau \cdot \int_0^2 H(Y(\xi) - Y(-\xi + 2\sigma))\, d\sigma.$$

It was proved in [1], section 1, that every solution of (3,1), is simultaneously a solution of (5,1), [1]; therefore V_i (V_i are solutions of (6,2) (cf. Lemma 2,2)) are solutions of

$$\frac{\mathrm{d}Y}{\mathrm{d}\tau} = \mathrm{D}_\tau F_0^*(Y, \tau)\,. \tag{23,4}$$

The following Lemma is a consequence of Lemma 6,4 and of the fact that I_ξ is a bounded linear map of C^1 into M.

Lemma 8,4. *Let y be a solution of* (5,1) ((7,1)) *in C^1 on $\langle \tau_1, \tau_2 \rangle$, $Y(\tau) = I_\xi\, y(\tau)$. Then Y is a solution of* (21,4) ((23,4)) *in M on $\langle \tau_1, \tau_2 \rangle$.*

Let $G^* \subset M$ be the set of such W, that $w = (\mathrm{d}/\mathrm{d}\xi)\, W$ exists and is continuous and that $\|w\|_{L_1} = \int_0^2 |w(\xi)|\, \mathrm{d}\xi \leqq R + 1 = 2a + 13$ (cf. Lemma 4,3). Obviously $\|W\|_M \leqq \leqq R + 1$ for $W \in G^*$.

Lemma 9,4. *$Y(\tau)$, $V_i(i\varepsilon + \tau) \in G^*$ for $\tau \geqq 0$, $0 < \varepsilon \leqq \varepsilon_0$ (ε_0 being small enough).*

Proof. It follows from Theorem 2,3 that $\|y(j\varepsilon)\|_{L_1}$, $\|v_i((i + j)\, \varepsilon)\| \leqq R$, $i, j = = 0, 1, 2, \ldots$. (10,2) implies that $\|y(j\varepsilon + \tau)\|_{L_1}$, $\|v_i((i + j)\, \varepsilon + \tau)\|_{L_1} \leqq R + 1$ for $0 \leqq \tau \leqq \varepsilon$. Lemma 9,4 holds.

Lemma 10,4. *For every $\zeta > 0$ there exists such an $\varepsilon_0 > 0$ that $\|\int_{\tau_1}^{\tau_2} [f(y, \tau, \varepsilon) - - f_0(y)]\, \mathrm{d}\tau\|_{L_1} \leqq \zeta$, if $y \in G^*$, $0 < \varepsilon \leqq \varepsilon_0$, $\tau_1, \tau_2 \in E_1$.*

Proof. As $\int_{\tau_1}^{\tau_1 + j\varepsilon} [f(y, \tau, \varepsilon) - f_0(y)]\, \mathrm{d}\tau = 0$, $j = \ldots -1, 0, 1, \ldots$, it may be supposed that $0 < \tau_2 - \tau_1 < \varepsilon$. In this case one verifies easily that $\|\int_{\tau_1}^{\tau_2} [f(y, \tau, \varepsilon) - - f_0(y)]\, \mathrm{d}\tau\|_{L_1} \leqq K_{23}\varepsilon$ and Lemma 10,4 holds.

Lemma 11,4. *There exists a function $\chi_1(\varepsilon, \mathrm{T})$ defined for $\varepsilon > 0$, $\mathrm{T} > 0$, non-decreasing in ε, that $\chi_1(\varepsilon, T) \to 0$ with $\varepsilon \to 0$ and that $\|y(i\varepsilon + \tau) - v_i(i\varepsilon + \tau)\|_{L_1} \leqq \leqq \chi_1(\varepsilon, T)$ for $i = 0, 1, 2, \ldots$, $0 \leqq \tau \leqq T$, $0 < \varepsilon \leqq \varepsilon_0$.*

Proof. v_i being a solution of (7,1) in C^1 is simultaneously a solution of (7,1) in L_1 and similarly y_i is a solution of (5,1) in L_1. According to Lemma 9,4 $y(\tau)$, $v_i(i\varepsilon + \tau) \in \in G^*$ for $\tau \geqq 0$. From the special form of f and f_0 (cf. (6,1) and (8,1)) and from $\|Y\|_M \leqq \frac{1}{2}\|y\|_{L_1}$ (cf. (4,1)) it may be verified that there exists a $K_{24} > 0$ such that $\|f(y, \tau, \varepsilon)\|_{L_1} \leqq K_{24}$, $\|f(y + u, \tau, \varepsilon) - f(y, \tau, \varepsilon)\|_{L_1} \leqq K_{24}\|u\|_{L_1}$ and $\|f(y + u + + v, \tau, \varepsilon) - f(y + u, \tau, \varepsilon) - f(y + v, \tau, \varepsilon) + f(y, \tau, \varepsilon)\|_{L_1} \leqq K_{24}\|u\|_{L_1}\, \|v\|_{L_1}$ if y, $y + + u$, $y + v$, $y + u + v \in G^*$, $\tau \in E_1$, $\varepsilon > 0$ and that the same inequalities hold, if f is replaced by f_0. Lemma 11,4 follows from Lemma 8,4 and Theorem 1,1 [1].

Let us fix a $T > 1$ that

$$(1 + K_{19}) \exp\left(-\tfrac{1}{2}\varkappa_5 T\right) < \tfrac{1}{2}\,. \tag{24,4}$$

The principal part in the proof that $\lim_{i\to\infty} Y(\xi, \tau + i\varepsilon)$ exists for $\xi \neq \vartheta + j$, $j = \dots -1, 0, 1, \dots$ as it was stated in Theorem 2,1 is played by an estimate for

$$(23',4) \qquad \|Y(j\varepsilon + \tau) - Y(i\varepsilon + \tau) - V_j(j\varepsilon + \tau) + V_i(i\varepsilon + \tau)\|_\eta, \quad \tau \in \langle 0, T\rangle .$$

(cf. Lemma 17,4). According to Lemma 7,4 $Y(\tau)$ is a solution of (21,4), $V_i(\tau)$, $V_j(\tau)$ are solutions of (23,4) in M. As $f(y, \tau, \varepsilon)$ is periodic in τ with the period ε, and f_0 is independent on τ, Y_i and Y_j defined by $Y_i(\tau) = Y(i\varepsilon + \tau)$, $Y_j(\tau) = Y(j\varepsilon + \tau)$ are solutions of (21,4) and W_i, W_j defined by $W_i(\tau) = V_i(i\varepsilon + \tau)$, $W_j(\tau) = V_j(j\varepsilon + \tau)$ are solutions of (23,4) in M, $Y_i(0) = \tilde{Y}_i = W_i(0)$, $Y_j(0) = \tilde{Y}_j = W_j(0)$. The required estimate is based on the following formula

$$(25,4) \qquad Y_j(t) - Y_i(t) - W_j(t) + W_i(t) =$$

$$= \int_0^t \mathrm{D}_\sigma[F^*(Y_j(\tau); \sigma, \varepsilon) - F^*(W_j(\tau) - W_i(\tau) + Y_i(\tau), \sigma, \varepsilon)] +$$

$$+ \int_0^t \mathrm{D}_\sigma[F^*(W_j(\tau) - W_i(\tau) + Y_i(\tau), \sigma, \varepsilon) - F^*(Y_i(\tau)\, \sigma, (\varepsilon) -$$

$$- F^*(W_j(\tau), \sigma, \varepsilon) + F^*(W_i(\tau), \sigma, \varepsilon)] +$$

$$+ \int_0^t \mathrm{D}_\sigma[F^*(W_j(\tau), \sigma, \varepsilon) - F^*(W_i(\tau), \sigma, \varepsilon) - F_0^*(W_j(\tau), \sigma, \varepsilon) + F_0^*(W_i(\tau), \sigma, \varepsilon)] .$$

Let $|h(\xi)| \leqq K_{25}$ for $|\xi| \leqq 2\mu_3$.

Lemma 12,4. *Let* $\|\tilde{y}\|_M \leqq D$, $0 < \varepsilon \leqq \varepsilon_0 \leqq 1$, $4\varepsilon_0 K_2 \leqq 1$. *Put* $N_2 = 8K_{25}D\varkappa_2^{-1} + N_1 + 2$. *Then*

$$(26,4) \qquad |y(\xi, \tau)| \leqq 2 \quad \text{if} \quad \xi \in J(\eta), \quad \tau \geqq N_2 ,$$

$$(27,4) \qquad |v_i(\xi, i\varepsilon + \tau)| \leqq 2 \quad \text{if} \quad \xi \in J(\eta), \quad i\varepsilon + \tau \geqq N_2 .$$

Proof. As y is a solution of $\mathrm{d}b/\mathrm{d}\tau = g(b, \tau)$ in M, g being defined by $g(b, \tau)(\xi) = \frac{1}{2}h(Y(\xi, \tau) - Y(-\xi + 2\tau/\varepsilon, \tau))(b(\xi) - b(-\xi + 2\tau/\varepsilon))$ and as $\|g(b, \tau)\|_M \leqq K_{25}\|b\|_M$, it follows that $\|y(\tau)\|_M \leqq D\exp(K_{25}\tau)$. Let i_1 be the least integer such that $i_1\varepsilon \geqq N_1$; then $\|y(i\varepsilon)\| \leqq D\exp(K_{25}(N_1 + 1))$. It follows from Lemma 1,4, (i) that $\mu_2 \leqq |Y(\xi, i_1\varepsilon)| \leqq \mu_3$ if $\xi \in J(\eta)$; according to (29,3) $\mu_2 \leqq |Y(\xi, j\varepsilon)| \leqq \mu_3$, if $\xi \in J(\eta)$, $j = i_1, i_1 + 1, \dots$ and Lemma 5,3, (ii) implies that

$$(28,4) \qquad |y(\xi, j\varepsilon)| \leqq \max(D\exp(K_{25}(N_1 + 1))(1 - \tfrac{1}{8}\varkappa_2\varepsilon)^{j-i_1}, 1) ,$$

$$j = i_1, i_1 + 1, \dots .$$

If $j\varepsilon \geqq N_2 - 1$, then $j - i_1 \geqq \exp(K_{25}(N_1 + 1)\, D \,.\, 8\varkappa_2^{-1}\varepsilon^{-1}$ and (cf. (7,4)) $(1 - \frac{1}{8}\varkappa_2\varepsilon)^{j-i_1} \leqq [(1 + \frac{1}{8}\varkappa_2\varepsilon)\exp(K_{25}(N_1 + 1))\, D \,.\, 8\varkappa_2^{-1}\varepsilon^{-1}]^{-1} \leqq [1 + \exp(K_{25}(N_1 +$

$+ 1)) D]^{-1}$. Hence and from (28,4) $|y(\xi, j\varepsilon)| \leqq 1$ for $j = i_1, i_1 + 1, \ldots$, $\xi \in J(\eta)$. If $\xi \in J(\eta)$, $j\varepsilon \leqq \tau < (j + 1)\varepsilon$, $j = i_1, i_1 + 1, \ldots$, then $|y(\xi, \tau)| \leqq 1 + 4\varepsilon K_2 \leqq 2$ and (26,4) is fulfilled. (27,4) may be proved in an analogous manner, as all necessary Lemmas and Theorems 1,3 and 2,3 are valid for solutions of (5,1) and (7,1) as well.

