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REMARKS ON THE KERNEL OF A MATRIX SEMIGROUP

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The intersection of all the (two-sided) ideals of a semigroup S is, if not empty, an ideal K of S contained in every ideal of S. K is a simple subsemigroup of S, called the kernel of S ([1], p. 67). In 1928, SUSCHKEWITSCH [9] showed that every finite semigroup has a kernel and determined the structure thereof. Ever since then, the questions of the existence and structure of the kernel of a semigroup have been in the forefront of the development of the theory of semigroups.

The purpose of the present paper is to investigate these questions for a semigroup of matrices of finite degree over a field.

If S is a matrix semigroup, we shall let $m(S)$ denote the set of elements of S of minimal rank. Since the rank of the product of two matrices does not exceed that of either factor, $m(S)$ is clearly an ideal of S. If S contains a kernel K, then clearly $K \subseteq m(S)$. We shall show that if S is pseudo-invertible, i.e., some power of every element of S lies in a subsgroup of S, then $m(S)$ is a minimal ideal. It follows then from a theorem of MUNN [7] that $m(S)$ is completely simple in the sense of Rees (see [1] § 2.7). Moreover, if S is any matrix semigroup having a completely simple kernel K, then we must have $K = m(S)$.

An example is given in § 2 of a matrix semigroup S having a kernel K (necessarily not completely simple) such that $K \neq m(S)$. This example also answers negatively a question raised by CLIFFORD and PRESTON [1, p. 70], namely: *if K is the kernel of a semigroup S, is every left ideal of K also a left ideal of S?*

The question of the structure of the kernel is tantamount to that of the structure of a simple semigroup. Consequently, for us this question takes the form: *what is the structure of a simple semigroup of matrices (of finite degree)?* As we shall see the possibilities are rather limited.

First we note, from considerations of rank, that any idempotent element of a simple matrix semigroup K is primitive relative to K; hence if K contains an idempotent, it is completely simple (by definition). This excludes, for example, a wide class of non-completely simple (inverse) semigroups studied by Clifford [2]. In particular, for an example like that in § 2, K cannot contain an idempotent.

We prove that a simple matrix semigroup K is always contained in a completely simple semigroup S of matrices of the same degree as those of K. It follows easily from this, that if K contains a minimal left (or right) ideal, then it must be completely simple. This excludes those simple semigroups containing minimal left (but not minimal right) ideals, studied by MARIANNE TEISSER [11], R. CROISOT [3], and T. SAITÔ and S. HORI [8].

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Preliminaries. By M_n we shall denote the multiplicative semigroup of the algebra of all matrices of degree n over an arbitrary field. We shall also regard M_n as the semigroup of all linear transformations on an n-dimensional vector space. The product ab of two elements of M_n will be their composition, first a and then b. By the rank $\varrho(a)$ of a is meant the dimension of the range of a.

If $S \subseteq M_n$ is a subsemigroup, we let $m(S) = \{x \in S : \varrho(x) = \min_{y \in S} \varrho(y)\}$. If e is an idempotent of S, we shall let $H(e) = H(e, S)$ denote the maximal subgroup of S containing e.

Throughout this paper we shall assume a knowledge of the results and notation of [1, chapters, 1, 2 and 3] concerning completely o-simple semigroups.

1. We state first without proof the following fundamental lemma due to Suschkewitsch; a proof of which may be found in [6, p. 273].

1.1. Lemma. *If $a \in M_n$, then a lies in a subgroup of M_n if and only if $\varrho(a) = \varrho(a^2)$.*

Moreover, some power of every element of M_n has this property.

1.2. We shall say that a semigroup S is *pseudo-invertible* if some power of every element of S lies in a subgroup of S. (The notion of pseudo-invertibility is due to DRAZIN [4], and the above characterization is that of Munn [7].) Thus the last sentence of 1.1 states that M_n is pseudo-invertible.

1.3. Let J_i $(i = 0, 1, 2, \ldots, n)$ denote the elements of rank not exceeding i in M_n. It is well known (see e.g. [5], p. 162) that these are the only ideals of M_n. From [1, § 2.6] one readily deduces that the Rees factor semigroup J_i/J_{i-1} $(i = 1, 2, \ldots, n)$ are o-simple. Clearly each factor is pseudo-invertible since M_n is. Thus by a result of Munn (see [1] p. 82 or [7], Theorem 2) each factor J_i/J_{i-1} is completely o-simple.

1.4. Lemma. *If e, f and g are idempotents in M_n such that*

(i) $\varrho(e) = \varrho(f) = \varrho(g)$
(ii) $ef \in H(g, M_n)$

then $\{g, f\}$ is a left zero semigroup and $\{g, e\}$ is a right zero semigroup.

Proof. By 1.3 it suffices to prove that if e, f and g are idempotents in a completely o-simple semigroup S such that $ef \in H(g, S)$ then the conclusions of the lemma hold. Now applying the Rees representation theorem for completely o-simple semigroups [1, chapter 3] one may verify in a straightforward manner that $\{g, f\}$ and $\{g, e\}$ are, respectively, left and right zero subsemigroups of S.

1.5. Theorem. *If S is a pseudo-invertible semigroup of matrices, then $m(S)$ is a completely simple minimal ideal of S.*

Proof. Let $K = m(S)$. We first establish that K is a union of groups. Let $x \in K$, then $\varrho(x) = \varrho(x^2)$; hence by 1.1 x lies in a subgroup of M_n, say $H(e, M_n)$ for some idempotent e in M_n. Since S is pseudo-invertible, some power of x, say x^q, lies in a subgroup $H(f, S)$ of S, where $f = f^2 \in S$. Since $x^q \in H(e, M_n) \cap H(f, S)$, it is well known [1, p. 22] that we must have $f = e$ and $H(f, S) \subseteq H(e, M_n)$. Hence $exe = x$, and if y is the inverse of x^q in $H(e, S)$ then $y' = x^{q-1}y$ is a left inverse and $y'' = yx^{q-1}$ is a right inverse for x, both of which are in S. It follows that $y' = y''$ and that $x \in H(e, S)$. Since $H(e, S)$ meets K and K is an ideal, $H(e, S) \subseteq K$. Therefore K is a union of groups.

Now let I be any ideal of S. We wish to show that $K \subseteq I$. Let x be in K. x lies in some subgroup $H(e, S)$ of K. It is clear that $I \cap K$ is an ideal and is a union of groups since K is. Let f be an idempotent in $I \cap K$. Let $H(g, S)$ be the subgroup of $I \cap K$ which contains fe. Since e, f, and g are in K we have $\varrho(e) = \varrho(f) = \varrho(g)$. It now follows from 1.4 that $\{e, g\}$ is a left zero semigroup. In particular $e = eg$. Since $g \in I$, e must be also. Therefore $x = ex \in I$, implying $K \subseteq I$. Now by a theorem of Munn (Theorem 2.55, p. 82 of [1]) it follows that since K is simple and pseudo-invertible, K is completely simple, q.e.d.

1.6. Corollary. *If S is a finite semigroup of matrices, then $m(S)