Lemma 13,4.

$$(29,4) \qquad \left\| \int_0^\tau \mathrm{D}_\sigma [F_\alpha^*(W_j(\tau) - W_i(\tau) + Y_i(\tau), \sigma, \varepsilon) - F_\alpha^*(Y_i(\tau), \sigma, \varepsilon) - - F_\alpha^*(W_j(\tau), \sigma, \varepsilon) + F_\alpha^*(W_i(\tau)\, \sigma, \varepsilon)] \right\|_\eta \leqq$$

$$\leqq K_{33}(\eta + \|\tilde{Y}_i - \tilde{Y}_j\|_\eta \, \chi_1(\varepsilon, T)), \quad \alpha = 1, 2, \quad \varepsilon_i, \varepsilon_j \geqq N_2 .$$

Proof. Put $\tau_m = tm/n$, $m = 0, 1, \ldots, n$. According to the definition of the integral $\int_{\alpha_1}^{\beta_1} \mathrm{D}_\sigma F(v(\tau), \sigma)$ (cf. [1], section 1) the integral on the right hand side of (29,4) is the limit in M for $n \to \infty$ of the expressions

$$(30,4) \qquad \sum_{m=0}^{n-1} [F_\alpha^*(W_j(\tau_m) - W_i(\tau_m) + Y_i(\tau_m), \tau_{m+1}, \varepsilon) - - F_\alpha^*(Y_i(\tau_m), \tau_{m+1}, \varepsilon) - F_\alpha^*(W_j(\tau_m), \tau_{m+1}, \varepsilon) + + F_\alpha^*(W_i(\tau_m), \tau_{m+1}, \varepsilon) - F_\alpha^*(W_j(\tau_m) - W_i(\tau_m) + Y_i(\tau_m), \tau_m, \varepsilon) + + F_\alpha^*(Y_i(\tau_m), \tau_m, \varepsilon) + F_\alpha^*(W_j(\tau_m), \tau_m, \varepsilon) - F_\alpha^*(W_i(\tau_m), \tau_m, \varepsilon)] .$$

Put

$$U(\tau) = W_j(\tau) - W_i(\tau), \quad Z(\tau) = Y_i(\tau) - W_i(\tau), \quad \Psi(\xi, \lambda, \beta) =$$
$$= \frac{1}{2} \int_{\tau_m}^{\tau_{m+1}} H(W_i(\xi, \tau) - W_i(-\xi + 2\tau/\varepsilon, \tau) + \lambda\, U(\xi, \tau) - \lambda\, U(-\xi + 2\tau/\varepsilon, \tau) + + \beta\, Z(\xi, \tau) - \beta\, Z(-\xi + 2\tau/\varepsilon, \tau)\, \mathrm{d}\tau .$$

Then the value of the bracket in (30,4) at ξ for $\alpha = 1$ is equal to

$$\Psi(\xi, 1, 1) - \Psi(\xi, 0, 1) - \Psi(\xi, 1, 0) + \Psi(\xi, 0, 0) = \frac{\partial^2 \Psi}{\partial \lambda\, \partial \beta}(\xi, \lambda, \beta) ,$$
$$0 < \lambda, \quad \beta < 1 .$$

$$(31,4) \qquad \frac{\partial^2 \Psi}{\partial \lambda\, \partial \beta}(\xi, \lambda, \beta) = \frac{1}{2} \int_{\tau_m}^{\tau_{m+1}} h'(\cdot/.)\, [U(\xi, \tau) - U(-\xi + 2\tau/\varepsilon, \tau)] \, . \, [Z(\xi, \tau) - Z(-\xi + 2\tau/\varepsilon, \tau)]\, \mathrm{d}\tau .$$

According to Lemma 10,4 (as $\|Z\|_M \leqq \frac{1}{2}\|Z\|_{L_1}$, $Z = I_\xi z$)

$$|Z(\xi, \tau)| \leqq \tfrac{1}{2}\chi_1(\varepsilon, \tau) . \tag{32,4}$$

From

$$W_l(\tau) = W_l(\hat{\tau}) + \int_{\hat{\tau}}^{\tau} f_0^*(W_l(\sigma))\, d\sigma , \quad l = i, j , \quad 0 \leqq \tilde{\tau} \leqq \tau$$

one obtains that

$$U(\tau) = U(\hat{\tau}) + \int_{\hat{\tau}}^{\tau}\int_0^1 \left[\frac{\partial f_0^*}{\partial Y}(W_i(\sigma) + \lambda\, U(\sigma))\, U(\sigma)\right] d\lambda\, d\sigma . \tag{33,4}$$

$$\left[\frac{\partial f_0^*}{\partial Y}(W_i(\sigma) + \lambda\, U(\sigma))\, U(\sigma)\right](\xi) =$$

$$= \frac{1}{4}\int_0^2 h(W_i(\xi, \sigma) - W_i(-\xi + 2\gamma, \sigma) + \lambda\, U(\xi, \sigma) - \lambda\, U(-\xi + 2\gamma, \sigma)) \,.$$

$$. \left[U(\xi, \sigma) - U(-\xi + 2\gamma, \sigma)\right] d\gamma .$$

Let $\xi_1 \in J(\eta)$. As $|W_l(\xi, \sigma)| \leqq R + 1$, $l = i, j$ (Theorem 2,3), $U = W_j - W_i$, $\int_0^2 |U(-\xi + 2\gamma, \sigma)|\, d\gamma \leqq 2\|U(\sigma)\|_\eta + 4\eta \,.\, 2(R + 1)$, it follows that

$$\left|\left[\frac{\partial f_0^*}{\partial Y}(W_i(\sigma) + \lambda\, U(\sigma))\, U(\sigma)\right](\xi_1)\right| \leqq K_{25}[\|U(\sigma)\| + \eta] \tag{34,4}$$

and (33,4) implies that $\|U(\tau)\|_\eta \leqq \|U(\hat{\tau})\|_\eta + K_{25} \int_{\hat{\tau}}^{\tau} [\|U(\sigma)\|_\eta + \eta]\, d\sigma$. Hence (by Gronwal's Lemma) $\|U(\tau)\|_\eta \leqq K_{26}[\|U(\hat{\tau})\|_\eta + \eta]$, $0 \leqq \hat{\tau} \leqq \tau \leqq \hat{\tau} + T$ and

$$\|U(\tau)\|_\eta \leqq K_{26}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta^1 + \eta] , \quad 0 \leqq \tau \leqq T , \tag{35,4}$$

as $\|U(0)\|_\eta = \|\tilde{Y}_j - \tilde{Y}_i\|_\eta$. From (33,4), (34,4) and (35,4) one obtains

$$\|U(\tau) - U(\hat{\tau})\|_\eta \leqq K_{25}K_{26}(\tau - \hat{\tau})\,[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta] , \quad 0 \leqq \hat{\tau} \leqq \tau \leqq T . \tag{36,4}$$

As $|W_l(\xi, \sigma)| \leqq R + 1$, $l = i, j$, there exists K_{27} such that

$$|h'(W_i(\xi, \tau) - W_i(-\xi + 2\tau/\varepsilon, \tau) + \lambda\, U(\xi, \tau) - \lambda\, U(-\xi + 2\tau/\varepsilon, \tau))| \leqq K_{27}$$

and the sum (30,4) is majorized by (cf. (31,4), (32,4))

$$\tfrac{1}{2}K_{27}\left[\int_0^T |U(\xi, \tau)|\, d\tau + \int_0^T |U(-\xi + 2\tau/\varepsilon, \tau)|\, d\tau\right] \chi_1(\varepsilon, T) .$$

Taking into account that $|U(-\xi + 2\tau/\varepsilon, \tau)| \leqq 2(R + 1)$ and $|U(-\xi + 2\tau/\varepsilon, \tau)| \leqq$

$\leqq \|U(\tau)\|_\eta$, if $-\xi + 2\tau/\varepsilon \in J(\eta)$, one obtains that

$$\int_0^T |U(-\xi + 2\tau/\varepsilon, \tau)| \,\mathrm{d}\tau \leqq \int_0^T \|U(\tau)\|_\eta \,\mathrm{d}\tau + (2R + 1)\, 2\eta(T + 1)\,.$$

Therefore (35,4) implies that for $\xi \in J(\eta)$ the sum (30,4) may be majorized by $K_{28}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta]$ and Lemma 13,4 holds for $\alpha = 1$, as the sum (30,4) is arbitrarily close to the integral in (29,4), if $\sigma = t/m$ is sufficiently small.

For $\alpha = 2$ put $\sigma = t/n$,

$$\Psi_m(\xi, \beta, \gamma, \lambda) = \tfrac{1}{2}\varepsilon I_\xi\, H(W_i(\xi, \tau_m) - W_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \beta\, U(\xi, \tau_m) - \\ - \beta\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \gamma\, Z(\xi, \tau_m) - \gamma\, Z(\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m))\,.$$

The value of the sum in (30,4) at ξ is equal to

$$\sum_{m=0}^{n-1} \{\Psi(\xi, 1, 1, 1) - \Psi_m(\xi, 0, 1, 1) - \Psi_m(\xi, 1, 0, 1) - \Psi_m(\xi, 1, 1, 0) + \\ + \Psi_m(\xi, 1, 0, 0) + \Psi_m(\xi, 0, 1, 0) + \Psi_m(\xi, 0, 0, 1) - \Psi_m(\xi, 0, 0, 0)\} = \\ = \sum_{m=0}^{n-1} \frac{\partial^3 \Psi_m}{\partial\lambda\, \partial\beta\, \partial\gamma}(\xi, \beta, \gamma, \lambda)$$

($0 < \beta, \gamma, \lambda < 1$ being fixed numbers).

$$\frac{\partial^3 \Psi}{\partial\lambda\, \partial\beta\, \partial\gamma} = -\sigma\, I_\xi\{h''(\cdot/.)(U(\xi, \tau_m) - U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\,(Z(\xi, \tau_m) - \\ - Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m))\,(w_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \\ + \beta u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \gamma z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m))\} + \\ + \sigma I_\xi\{h'(\cdot/.)\,[u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \\ + U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)]\} - \\ - \sigma\, I_\xi\{h'(\cdot/.)\,[u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, Z(\xi, \tau_m) + \\ + U(\xi, \tau_m)\, z\,(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)]\} = A_1 + A_2 + A_3\,.$$

A_2 and A_3 may be transformed in the following way (according to the formula $I_\xi(aB + Ab) = AB - \frac{1}{2}\int_0^2 AB\,\mathrm{d}\xi,\, I_\xi a = A,\, I_\xi b = B)$:

$$A_2 = -\sigma h'(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \\ + \frac{1}{2}\sigma \int_0^2 h'(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\,\mathrm{d}\xi + \\ + \sigma I_\xi\{h''(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\,[w_i(\xi, \tau_m) + \\ + w_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \beta u(\xi, \tau_m) + \beta u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \\ + \gamma z(\xi, \tau_m) + \gamma z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)]\}\,,$$

$$A_3 = \sigma h'(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, Z(\xi, \tau_m) -$$
$$- \frac{1}{2}\sigma \int_0^2 h'(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, Z(\xi, \tau_m)\, d\xi -$$
$$- \sigma I_\xi\{h'(\cdot/.)\, [U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, z(\xi, \tau_m) +$$
$$+ U(\xi, \tau_m)\, z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)]\} -$$
$$- \sigma I_\xi\{h''(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, Z(\xi, \tau_m)\,.$$
$$\cdot [w_i(\xi, \tau_m) + w_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \beta\, u(\xi, \tau_m) +$$
$$+ \beta\, u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \gamma\, v(\xi, \tau_m) + \gamma\, v(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)]\}\,.$$

Hence we obtain

(37,4) $$\frac{\partial^3 \Psi}{\partial\lambda\, \partial\beta\, \partial\gamma}(\xi, \lambda, \beta, \gamma) = A_1 + A_4 - \sigma\, h'(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\,.$$
$$\cdot [Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) - Z(\xi, \tau_m)] + \frac{1}{2}\sigma \cdot \int_0^2 h'(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\,.$$
$$\cdot [Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) - Z(\xi, \tau_m)]\, d\xi -$$
$$- \sigma I_\xi\{h'(\cdot/.)\, [U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, z(\xi, \tau_m) +$$
$$+ U(\xi, \tau_m)\, z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)]\}\,,$$
$$A_4 = -\sigma I_\xi\{h''(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\,.$$
$$\cdot [-Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + Z(\xi, \tau_m)]\,.$$
$$\cdot [w_i(\xi, \tau_m) + w_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \beta\, u(\xi, \tau_m) +$$
$$+ \beta\, u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \gamma\, z(\xi, \tau_m) + \gamma\, z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)]\}\,,$$
$$A_1 + A_4 = A_5 + A_6\,,$$
$$A_5 = -\sigma I_\xi\{h''(\cdot/.)\, U(\xi, \tau_m)\, [Z(\xi, \tau_m) - Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)]\,.$$
$$\cdot [w_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \beta\, u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) +$$
$$+ \gamma\, v(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)]\}\,,$$
$$A_6 = -\sigma I_\xi\{h''(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\,.$$
$$\cdot [Z(\xi, \tau_m) - Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)]\,.$$
$$\cdot [w_i(\xi, \tau_m) + \beta\, u(\xi, \tau_m) + \gamma\, v(\xi, \tau_m)]\}\,.$$

The value of the sum (30,4) at ξ is equal to the sum of the expressions (37,4) for $m = 0, 1, \ldots, n - 1$. Obviously (cf. (32,4))

(38,4) $$|A_6| \leqq \sigma K_{29} \chi_1(\varepsilon, T) \int_0^2 |U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)|\,.$$
$$\cdot |w_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \beta\, u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) +$$
$$+ \gamma\, z(-\xi + 2(\tau + \lambda\sigma)/\varepsilon, \tau_m)|\, d\xi\,.$$

It follows from Theorem 2,3 that $\|y(\tau)\|_{L_1}$, $\|v_i(i\varepsilon + \tau)\|_{L_1} \leqq R + 1$ for $\tau \geqq 0$, $i = 0, 1, 2 \dots$. Hence

(39,4) $$\int_0^2 |w_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \beta\, u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \\ + \gamma\, z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)| \mathrm{d}\xi \leqq 5R + 5\,,$$

(40,4) $$\|U(\tau)\|_M = \|W_j(\tau) - W_i(\tau)\|_M \leqq 2R + 2\,.$$

From (36,4) one obtains

(41,4) $$\int_{\langle 0,2\rangle \cap J_c(\eta)} |U(\xi, \tau_m)|\, |w_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \\ + \beta\, u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \gamma\, z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)|\, \mathrm{d}\xi \leqq \\ \leqq (5R + 5)\, K_{26}[\|\tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta]\,,$$

it follows from Lemma 12,4 that

(42,4) $$|w_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)| \leqq 2\,, \quad |u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)| \leqq 4\,, \\ |z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)| \leqq 4 \quad \text{if} \quad i\varepsilon\,, \quad j\varepsilon \geqq N_2\,, \quad -\xi + 2(\tau_m + \lambda\sigma)/\varepsilon \in J(\eta)\,.$$

Therefore if $-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon \in J(\eta)$ for $\xi \in J_c(\eta)$, then

(43,4) $$\int_{\langle 0,2\rangle \cap J_c(\eta)} |U(\xi, \tau_m)|\, |w_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \\ + \beta\, u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \gamma\, z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)|\, \mathrm{d}\xi \leqq \\ \leqq 4\eta(2R + 2)\,.\,10 = 80\,.\,\eta(R + 1)$$

and in any case

(44,4)

$$\int_{\langle 0,2\rangle \cap J_c(\eta)} |U(\xi, \tau_m)|\, |w_i(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \beta\, u(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \\ + \gamma\, z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)|\, \mathrm{d}\xi \leqq (2R + 2)(5R + 5) = 10(R + 1)^2\,.$$

The condition that $-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon \in J(\eta)$ for $\xi \in J_c(\eta)$ is equivalent to $(2(\tau_m + \lambda\sigma)/\varepsilon - \vartheta - \eta,\ 2(\tau_m + \lambda\sigma)/\varepsilon - \vartheta + \eta) \cap \bigcup_{i=-\infty}^{\infty} (i + \vartheta - \eta,\ i + \vartheta + \eta) = \emptyset$, i.e. $2\tau_m/\varepsilon \notin \bigcup_{i=-\infty}^{\infty} (i + 2\vartheta - 2\eta - \lambda\sigma/\varepsilon,\ i + 2\vartheta + 2\eta - \lambda\sigma/\varepsilon)$. (38,4), (41,4), (43,4) and (44,4) imply that

$$(45,4) \qquad |A_6| \leqq \sigma K_{29}\chi_1(\varepsilon, \tau) \{5(R + 1) K_{26}[\| \tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta] + 10(R + 1)^2\},$$
$$m = 0, 1, 2, \ldots, n - 1$$

and

$$(46,4) \quad |A_6| \leqq \sigma K_{29}\, \chi_1(\varepsilon, T) \{5(R + 1) K_{26}[\| \tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta] + 80(R + 1)\, \eta\}$$

if $2\tau_m/\varepsilon \notin \bigcup_{i=-\infty}^{\infty} (i + 2\vartheta - 2\eta - \lambda\sigma/\varepsilon,\ i + 2\vartheta + 2\eta - \lambda\sigma/\varepsilon)$. Assume in addition that $\sigma = \eta/l$, l being a positive integer, i.e. that $t = \eta \,.\, n/l$. As $\tau_m = m\sigma = mt/n$, the number of points $2\tau_m/\varepsilon$ which belong to an interval $(i + 2\vartheta - 2\eta - \lambda\sigma/\varepsilon, i + 2\vartheta + 2\eta - \lambda\sigma/\varepsilon)$ does not exceed $\varepsilon\sigma^{-1} \,.\, 2\eta$ and the number of intervals $(i + 2\vartheta - 2\eta - \lambda\sigma/\varepsilon, i + 2\vartheta + 2\eta - \lambda\sigma/\varepsilon)$, $i = \ldots -1, 0, 1, \ldots$ which contain some points $2\tau_m/\varepsilon = 2\sigma m/\varepsilon$, $m = 0, 1, \ldots, n - 1$, does not exceed $(2T + 1)/\varepsilon$. Therefore the number of numbers m for which we have to use (45,4) doe snot exceed $2(2T + 1)\, \sigma^{-1}\eta$; for the remaining numbers in (46,4) is used;

$$(47,4) \quad \sum_{m=0}^{n-1} |A_6| \leqq K_{29}\, \chi_1(\varepsilon, T) \{5(R + 1) K_{26}[\| \tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta] + 10(R + 1)^2\} \,.$$
$$.\, 2(2T + 1)\, \eta + K_{29}\, \chi_1(\varepsilon, T) \{5(R + 1) K_{26}[\| \tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta] + 80(R + 1)\, \eta\}\, T \leqq$$
$$\leqq K_{30}\, \chi_1(\varepsilon, T)\, [\| \tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta]\,.$$

(47,4) was proved for $t = \eta n/l$, n, l being positive integers, $0 < t \leqq T$; as A_6 depends continuously on t $(\tau_m = tm/n)$, (47,4) holds foı all t, $0 \leqq t \leqq T$. Similarly one obtains

$$(48,4) \qquad \sum_{m=0}^{n-1} |A_5| \leqq K_{30}\, \chi_1(\varepsilon, T)\, [\| \tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta]\,.$$

Let $|h'(\xi)| \leqq K_{31}$ for $|\xi| \leqq 2\mu_3$. From (36,4) and Lemma 1,4 one obtains that

$$(49,4) \quad |\sigma h'(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, [Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) - Z(\xi, \tau_m)]| \leqq$$
$$\leqq \sigma K_{31} K_{26}[\| \tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta]\, 2\chi_1(\varepsilon, T)$$

if $-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon \in J(\eta)$, i.e. if $2\tau_m/\varepsilon \notin \bigcup_{i=-\infty}^{\infty} (i + \vartheta + \xi - \eta,\ i + \vartheta + \xi + \eta)$. As $|U|_M \leqq 2(R + 1)$,

$$(50,4) \quad |\sigma h'(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, [Z(-\xi + 2(\tau_m + \lambda\sigma, \tau_m) - Z(\xi, \tau_m)]| \leqq$$
$$\leqq \sigma K_{31}\, 2(R + 1)\, 2\chi_1(\varepsilon, T)\,.$$

Assuming in addition that $\sigma = \eta/l$, i.e. $t = \eta n/l$, l being a positive integer, the number of m's such that $2\tau_m/\varepsilon \in \bigcup_{i=-\infty}^{\infty} (i + \vartheta + \xi - \eta, i + \vartheta + \xi + \eta)$ may be estimated in a similar way as above by $(2T + 1)\,\sigma^{-1}\eta$ and from (49,4) and (50,4) one obtains

$$\Big|\sum_{m=0}^{n-1} \sigma h'(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\,[Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) - Z(\xi, \tau_m)]\Big| \leqq$$
$$\leqq K_{31}K_{26}[\|\tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta]\, 2\chi_1(\varepsilon, T)\, T +$$
$$+ \sigma K_{31}\, 2(R + 1)\, 2\chi_1(\varepsilon, T)\,(2T + 1)\,\sigma^{-1}\eta \leqq$$
$$\leqq K_{32}[\|\tilde{Y}_i - \tilde{Y}_j\|_\eta\, \chi_1(\varepsilon, T) + \eta]\,.$$

Hence

(51,4)
$$\Big|\sum_{m=0}^{n-1} \frac{1}{2}\int_0^2 \sigma h'(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\,.$$
$$.\,[Z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) - Z(\xi, \tau_m)]\, \mathrm{d}\xi\Big| \leqq$$
$$\leqq K_{32}[\|\tilde{Y}_i - \tilde{Y}_j\|_\eta\, \chi_1(\varepsilon, T) + \eta]\,.$$

As the left hand side of (51,4) depends continuously on t ($\tau_m = tm/n$), the assumption that $t = \eta n/l$ may be omitted and (51,4) holds for all t, $0 \leqq t \leqq T$.

(52,4)
$$|I_\xi\{h'(\cdot/.)\, U(\xi, \tau_m)\, z(-\xi + (2\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\}| \leqq$$
$$\leqq \frac{1}{2} K_{31} \int_0^2 |U(\xi, \tau_m)|\, |z(-\xi + (2\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, \mathrm{d}\xi\,.$$

According to (36,4) and Lemma 11,4

(53,4)
$$\int_{\langle 0,2\rangle \cap J(\eta)} |U(\xi, \tau_m)|\, |z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)|\, \mathrm{d}\xi \leqq$$
$$\leqq K_{26}[\|\tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta]\, \chi_1(\varepsilon, T)\,.$$

According to Lemma 12,4 $|z(-\xi + (2\tau_m + \lambda\sigma)/\varepsilon, \tau_m)| \leqq 4$ if $-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon \in$ $\in J(\eta)$, $i\varepsilon \geqq N_2$. Therefore (as $\|U\|_M \leqq 2(R + 1)$)

(54,4)
$$\int_{\langle 0,2\rangle \cap J_c(\eta)} |U(\xi, \tau_m)|\, |z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)|\, \mathrm{d}\xi \leqq$$
$$\leqq 4\eta\, 2(R + 1)\, 4 = 32(R + 1)\,\eta$$

if $-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon \in J(\eta)$ for $\xi \in J_c(\eta)$, i.e. if

$$2\tau_m/\varepsilon \notin \bigcup_{i=-\infty}^{\infty} (i + 2\vartheta - 2\eta - \lambda\sigma/\varepsilon \;\; i + 2\vartheta + 2\eta - \lambda\sigma/\varepsilon)\,.$$

From Lemma 11,4 (and $\|U\|_M \leqq 2(R + 1)$) one obtains

(55,4)
$$\int_{\langle 0,2\rangle \cap J_c(\eta)} |U(\xi, \tau_m)|\, |z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)|\, \mathrm{d}\xi \leqq 2(R + 1)\, \chi_1(\varepsilon, T)\,.$$

(52,4)–(55,4) imply that

$$(56,4)\qquad \int_0^2 |U(\xi, \tau_m)|\, |z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)|\, d\xi \leqq$$
$$\leqq K_{26}[\|\tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta]\, \chi_1(\varepsilon, T) + 2(R + 1)\, \chi_1(\varepsilon, T)\,.$$

$$(57,4)\qquad \int_0^2 |U(\xi, \tau_m)|\, |z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)|\, d\xi \leqq$$
$$\leqq K_{26}[\|\tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta]\, \chi_1(\varepsilon, T) + 32(R + 1)\, \eta\,,$$

if $2\tau_m/\varepsilon \notin \bigcup_{i=-\infty}^{\infty} (i + 2\vartheta - 2\eta - \lambda\sigma/\varepsilon,\ i + 2\vartheta + 2\eta - \lambda\sigma/\varepsilon)$. Estimating the number of m's, $m = 0,1,\ldots, n - 1$ such that $2\tau_m/\varepsilon \in \bigcup_{i=-\infty}^{\infty} (i + 2\vartheta - 2\eta - \lambda\sigma/\varepsilon,\ i + 2\vartheta + 2\eta - \lambda\sigma/\varepsilon)$ (having introduced the additional assumption that $\sigma = \eta/l$, i.e. $\tau = \eta n/l$ (l being a positive integer), which will be omitted after (58,4)) in the same way as in the proof of (47,4) by $2(2T + 1)\,\sigma^{-1}\eta$ one obtains from (56,4) and (57,4)

$$(58,4)\qquad \Big|\sum_{m=0}^{n-1} \sigma I_\xi\{h'(\cdot/.)\, U(\xi, \tau_m)\, z(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\}\Big| \leqq$$
$$\leqq [K_{26}[\|\tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta]\, \chi_1(\varepsilon, T) + 32(R + 1)\, \eta]\, T +$$
$$+ [K_{26}[\|\tilde{Y}_i - \tilde{Y}_j\|_\eta + \eta]\, \chi_1(\varepsilon, T) + 2(R + 1)\, \chi_1(\varepsilon, T)]\, z(2T + 1)\, \eta \leqq$$
$$\leqq K_{32}[\|\tilde{Y}_i - \tilde{Y}_j\|_\eta\, \chi_1(\varepsilon, T) + \eta]\,.$$

Similarly

$$(59,4)\qquad \Big|\sum_{m=0}^{n-1} \sigma I_\xi(h'(\cdot/.)\, U(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\, z(\xi, \tau_m)\}\Big| \leqq$$
$$\leqq K_{32}[\|\tilde{Y}_i - \tilde{Y}_j\|_\eta\, \chi_1(\varepsilon, T) + \eta]\,.$$

As the value of the sum (30,4) at ξ is equal to the sum of the expressions (34,4) for $m = 0, 1, \ldots, n - 1$, and as the sum (30,4) is arbitrarily close to the integral in (29,4) if $\sigma = t/n$ is sufficiently small, it follows from (47,4), (48,4), (51,4), (52,4), (58,4) and (59,4) that Lemma 13,1 holds for $\alpha = 2$, too.

Lemma 14,4. *Put* $\bar{U}(\tau) = W_j(\tau) - W_i(\tau) - Y_j(\tau) + Y_i(\tau)$ *and assume that* $\varepsilon i, \varepsilon_j \geqq \geqq N_2$. *Then*

$$(60,4)\qquad \left\|\int_0^t \mathrm{D}_\sigma[F^*_\alpha(Y_j(\tau) + \bar{U}(\tau), \sigma, \varepsilon) - F^*_\alpha(Y_j(\tau), \sigma, \varepsilon)]\right\|_\eta \leqq$$
$$\leqq K_{34}\left[\int_0^\tau \|\bar{U}(\tau)\|_\eta\, d\tau + \eta\right], \quad \alpha = 1, 2\,.$$

Proof. For $n > 0$ put $\sigma = t/n$, $\tau_m = m\sigma$, $m = 0, 1, \ldots, n$. Then the integral in (25,4) is the limit of

$$(61,4) \qquad \sum_{m=0}^{n-1} [F_\alpha^*(Y_j(\tau_m) + \bar{U}(\tau_m), \tau_m + \sigma, \varepsilon) - F_\alpha^*(Y_j(\tau_m) + \bar{U}(\tau_m), \tau_m, \varepsilon) - \\ - F_\alpha^*(Y_j(\tau_m), \tau_m + \sigma, \varepsilon) + F_\alpha^*(Y_j(\tau_m), \tau_m, \varepsilon)] .$$

Let $\alpha = 1$; put

$$\varphi_m(\xi, \lambda) = \int_{\tau_m}^{\tau_m + \sigma} H(Y_j(\xi) - Y_j(-\xi + 2\sigma_1/\varepsilon) + \lambda\, \bar{U}(\xi) - \lambda\, \bar{U}(-\xi + 2\sigma_1/\varepsilon))\, d\sigma_1 .$$

Then the value of the sum (61,4) at ξ is equal to

$$\sum_{m=0}^{n-1} (\varphi_m(\xi, 1) - \varphi_m(\xi, 0)) = \sum_{m=0}^{n-1} \frac{\partial \varphi_m}{\partial \lambda} (\xi, \lambda') , \quad 0 < \lambda' < 1 ,$$

$$\frac{\partial \varphi}{\partial \lambda} (\xi, \lambda) = \int_{\tau_m}^{\tau_m + \sigma} h(\cdot/.) [\bar{U}(\xi, \tau_m) - \bar{U}(-\xi + 2\sigma_1/\varepsilon, \tau_m)]\, d\sigma_1 .$$

Let $|h(\xi)| \leqq K_{32}$ for $|\xi| \leqq 18(R + 1)$. Let $\xi_1 \in J(\eta)$. As $|\bar{U}(\xi_1, \tau_m)| \leqq \|\bar{U}(\tau_m)\|_\eta$, $|\bar{U}(\xi, \tau_m)| \leqq 8(R + 1)$ (cf. Theorem 2,3) for $\xi \in E_1$, the value of the sum (61,4) at ξ_1 may be estimated by $K_{32} \int_0^t \Psi_1(\sigma_1)\, d\sigma_1$, Ψ_1 being defined by

$$(62,4) \quad \Psi_1(\sigma_1) = 2\|\bar{U}(\tau_m)\|_m , \qquad \text{if } \sigma_1 \in \langle \tau_m, \tau_{m+1}) , \quad -\xi_1 + 2\sigma_1/\varepsilon \in J(\eta) , \\ \Psi_1(\sigma_1) = \|\bar{U}(\tau_m)\|_\eta + 8(R + 1) , \quad \text{if } \sigma_1 \in (\tau_m, \tau_{m+1}) , \quad -\xi_1 + 2\tau_m/\varepsilon \notin J(\eta) .$$

If $\sigma = \eta/l$, i.e. if $\tau = \eta n/l$, the number of such m's that $-\xi + 2\tau_m/\varepsilon \notin J(\eta)$ may be estimated similarly as in the proof of Lemma 13,1 by $2(2T + 1)\, \sigma^{-1}\eta$; hence

$$(63,4) \quad \int_0^t \psi_1(\sigma_1)\, d\sigma_1 \leqq 2 \sum_{m=0}^{n-1} \|\bar{U}(\tau_m)\|_\eta + 16(R + 1) + (2T + 1)\, \eta , \quad 0 \leqq t \leqq T .$$

(The assumption that $t = \eta n/l$ may be dropped, as the left hand side of (63,4) is continuous in t.) (60,4) holds for $\alpha = 1$.

Let $\alpha = 2$. Put

$$\Phi_m(\xi, \lambda, \beta) = \tfrac{1}{2}\varepsilon I_\xi \{H(Y_j(\xi, \tau_m) - Y_j(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m) + \\ + \beta\, \bar{U}(\xi, \tau_m) - \beta\, \bar{U}(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m))\} .$$

The value of the sum (61,4) at ξ is equal to

$$\sum_{m=0}^{n-1} [\Phi_m(\xi, 1, 1) - \Phi_m(\xi, 1, 0) - \Phi_m(\xi, 0, 1) + \Phi_m(\xi, 0, 0)] = \\ = \sum_{m=0}^{n-1} \frac{\partial^2 \Phi_m}{\partial \lambda\, \partial \beta} (\xi, \lambda, \beta) , \quad 0 < \lambda, \beta < 1 .$$

$$\frac{\partial^2 \Phi_m}{\partial\lambda\,\partial\beta}(\xi,\lambda,\beta) = -\sigma I_\xi\{h'(\cdot/.)\,[\bar{U}(\xi,\tau_m) - \bar{U}(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon,\tau_m)]\,.$$
$$.\,[y_j(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon,\tau_m) + \beta\,\bar{u}(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon,\tau_m)]\} -$$
$$- \sigma I_\xi\{h(\cdot/.)\,\bar{u}(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon,\tau_m)\},$$

$\varepsilon > 0$ being fixed. Suppose that $\sigma \leqq \varepsilon\eta$. Similarly as in the case $\alpha = 1$ one obtains (cf. (63,4), (64,4))

$$(65{,}4)\qquad \sum_{m=0}^{n-1}\sigma|h(\cdot/.)\,\bar{U}(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon,\tau_m)| < K_{32}\int_0^T\psi_1(\sigma_1)\,\mathrm{d}\sigma_1 \leqq$$
$$\leqq K_{32}[2\sum_{m=0}^{n-1}\|\bar{U}(\tau_m)\|_\eta + 16(R+1)(2T+1)\,\eta]\,.$$

(65,4) implies

$$(66{,}4)\qquad \sum_{m=0}^{n-1}\frac{1}{2}\int_0^2\sigma|h(\cdot/.)\,\bar{U}(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon,\tau_m)|\,\mathrm{d}\xi \leqq$$
$$\leqq K_{32}[2\sum_{m=0}^{n-1}\|\bar{U}(\tau_m)\|_\eta + 16(R+1)(2T+1)\,\eta]\,.$$

Let $|h'(\xi)| \leqq K_{33}$ for $|\xi| \leqq 18(R+1)$; as $\|\bar{y}_j(\tau_m) + \beta\,\bar{u}(\tau_m)\|_{L_1} \leqq 5(R+1)$, $\|\bar{U}(\tau_m)\|_M \leqq 8(R+1)$, $y_j(\xi,\tau_m) + \beta\,\bar{u}(\xi,\tau_m)| \leqq 10$, if εi, $\varepsilon j \geqq N_2$, it follows that

$$(67{,}4)\qquad |\sigma I_\xi\{h'(\cdot/.)\,\bar{U}(\xi,\tau_m)\,[y_j(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon,\tau_m) +$$
$$+ \beta\,\bar{u}(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon,\tau_m)]\}| \leqq \sigma\,.\,40K_{33}(R+1)$$

and that

$$(68{,}4)\qquad |\sigma I_\xi\{h'(\cdot/.)\,\bar{U}(\xi,\tau_m)\,[y_j(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon,\tau_m) +$$
$$+ \beta\,\bar{u}(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon,\tau_m)]\}| \leqq$$
$$\leqq \sigma\left|\int_{\langle 0,2\rangle\cap J(\eta)}\right| + \sigma\left|\int_{\langle 0,2\rangle\cap J_c(\eta)}\right| \leqq \sigma K_{33}[\|\bar{U}(\tau_m)\|_\eta\,5(R+1) + 4\eta\,.\,8(R+1)\,10] \leqq$$
$$\leqq \sigma K_{33}\,.\,320(R+1)\,[\|\bar{U}(\tau_m)\|_\eta + \eta]\,,$$

if $-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon \in J(\eta)$ for $\xi \in J_c(\eta) \cap \langle 0, 2\rangle$, i.e. if $2\tau_m/\varepsilon \notin \bigcup_{i=-\infty}^{\infty} (i + 2\vartheta - 2\eta - \lambda\sigma/\varepsilon,\ i + 2\vartheta + 2\eta - \lambda\sigma/\varepsilon)$ (cf. the proof of (45,4)). Proceeding in the same way as above in the proof of (47,4) one obtains

$$(69{,}4)\qquad |\sum_{m=0}^{n-1}\sigma I_\xi\{h'(\cdot/.)\,\bar{U}(\xi,\tau_m)\,[y_j(-\xi + 2(\tau_m + \lambda\sigma/\varepsilon,\tau_m) +$$
$$+ \beta\,\bar{u}(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon,\tau_m)]\}| \leqq$$
$$\leqq K_{33}\,.\,320(R+1)\,[\sum_{m=0}^{n-1}\|\bar{U}(\tau_m)\|_\eta\,\sigma + T\eta] + 40K_{33}(R+1)\,.\,2(2T+1)\,\eta\,.$$

Similarly

$$(70,4)\quad \left|\sum_{m=0}^{n-1} \sigma I_\xi\{h'(\cdot/.)\ \bar{U}(-\xi + 2(\tau_m + \lambda\sigma)/\varepsilon, \tau_m)\,[y_j(\xi, \tau_m) + \beta\,\bar{u}(\xi, \tau_m)]\}\right| \leqq$$

$$\leqq K_{33}\,.\,320(R + 1)\left[\sum_{m=0}^{n-1} \|\bar{U}(\tau_m)\|_\eta\,\sigma + T\eta\right] + 40K_{33}(R + 1)\,.\,2(2T + 1)\,\eta\,.$$

(60,4) for $\alpha = 2$ follows from (65,4), (66,4), (69,4), (70,4), as sum (61,4) is arbitrarily close to the integral in (60,4) if σ is sufficiently small.

Lemma 15,4. *Let $i\varepsilon$, $j\varepsilon \geqq N_2$, $0 \leqq t \leqq T$. Then*

$$(71,4)\quad \left\|\int_0^t \mathrm{D}_\sigma[F_2^*(W_j(\tau), \sigma, \varepsilon) - F_2^*(W_i(\tau), \sigma, \varepsilon)]\right\|_M \leqq K_{50}[\varepsilon\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta]\,.$$

Proof. Like in the proof of Lemma 13,4 put $U(\tau) = W_j(\tau) - W_i(\tau)$. Put $\varphi(\xi, \beta) = \frac{1}{2}\varepsilon \int_0^t \mathrm{D}_\sigma I_\xi\{H(W_i(\xi, \tau) - W_i(-\xi + 2\sigma/\varepsilon, \tau) + \beta\,U(\xi, \tau) - \beta\,U(-\xi + 2\sigma/\varepsilon, \tau))\}$. The value of integral in (71,4) is equal to $\varphi(\xi, 1) - \varphi(\xi, 0) = (\partial\varphi/\partial\beta)(\xi, \beta)$, $0 < \beta < 1$ (cf. (20,4) and the definition of $\int_\alpha^\beta \mathrm{D}_\sigma F(x(\tau), \sigma)$, [1], section 1, after (51,1)). Put $\tau_m = m\varepsilon$, $m = 0, 1, \ldots$, let $\tau_n \leqq t < \tau_{n+1}$,

$$B_m = \tfrac{1}{2}\varepsilon \int_{\tau_m}^{\tau_{m+1}} \mathrm{D}_\sigma I_\xi\{h(\cdot/.)\,[U(\xi, \tau) - U(-\xi + 2\sigma/\varepsilon, \tau)]\}\,, \quad m = 0, 1, \ldots, n - 1\,,$$

$$B_n = \tfrac{1}{2}\varepsilon \int_{\tau_n}^{t} \mathrm{D}_\sigma I_\xi\{h(\cdot/.)\,[U(\xi, \tau) - U(-\xi + 2\sigma/\varepsilon, \tau)]\}\,,$$

$$h(\cdot/.) = h(W_i(\xi, \tau) - W_i(-\xi + 2\sigma/\varepsilon, \tau) + \beta\,U(\xi, \tau) + \beta\,U(-\xi + 2\sigma/\varepsilon, \tau))\,.$$

Then

$$(72,4)\quad \frac{\partial\varphi}{\partial\beta}(\xi, \beta) = \sum_{m=0}^{n} B_m\,.$$

As

$$\int_\alpha^\beta \mathrm{D}_\sigma F(x(\tau), \sigma) = \int_\alpha^\beta \frac{\partial F}{\partial\sigma}(x(\sigma), \sigma)\,\mathrm{d}\sigma$$

(cf. [1], section 1, after (52,1)), it follows that $\int_\alpha^\beta \mathrm{D}_\sigma F(x(\alpha), \sigma) = F(x(\alpha), \beta) - F(x(\alpha), \alpha)$ and

$$\int_\alpha^\beta \mathrm{D}_\sigma F(x(\tau), \sigma) = \int_\alpha^\beta \left[\frac{\partial F}{\partial\sigma}(x(\sigma), \sigma)\right]\mathrm{d}\sigma - \int_\alpha^\beta \frac{\partial F}{\partial\sigma}(x(\alpha), \sigma)\,\mathrm{d}\sigma + F(x(\alpha), \beta) - F(x(\alpha), \alpha)\,.$$

Apply the last formula to the integral defining B_m. As $U(-\xi+2\sigma/\varepsilon, \tau)$, $W_j(-\xi+2\sigma/\varepsilon, \tau)$ have the period ε in σ, one obtains

$$(73,4) \quad B_m = \int_{\tau_m}^{\tau_{m+1}} I_\xi\{-h'(\cdot\cdot/..)\,[w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)]\,.$$

$$.\,[U(\xi, \sigma) - U(-\xi + 2\sigma/\varepsilon, \sigma)] - h(\cdot\cdot/..)\, u(-\xi + 2\sigma/\varepsilon, \sigma)\} -$$

$$- \int_{\tau_m}^{\tau_{m+1}} I_\xi\{-h'(//)\,[w_i(-\xi + 2\sigma/\varepsilon, \tau_m) + \beta\, u(-\xi + 2\sigma/\varepsilon, \tau_m)]\,.$$

$$.\,[U(\xi, \tau_m) - U(-\xi + 2\sigma/\varepsilon, \tau_m] - h(//)\, u(-\xi + 2\sigma/\varepsilon, \tau_m)\}\,,$$

$$m = 0, 1, \ldots, n - 1\,,$$

$$h'(\cdot\cdot/..) = h'(W_i(\xi, \sigma) - W_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, U(\xi, \sigma) - \beta\, U(-\xi + 2\sigma/\varepsilon, \sigma))\,,$$

$$h'(//) = h'(W_i(\xi, \tau_m) - W_i(-\xi + 2\sigma/\varepsilon, \tau_m) + \beta\, U(\xi, \tau_m) - \beta\, U(-\xi + 2\sigma/\varepsilon, \tau_m)\,,$$

$h(\cdot\cdot/..)$, $h(//)$ having an analogous meaning.

$$I_\xi\{-h(\cdot\cdot/..)\, u(-\xi + 2\sigma/\varepsilon, \sigma)\} = h(\cdot\cdot/..)\, U(-\xi + 2\sigma/\varepsilon, \sigma) -$$

$$- I_\xi h'(\cdot\cdot/..)\,[w_i(\xi, \sigma) + w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(\xi, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)]\,.$$

Applying the last formula and analogous formula for $I_\xi(-h(//)\, u(-\xi + 2\sigma/\varepsilon, \tau_m))$ to (73,4) one obtains

$$(74,4) \quad B_m = \int_{\tau_m}^{\tau_{m+1}} [h(\cdot\cdot/..)\, U(-\xi + 2\sigma/\varepsilon, \sigma) - h(//)\, U(-\xi + 2\sigma/\varepsilon, \tau_m)]\, d\sigma -$$

$$- \int_{\tau_m}^{\tau_{m+1}} I_\xi\{h'(\cdot\cdot/..)\,[w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)]\, U(\xi, \sigma) -$$

$$- h'(//)\,[w_i(-\xi + 2\sigma/\varepsilon, \tau_m) + \beta\, u(-\xi + 2\sigma/\varepsilon, \tau_m)]\, U(\xi, \tau_m)\}\, d\sigma -$$

$$- \int_{\tau_m}^{\tau_{m+1}} I_\xi\{h'(\cdot\cdot/..)\,[w_i(\xi, \sigma) + \beta(u(\xi, \sigma))]\, U(-\xi + 2\sigma/\varepsilon, \sigma) -$$

$$- h'(//)\,[w_i(\xi, \tau_m) + \beta\, u(\xi, \tau_m)]\, U(-\xi + 2\sigma/\varepsilon, \tau_m)\} = B_{m,1} + B_{m,2} + B_{m,3}\,.$$

As W_i and W_j are solutions of (6,2) in M and as $\|f_0(\overline{Y})\|_M \leqq K_{34}$ for $\overline{Y} \in M$, $\|\overline{Y}\|_M \leqq$ $\leqq R + 1$, it follows that

$$(75,4) \qquad \|W_l(\tau) - W_l(\hat{\tau})\|_M \leqq K_{34}(\tau - \hat{\tau})\,, \quad l = i, j\,, \quad 0 \leqq \hat{\tau} \leqq \tau\,.$$

Taking into account that $\|U(\sigma)\|_M \leqq 2(R + 1)$, $|h(\cdot\cdot/..)|$, $|h(//)| \leqq K_{35}$, $|h'(\xi)| \leqq K_{36}$

for $\xi\langle -R-1, R+1\rangle$, one obtains from (74,4), (36,4), (35,4) and (75,4) that

$$|B_{m,1}| \leqq \int_{\tau_m}^{\tau_{m+1}} |h(\cdot\cdot/..) - h(//)| \, |U(-\xi + 2\sigma/\varepsilon, \sigma)| \, d\sigma +$$

$$+ \int_{\tau_m}^{\tau_{m+1}} |h(//)| \, |U(-\xi + 2\sigma/\varepsilon, \sigma) - U(-\xi + 2\sigma/\varepsilon, \tau_m)| \, d\sigma \leqq$$

$$\leqq \varepsilon^2 K_{36} K_{34} K_{26}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta] + K_{35} \int_{\tau_m}^{\tau_{m+1}} \Psi_2(\sigma) \, d\sigma ,$$

Ψ_2 being defined by $\Psi_2(\sigma) = \varepsilon K_{25} K_{26}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta]$, if $-\xi + 2\sigma/\varepsilon \in J(\eta)$, $\Psi_2(\sigma) = 2(R+1)$ if $-\xi + 2\sigma/\varepsilon \in J_c(\eta)$. Therefore

$$\int_{\tau_m}^{\tau_{m+1}} \Psi_2(\sigma) \, d\sigma \leqq \varepsilon^2 K_{25} K_{26}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta] + \varepsilon\eta \, . \, 4(R+1)$$

and

(76,4) $$|B_{m,1}| \leqq \varepsilon K_{37}[\varepsilon \|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta] ,$$

(77,4) $$|I_\xi\{h'(\cdot\cdot/..) [w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta \, u(-\xi + 2\sigma/\varepsilon, \sigma)] \, .$$

$$. \, [U(\xi, \sigma) - U(\xi, \tau_m)]\}| \leqq$$

$$\leqq K_{36} \int_0^2 |w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta \, u(-\xi + 2\sigma/\varepsilon, \sigma)| \, .$$

$$. \, |U(\xi, \sigma) - U(\xi, \tau_m)| \, d\xi \, .$$

Taking into account that $\|w_i(\sigma) + \beta \, u(\sigma)\|_{L_1} \leqq R + 1$ (cf. Theorem 2,3), $|w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta \, u(-\xi + 2\sigma/\varepsilon, \sigma)| \leqq 2$, if $-\xi + 2\sigma/\varepsilon \in J(\eta)$ (cf. Lemma 12,4), $|U(\xi, \sigma) - U(\xi, \tau_m)| \leqq \varepsilon K_{25} K_{26}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta]$ if $\xi \in J(\eta)$, $\sigma \in \langle \tau_m, \tau_{m+1} \rangle$ (cf. (36,4)) and $\|U(\sigma) - U(\tau_m)\|_M \leqq 2\varepsilon K_{34}$ (cf. 75,4)), one obtains that

$$\int_0^2 |w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta \, u(-\xi + 2\sigma/\varepsilon, \sigma)| \, |U(\xi, \sigma) - U(\xi, \tau_m)| \, d\xi \leqq (R+1) \, . \, 2\varepsilon K_{34} ,$$

and that if $-\xi + 2\sigma/\varepsilon \in J(\eta)$ for $\xi \in J_c(\eta)$, i.e. if $2\sigma/\varepsilon \notin \bigcup_{i=-\infty}^{\infty} (i + 2\vartheta - 2\eta, \; i + 2\vartheta + 2\eta)$, then

$$\int_0^2 |w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta \, u(-\xi + 2\sigma/\varepsilon, \sigma)| \, |U(\xi, \sigma) - U(\xi, \tau_m)| \, d\xi \leqq$$

$$\leqq \int_{\langle 0,2 \rangle \cap J(\eta)} + \int_{\langle 0,2 \rangle \cap J_c(\eta)} \leqq \varepsilon K_{25} K_{26}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta](R+1) + 4\eta \, . \, 2 \, . \, 2\varepsilon K_{34} \, .$$

Therefore

$$\int_{\tau_m}^{\tau_{m+1}} \int_0^2 \left|w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)\right| \left|U(\xi, \sigma) - U(\xi, \tau_m)\right| \mathrm{d}\xi\, \mathrm{d}\sigma \leqq$$

$$\leqq \varepsilon^2 K_{25} K_{26} [\| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] (R + 1) + \varepsilon^2 \eta \,.\, 16 K_{34} + 2\varepsilon^2 (R + 1) K_{34}$$

and from (77,4) one obtains

(78,4)
$$\left| \int_{\tau_m}^{\tau_{m+1}} I_\xi\{h'(\cdot\cdot/..) [w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)] \,.\right.$$

$$\left. .\, [U(\xi, \sigma) - U(\xi, \tau_m)]\} \,\mathrm{d}\sigma \right| \leqq \varepsilon^2 K_{38} [\| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] \,.$$

Let $|h''(\xi)| \leqq K_{39}$ for $|\xi| \leqq 4(R + 1)$.

(79,4)
$$\left|I_\xi[h'(\cdot\cdot/..) - h'(//)] [w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \right.$$

$$\left. + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)] \, U(\xi, \tau_m)\}\right| \leqq$$

$$\leqq K_{39} \int_0^2 \left|(1 - \beta) (W_i(\xi, \sigma) - W_i(-\xi + 2\sigma/\varepsilon, \sigma) - W_i(\xi, \tau_m) + W_i(-\xi + 2\sigma/\varepsilon, \tau_m)\right| \,.$$

$$.\, \left|U(\xi, \tau_m)\right| \left|w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)\right| \mathrm{d}\xi \leqq$$

$$\leqq \varepsilon K_{39} K_{34} \,.\, 2(R + 1)(R + 1)$$

(cf. (75,4)).

If $-\xi + 2\sigma/\varepsilon \in J(\eta)$ for $\xi \in J_c(\eta)$, i.e. if $2\sigma/\varepsilon \notin \bigcup_{i=-\infty}^{\infty} (i + 2\vartheta - 2\eta,\ i + 2\vartheta + 2\eta)$ then

(80,4)
$$\left|I_\xi\{[h'(\cdot\cdot/..) - h'(//)] [w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \right.$$

$$\left. + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)] \, U(\xi, \tau_m)\}\right| \leqq$$

$$\leqq K_{39} \left[\int_{\langle 0,2 \rangle \cap J(\eta)} + \int_{\langle 0,2 \rangle \cap J_c(\eta)} \right] \leqq K_{39} \,.\, 2K_{34} \varepsilon \|U(\tau_m)\|_\eta \,.\, 2(R + 1) +$$

$$+ K_{39} K_{34} \varepsilon \,.\, 2(R + 1) \,.\, 2 \,.\, 4\eta \leqq \varepsilon \,.\, 4(R + 1) K_{39} K_{34} K_{26} [\| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] +$$

$$+ \varepsilon \eta \,.\, 16(R + 1) K_{39} K_{34} \leqq \varepsilon K_{40} [\| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] \,.$$

(79,4) and (80,4) imply that

$$(81,4) \quad \left| \int_{\tau_m}^{\tau_{m+1}} I_\xi\{[h'(\cdot\cdot/..) - h'(//)] [w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \right.$$
$$\left. + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)] \, U(\xi, \tau_m)\} \, d\sigma \right| \leqq$$
$$\leqq \varepsilon^2 K_{40}[\| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] + 8\eta\varepsilon^2 K_{39} K_{34}(R + 1)^2 \leqq$$
$$\leqq \varepsilon^2 K_{41}[\| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] .$$

It follows from $\|U(\tau_m)\|_M \leqq 2(R + 1)$, Lemma 8,2 and Theorem 2,3

$$(82,4) \quad |I_\xi\{h'(//) [w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma) -$$
$$- w_i(-\xi + 2\sigma/\varepsilon, \tau_m) - \beta\, u(-\xi + 2\sigma/\varepsilon, \tau_m)] \,.$$
$$. \, U(\xi, \tau_m)\}| \leqq K_{36} \int_0^2 |(1 - \beta) [w_i(-\xi + 2\sigma/\varepsilon, \sigma) - w_i(-\xi + 2\sigma/\varepsilon, \tau_m)] +$$
$$+ \beta[w_j(-\xi + 2\sigma/\varepsilon, \sigma) - w_j(-\xi + 2\sigma/\varepsilon, \tau_m)]| \, |U(\xi, \tau_m)| \, d\sigma \leqq$$
$$\leqq K_{36} \,.\, 2(R + 1)\, \varepsilon \,.\, 2K_2[(1 - \beta) \|w_i(\tau_m)\|_{L_1} + \beta \|w_j(\tau_m)\|_{L_1} \leqq$$
$$\leqq \varepsilon \,.\, 4K_{36} K_2 (R + 1)^2 .$$

If $-\xi + 2\sigma/\varepsilon \in J(\eta)$ for $\xi \in J_c(\eta)$, i.e. if $2\sigma/\varepsilon \notin \bigcup_{i=-\infty}^{\infty} (i + 2\vartheta - 2\eta,\ i + 2\vartheta + 2\eta)$, there

$$(83,4) \quad |I_\xi\{h'(//) [w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon) - w_i(-\xi + 2\sigma/\varepsilon, \tau_m) -$$
$$- \beta\, u(-\xi + 2\sigma/\varepsilon, \tau_m)] \, U(\xi, \tau_m)| \leqq K_{36} \left(\int_{\langle 0,2 \rangle \cap J(\eta)} + \int_{\langle 0,2 \rangle \cap J_c(\eta)} \right) \leqq$$
$$\leqq K_{36} \,.\, \|U(\tau_m)\|_\eta \, \varepsilon \,.\, 2K_2[(1-\beta) \|w_i(\tau_m)\|_{L_1} + \beta \|w_j(\tau_m)\|_{L_1}] + K_{36} \,.\, 4\eta \,.\, 4 \,.\, 2(R + 1) \leqq$$
$$\leqq \varepsilon \,.\, 2K_{36} K_{26} K_2 (R + 1) \, [\| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] + \eta \,.\, 32(R + 1) \, K_{36} =$$
$$= K_{42}[\varepsilon \| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] .$$

(82,4) and (83,4) imply that

$$(84,4) \quad \left| \int_{\tau_m}^{\tau_{m+1}} I_\xi\{h'(//) [w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma) - w_i(-\xi + 2\sigma/\varepsilon, \tau_m) + \right.$$
$$\left. + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)] \, U(\xi, \tau_m)\} \, d\sigma \right| \leqq$$
$$\leqq \varepsilon K_{42}[\varepsilon \| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] + \varepsilon^2 \eta \,.\, 16 K_{36} K_2 (R + 1)^2 \leqq \varepsilon K_{43}[\varepsilon \| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] .$$

It follows from (78,4), (81,4) and (84,4) that

$$(85,4) \quad |B_{m,2}| \leqq \varepsilon K_{44}[\varepsilon \| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] , \quad m = 0, 1, \ldots, n - 1 .$$

As $|B_{m,3}|$ may be estimated in the same way as $|B_{m,2}|$, one obtains (cf. (76,4), (74,4))

(86,4) $$|B_m| \leqq \varepsilon K_{45}[\varepsilon \| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] .$$

The formula

$$\int_\alpha^\beta \mathrm{D}_\sigma F(x(\tau), \sigma) = \int_\alpha^\beta \frac{\partial F}{\partial \sigma}(x(\sigma), \sigma)\, \mathrm{d}\sigma$$

applied to B_n gives

(87,4) $$B_n = \int_{\tau_n}^t I_\xi\{-h'(\cdot\cdot/..)\,[w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)] \,.$$
$$. [U(\xi, \sigma) - U(-\xi + 2\sigma/\varepsilon, \sigma)] - h(\cdot\cdot/..)\, u(-\xi + 2\sigma/\varepsilon, \sigma)\}\, \mathrm{d}\sigma =$$
$$= \int_{\tau_n}^t h(\cdot\cdot/..)\, U(-\xi + 2\sigma/\varepsilon, \sigma)\, \mathrm{d}\sigma - \int_{\tau_n}^t I_\xi\{h'(\cdot\cdot/..)\,[w_i(-\xi + 2\sigma/\varepsilon, \sigma) +$$
$$+ \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)]\, U(\xi, \sigma)\}\, \mathrm{d}\sigma -$$
$$- \int_{\tau_n}^t I_\xi\{h'(\cdot\cdot/..)\,[w_i(\xi, \sigma) + \beta\, u(\xi, \sigma)]\, U(-\xi + 2\sigma/\varepsilon, \sigma)\}\, \mathrm{d}\sigma .$$

As $|h(\cdot\cdot/..)| \leqq K_{35}$, $|U(-\xi + 2\sigma/\varepsilon, \sigma)| \leqq 2(R + 1)$ and $|U(-\xi + 2\sigma/\varepsilon, \sigma)| \leqq \leqq K_{26}[\| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta]$, if $-\xi + 2\sigma/\varepsilon \in J(\eta)$, $0 \leqq \sigma \leqq T$ (cf. (35,4)), $\tau_n \leqq t < \tau_n + \varepsilon$, $t \leqq T$, it follows that

(88,4) $$\left| \int_{\tau_n}^t h(\cdot/.)\, U(-\xi + 2\sigma/\varepsilon, \sigma)\, \mathrm{d}\sigma \right| \leqq \varepsilon K_{35} K_{26}[\| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta] +$$
$$+ 4\eta\varepsilon K_{35} \,.\, 2(R + 1) \leqq K_{46}(\varepsilon \| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta) .$$

As $\|w_i(\sigma) + \beta\, u(\sigma)\|_{L_1} \leqq R + 1$, $\|U(\sigma)\|_M \leqq 2(R + 1)$, $h'(\cdot\cdot/..) \leqq K_{36}$, it follows that

(89,4) $$|I_\xi h'(\cdot\cdot/..)\,[w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)]\, U(\xi, \sigma)| \leqq$$
$$\leqq K_{36} \int_0^2 |w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)|\, |U(\xi, \sigma)|\, \mathrm{d}\xi \leqq K_{36} \,.\, 2(R + 1)^2 .$$

Taking into account that $|w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)| \leqq 2$, if $-\xi + + 2\sigma/\varepsilon \in J(\eta)$, one obtains that

(90,4) $$|I_\xi\{h'(\cdot\cdot/..)\,[w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)]\, U(\xi, \sigma)\}| \leqq$$
$$\leqq K_{36} \left[\int_{\langle 0,2 \rangle \cap J(\eta)} + \int_{\langle 0,2 \rangle \cap J_c(\eta)} \right] \leqq$$
$$\leqq K_{36} K_{26}[\| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta](R + 1) + K_{36} \,.\, 4\eta \,.\, 2 \,.\, 2(R + 1) \leqq K_{47}(\| \tilde{Y}_j - \tilde{Y}_i \|_\eta + \eta) ,$$

if $-\xi + 2\sigma/\varepsilon \in J(\eta)$ for $\xi \in J_c(\eta)$, i.e. if $2\sigma/\varepsilon \notin \bigcup_{i=-\infty}^{\infty} (i + 2\vartheta - 2\eta,\ i + 2\vartheta + 2\eta)$.

(89,4) and (90,4) imply that

$$(91,4)\quad \left|\int_{\tau_m}^{t} I_\xi\{h'(\cdot\cdot/..)\,[w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)]\, U(\xi, \sigma)\}\right| \leqq$$

$$\leqq \varepsilon K_{47}(\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta) + 4\varepsilon\eta K_{36}\,.\,2(R + 1)^2 \leqq \varepsilon K_{48}(\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta)\,.$$

As

$$\left|\int_{\tau_m}^{t} I_\xi\{h'(\cdot\cdot/..)\,[w_i(\xi, \sigma) + \beta\, u(\xi, \sigma)]\, U(-\xi + 2\sigma/\varepsilon, \sigma)]\,\mathrm{d}\sigma\right|$$

may be estimated in the same way as

$$\left|\int_{\tau_m}^{t} I_\xi\{h'(\cdot\cdot/..)\,[w_i(-\xi + 2\sigma/\varepsilon, \sigma) + \beta\, u(-\xi + 2\sigma/\varepsilon, \sigma)]\, U(\xi, \sigma)\}\right|,$$

it follows from (87,4), (88,4) and (91,4) that

$$(92,4)\qquad |B_n| \leqq K_{49}(\varepsilon\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta)\,, \quad (0 < \varepsilon \leqq \varepsilon_0 \leqq 1)\,.$$

Lemma 15,4 is a consequence of (86,4) and (92,4).

Lemma 16,4. *If* $\varepsilon_i, \varepsilon j \geqq N_2$, $0 \leqq t \leqq T$, $0 < \varepsilon \leqq \varepsilon_0$, *then*

(93,4)

$$\left\|\int_0^t \mathrm{D}_\sigma[F_1^*(W_j(\tau), \sigma, \varepsilon) - F_1^*(W_i(\tau), \sigma, \varepsilon) - F_0^*(W_j(\tau), \sigma, \varepsilon) + F_0^*(W_i(\tau), \sigma, \varepsilon)]\right\|_\eta \leqq$$

$$\leqq \varepsilon K_{54}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta]\,.$$

Proof. For $Y \in M$ put

$$f_1^*(Y, \tau, \varepsilon) = \frac{\partial F_1^*}{\partial \tau}(y, \tau, \varepsilon)\,,$$

i.e.

$$f_1^*(Y, \tau, \varepsilon)(\xi) = \tfrac{1}{2}H(Y(\xi) - Y(-\xi + 2\tau/\varepsilon))\,,$$

$$\varphi(\xi, \lambda) = \int_0^t [f_1^*(W_i(\sigma) + \lambda\, U(\sigma), \sigma, \varepsilon)(\xi) - f_0^*(W_i(\sigma) + \lambda\, U(\sigma))(\xi)]\,\mathrm{d}\sigma,$$

$$U(\sigma) = W_j(\sigma) - W_i(\sigma),$$

f_0^* being defined by (5,2). Then the value of the integral in (93,4) at ξ is equalto (cf. (20,4), (22,4) and

$$\int_\alpha^\beta \mathrm{D}_\sigma F(x(\tau), \sigma) = \int_\alpha^\beta \frac{\partial F}{\partial \sigma}(x(\sigma), \sigma)\,\mathrm{d}\sigma,\ \varphi(\xi, 1) - \varphi(\xi, 0) = \frac{\partial \varphi}{\partial \lambda}(\xi, \lambda)\,, \quad 0 < \lambda < 1\,,$$

(94,4) $$\frac{\partial\varphi}{\partial\lambda}(\xi,\lambda) = \int_0^t \left[\left(\frac{\partial f_1^*}{\partial Y}(W_i(\sigma) + \lambda\, U(\sigma), \sigma, \varepsilon)\, U(\sigma)\right)(\xi) - \right.$$
$$\left. - \left(\frac{\partial f_0^*}{\partial Y}(W_i(\sigma) + \lambda\, U(\sigma))\, U(\sigma)\right)(\xi)\right] d\sigma\,,$$

(95,4) $$\left(\frac{\partial f_1^*}{\partial Y}(W_i(\tau) + \lambda\, U(\tau), \sigma, \varepsilon)\, U(\tau)\right)(\xi) =$$
$$= \tfrac{1}{2} h(W_i(\xi,\tau) - W_i(-\xi + 2\sigma/\varepsilon, \tau) + \lambda\, U(\xi,\tau) - \lambda\, U(-\xi + 2\sigma/\varepsilon, \tau))\,.$$
$$.\,[U(\xi,\tau) - U(-\xi + 2\sigma/\varepsilon, \tau)]\,,$$

(96,4) $$\left(\frac{\partial f_0^*}{\partial Y}(W_i(\tau) + \lambda\, U(\tau), \sigma)\, U(\tau)\right)(\xi) =$$
$$= \frac{1}{4}\int_0^2 h(W_i(\xi,\tau) - W_i(-\xi + 2\sigma, \tau) + \lambda\, U(\xi,\tau) - \lambda\, U(-\xi + 2\sigma, \tau))\,.$$
$$.\,[U(\xi,\tau) - U(-\xi + 2\sigma, \tau)]\, d\sigma\,.$$

Put $\tau_m = m\varepsilon$, $m = 0, 1, 2, \ldots$, $\tau_n \leqq t < \tau_{n+1}$,

$$B_m = \int_{\tau_m}^{\tau_{m+1}} \left[\left(\frac{\partial f_1^*}{\partial Y}(W_i(\sigma) + \lambda\, U(\sigma), \sigma, \varepsilon)\, U(\sigma)\right)(\xi) - \right.$$
$$\left. - \left(\frac{\partial f_0^*}{\partial Y}(W_i(\sigma) + \lambda\, U(\sigma))\, U(\sigma)\right)(\xi)\right] d\sigma\,, \quad m = 0, 1, \ldots, n-1\,,$$
$$B_n = \int_{\tau_n}^{t} \left[\left(\frac{\partial f_1^*}{\partial Y}(W_i(\sigma) + \lambda\, U(\sigma), \sigma, \varepsilon)\, U(\sigma)\right)(\xi) - \right.$$
$$\left. - \left(\frac{\partial f_0^*}{\partial Y}(W_i(\sigma) + \lambda\, U(\sigma))\, U(\sigma)\right)(\xi)\right] d\sigma\,.$$

As $|h(\gamma)| \leqq K_{50}$ if $|\gamma| \leqq \mu_3$, it follows from (35,4), (95,4) and (96,4) that

(97,4) $$|B_n| \leqq \varepsilon K_{51}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta] \quad \text{if} \quad \xi \in J(\eta)\,.$$

It follows from the definitions of f_1^* and f_0^* (cf. (5,2)) that

$$\int_{\tau_m}^{\tau_{m+1}} [f_1^*(W_i(\tau_m) + \lambda\, U(\tau_m), \sigma, \varepsilon) - f_0^*(W_i(\tau_m) + \lambda\, U(\tau_m))]\, d\sigma = 0\,,$$

Hence

$$\int_{\tau_m}^{\tau_{m+1}} \left[\left(\frac{\partial f_1^*}{\partial Y}(W_i(\tau_m) + \lambda\, U(\tau_m), \sigma, \varepsilon)\, U(\tau_m)\right)(\xi) - \right.$$
$$\left. - \left(\frac{\partial f_0^*}{\partial Y}(W_i(\tau_m) + \lambda\, U(\tau_m))\, U(\tau_m)\right)(\xi)\right] d\sigma = 0$$

and

$$B_m = \int_{\tau_m}^{\tau_{m+1}} \left[\left(\frac{\partial f_1^*}{\partial Y} (W_i(\sigma) + \lambda\, U(\sigma), \sigma, \varepsilon))\, U(\sigma)\right)(\xi) - \right.$$

$$\left. - \left(\frac{\partial f_1^*}{\partial Y} (W_i(\tau_m) + \lambda\, U(\tau_m), \sigma, \varepsilon)\, U(\tau_m)\right)(\xi)\right] \mathrm{d}\sigma -$$

$$- \int_{\tau_m}^{\tau_{m+1}} \left[\left(\frac{\partial f_0^*}{\partial Y} (W_i(\sigma) + \lambda\, U(\sigma))\, U(\sigma)\right)(\xi) - \right.$$

$$\left. - \left(\frac{\partial f_0^*}{\partial Y} (W_i(\tau_m) + \lambda\, U(\tau_m))\, U(\tau_m)\right)(\xi)\right] \mathrm{d}\sigma = B_{m,1} + B_{m,2}\,.$$

As $|h(\gamma)| \leqq K_{50}$, $|h'(\gamma)| \leqq K_{52}$ for $|\gamma| \leqq \mu_3$, and as $\|(1 - \lambda)(W_i(\sigma) - W_i(\tau_m)) + \lambda(W_j(\sigma) - W_j(\tau_m)\|_M \leqq \varepsilon K_{34}$ according to (75,4), it follows from (95,4), (35,4) and (36,4) that $|B_{m,1}| \leqq \varepsilon^2 K_{53}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta]$ if $\xi \in J(\eta)$. $|B_{m,2}|$ may be estimated in the same manner; hence

(98,4) $$|B_m| \leqq 2\varepsilon^2 K_{53}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta] \quad \text{of} \quad \xi \in J(\eta) \quad \text{and}$$

Lemma 16,4 follows from (97,4) and (98,4).

The following Lemma is the required estimate of (23′,4).

Lemma 17,4. *If* $i\varepsilon, j\varepsilon \geqq N_2$, $0 \leqq \tau \leqq \tau$, $0 < \varepsilon \leqq \varepsilon_0$, *then*

(99,4) $$\|Y(j\varepsilon + \tau) - Y(i\varepsilon + \tau) - V_j(j\varepsilon + \tau) + V_i(i\varepsilon, \tau)\|_\eta \leqq$$

$$\leqq K_{56}[\|\tilde{Y}_j - \tilde{Y}_i\|_\eta (\varepsilon + \chi_1(\varepsilon, T)) + \eta]\,.$$

Proof. The right hand side of (99,4) is equal to $\|Y_j(\tau) - Y_i(\tau) - W_j(\tau) + W_i(\tau)\|_\eta$ (cf. the definition of Y_j, Y_i, W_j, W_i after (24,4)), (25,4) implies that

(100,4) $$\|Y_j(t) - Y_i(t) - W_j(t) + W_i(t)\|_\eta \leqq$$

$$\leqq \left\|\int_0^t \mathrm{D}_\sigma[F^*(Y_j(\tau), \sigma, \varepsilon) - F^*(W_j(\tau) - W_i(\tau) + Y_i(\tau), \sigma, \varepsilon)]\right\|_\eta +$$

$$+ \left\|\int_0^t \mathrm{D}_\sigma[F^*(W_j(\tau) - W_i(\tau) + Y_i(\tau), \sigma, \varepsilon) - F^*(Y_i(\tau), \sigma, \varepsilon) - \right.$$

$$\left. - F^*(W_j(\tau), \sigma, \varepsilon) + F^*(W_i(\tau), \sigma, \varepsilon)]\right\|_\eta +$$

$$+ \left\|\int_0^t \mathrm{D}_\sigma[F^*(W_j(\tau)\, \sigma, \varepsilon) - F^*(W_i(\tau), \sigma, \varepsilon) - F_0^*(W_j(\tau), \sigma) + F_0^*(W_i(\tau), \sigma)]\right\|_\eta\,.$$

According to Lemma 14,4

$$(101,4)\qquad \left\| \int_0^t \mathrm{D}_\sigma[F^*(Y_j(\tau), \sigma, \varepsilon) - F^*(W_j(\tau) - W_i(\tau) + Y_i(\tau), \sigma, \varepsilon)] \right\|_\eta \leqq$$
$$\leqq 2K_{34}\left[\int_0^t \| Y_j(\tau) - Y_i(\tau) - W_j(\tau) + W_i(\tau)\|_\eta \,\mathrm{d}\tau + \eta\right].$$

From Lemmas 16,4 and 15,4 one obtains

$$(102,4)\qquad \left\| \int_0^t \mathrm{D}_\sigma[F^*(W_j(\tau), \sigma, \varepsilon) - F^*(W_i(\tau), \sigma, \varepsilon) - F_0^*(W_j(\tau), \sigma) + F_0^*(W_i(\tau), \sigma)] \right\|_\eta \leqq$$
$$\leqq \varepsilon(K_{50} + K_{54})\,[\| \tilde{Y}_j - Y_i\|_\eta + \eta]\,.$$

Therefore (100,4), (101,4), (29,4) and (102,4) give

$$\| Y_j(t) - Y_i(t) - W_j(t) - W_i(t)\|_\eta \leqq$$
$$\leqq 2K_{34}\int_0^t \| Y_j(\tau) - Y_i(\tau) - W_j(\tau) + W_i(\tau)\|_\eta \,\mathrm{d}\tau + 2K_{34}\eta +$$
$$+ 2K_{33}(\eta + \| \tilde{Y}_i - \tilde{Y}_j\|_\eta \,\chi_1(\varepsilon, T)) + \varepsilon(K_{50} + K_{54})\,[\| \tilde{Y}_j - \tilde{Y}_i\|_\eta + \eta] \leqq$$
$$\leqq 2K_{34}\int_0^t \| Y_j(\tau) - Y_i(\tau) - W_j(\tau) + W_i(\tau)\|_\eta \,\mathrm{d}\tau + K_{55}[\| \tilde{Y}_i - \tilde{Y}_j\|_\eta (\varepsilon + \chi_1(\varepsilon, T)) + \eta]$$

and Lemma 17,4 follows by Gronwall's Lemma.

Lemma 18,4. *If* $j\varepsilon$, $i\varepsilon \geqq N_2$, $\frac{1}{2}T \leqq \tau \leqq T$ *(i, j being integers),* $0 < \varepsilon \leqq \varepsilon_0$, ε_0 *being small enough, then*

$$(103,4)\qquad \| Y_j(j\varepsilon + \tau) - Y_i(i\varepsilon + \tau)\|_\eta \leqq$$
$$\leqq \tfrac{3}{4}\| \tilde{Y}_j - \tilde{Y}_i\|_\eta + K_{58}\eta^{1/2}\,, \quad \tfrac{1}{2}T \leqq \tau \leqq T\,.$$

Proof. It follows from (99,4)

$$\| Y(j\varepsilon + \tau) - Y(i\varepsilon + \tau)\|_\eta \leqq \| V_j(j\varepsilon + \tau) - V_i(i\varepsilon + \tau)\|_\eta +$$
$$+ K_{56}(\varepsilon + \chi_1(\varepsilon, T))\,\| Y(j\varepsilon) - Y(i\varepsilon)\|_\eta + K_{56}\eta\,, \quad 0 \leqq \tau \leqq T\,.$$

As $\|\tilde{S}_j - \tilde{S}_i\|_M = \|\tilde{S}_j - \tilde{S}_i\|_\eta = \| \tilde{Y}_i - \tilde{Y}_j\|_\eta$ (cf. the definition of $\tilde{S}_i$ after (6,4)), (10,4), (15,4) and (7,4) imply that

$$\| V_j(j\varepsilon + \tau) - V_i(i\varepsilon + \tau)\|_M \leqq \| V_j(j\varepsilon + \tau) - S_j(j\varepsilon + \tau)\|_\eta +$$
$$+ \| S_j(j\varepsilon + \tau) - S_i(i\varepsilon + \tau)\|_\eta + \| S_i(i\varepsilon + \tau) - V_i(i\varepsilon + \tau)\|_\eta \leqq$$
$$\leqq K_{19}\exp(-\varkappa_5\tau)\,\| \tilde{Y}_i - \tilde{Y}_j\|_\eta + K_{57}\eta^{1/2}\,,$$
$$K_{57} = 8\mu_3 K_{22}\exp(K_{22}T)\,, \quad 0 \leqq \tau \leqq T\,.$$

Hence ($\eta < 1$)

$$\|Y(j\varepsilon + \tau) - Y(i\varepsilon + \tau)\|_\eta \leqq$$
$$\leqq \|\tilde{Y}_j - \tilde{Y}_i\|_\eta \left[K_{56}(\varepsilon + \chi_1(\varepsilon, T)) + K_{19} \exp(-\varkappa_5 \tau)\right] + (K_{57} + K_{56})\, \eta^{1/2},$$
$$0 \leqq \tau \leqq T.$$

Let ε_0 be so small that (cf. (24,4))

$$2K_{56}(\varepsilon_0 + \chi_1(\varepsilon_0, T)) + K_{19} \exp(-\tfrac{1}{2}\varkappa_5 T) = \tfrac{3}{4}.$$

Then (103,4) holds with $K_{58} = K_{57} + K_{56}$ and Lemma 18,4 is proved.

Lemma 19,4. *Let $L = L(\eta)$ be such a positive integer that*

$$(\tfrac{3}{4})^L (\mu_3 - \mu_2) \leqq 4K_{58}\eta^{1/2}, \quad N_3 = N_3(D, \eta) = N_2 + LT.$$

If εi, $\varepsilon j \geqq N_3$, then

(104,4) $$\|\tilde{Y}_j - \tilde{Y}_i\|_\eta \leqq 8K_{58}\eta^{1/2}.$$

Proof. Let $0 < \varepsilon \leqq \varepsilon_0$, $\varepsilon i, \varepsilon j \geqq N_3$. Let l be such a positive integer that $\frac{1}{2}T \leqq l\varepsilon \leqq T$. Put $j_1 = j - Ll$, $i_1 = i - Ll$. Obviously $\varepsilon i_1, \varepsilon j_1 \geqq N_2$. As $S_k(\xi, i\varepsilon) = Y_k(\xi, i\varepsilon) = \tilde{Y}_k(\xi)$ for $\xi \in J(\eta)$, $k = 0, 1, \ldots$, it follows from Lemma 1,4 that $\|\tilde{Y}_{j_1} - q^\backprime\|_\eta \leqq \mu_3 - \frac{1}{2}a$, $\|\tilde{Y}_{i_1} - q^\backprime\|_\eta \leqq \mu_3 - \frac{1}{2}a$; therefore $\|\tilde{Y}_{j_1} - \tilde{Y}_{i_1}\|_\eta \leqq \mu_3 - \mu_2$. It follows from Lemma 18,4 by induction that

$$\|\tilde{Y}_{j_1+kl} - \tilde{Y}_{i_1+kl}\|_\eta \leqq (\tfrac{3}{4})^k (\mu_3 - \mu_2) + 4(1 - (\tfrac{3}{4})^k)\, K_{58}\eta^{1/2}, \quad k = 0, 1, 2, \ldots.$$

Hence (104,4) follows for $k = L$ and Lemma 19,4 is proved.

Proof of Theorem 2,1. Let us choose γ, v, ϱ such that the conditions of Theorem 2,3 are satisfied. Let D be sufficiently large and choose a $\tilde{y} \in U(v, \gamma, \varrho)$, $\|\tilde{y}\|_M \leqq D$. (i) of Theorem 2,1 is consequence of Theorem 2,3.

Let $\varepsilon > 0$ be fixed. Let us choose an $\eta_1 > 0$; as $\|Z\|_{\eta_1} \leqq \|Z\|_\eta$ for $Z \in M$, $0 < \eta \leqq \eta_1$, it follows from Lemma 19,4 that $\|Y(j\varepsilon) - Y(i\varepsilon)\|_{\eta_1} \leqq 8K_{58}\eta^{1/2}$ if $\varepsilon i, \varepsilon j \geqq N_3(D, \eta)$. And it follows from Lemma 18,4 that

(105,4) $$\|Y(j\varepsilon + \tau) - Y(i\varepsilon + \tau)\|_{\eta_1} \leqq 8K_{58}\eta^{1/2} \quad \text{if} \quad \varepsilon i, \ \varepsilon j \geqq N_3(D, \eta),$$
$$\tfrac{1}{2}T \leqq \tau \leqq T.$$

Assuming that $4\varepsilon_0 < T$ one finds that $\|Y(j\varepsilon + \tau) - Y(i\varepsilon + \tau)\|_{\eta_1} \leqq 8K_{58}\eta^{1/2}$, if $\varepsilon i, \varepsilon j \geqq N_3(D, \eta) + \frac{1}{2}T$, $0 \leqq \tau \leqq \varepsilon$. Hence $Y(\xi, j\varepsilon + \tau)$ converges uniformly with $j \to \infty$ for $\xi \in J(\eta_1)$, $\tau \in \langle 0, \infty)$ and (ii) holds.

(iii) is a consequence of the fact that the convergence in (9,1) is uniform ($Y(\xi, \tau)$ being continuous). As $\|Y(j\varepsilon) - q^\backprime\|_\eta \leqq \mu_3 - \frac{1}{2}a < \frac{1}{2}\delta$ for $\varepsilon j \geqq N_1$ (cf. Lemma 1,4 and $Y(\xi, j\varepsilon) = \tilde{S}_i(\xi)$ for $\xi \in J(\eta)$) and as $\|Y(\tau) - Y(j\varepsilon)\|_M \leqq \varepsilon K_2(R + 2)$, (cf. Lemma

8,2, $\|y(i\varepsilon)\|_{L_1} \leqq R$, $\|Y(i\varepsilon)\| \leqq \frac{1}{2}\|y(i\varepsilon)\|_{L_1})$ for $j\varepsilon \leqq \tau \leqq (j + 1)\,\varepsilon$, $j = 0, 1, 2, \ldots$, it follows that $|Y(\xi, \tau + j\varepsilon) - q(\xi - \vartheta')| \leqq \frac{1}{2}\delta + \varepsilon K_2(R + 2)$ if $\xi \in J(\eta)$, $\varepsilon j \geqq N_1$, $\tau \geqq 0$ and (iv) follows from (9,1). Theorem 2,1 is proved.

Note 1,4. Observe that all estimates are independent on $\tilde{y}$, if $\tilde{y} \in U(v, \gamma, \varrho)$, $\|\tilde{y}\|_M \leqq D$ and that the constants K_{58}, K_2 and N_3 are independent on ε. Of course, the solution Y of (5,1) depends on $\tilde{y}$ and ε and so does Z (defined by (9,1)). Assertion (ii) of Theorem 2,1 may be strengthened (cf. (105,4)): The convergence in (9,1) is uniform in the following way: Let A be a subset of E_1 whose distance from the set $\{\vartheta + j\}$, $j = \ldots -1, 0, 1, \ldots$. Then for every $\beta_1 > 0$ there exists such a $B_1 > 0$ that

$$|Y(\xi, \tau + i\varepsilon) - Z(\xi, \tau)| \leqq \beta_1, \text{ if } \xi \in A,\ \tau \geqq 0,\ \tilde{y} \in U(v, \gamma, \varrho),\ \|\tilde{y}\|_M \leqq D,\ i\varepsilon \geqq B_1.$$

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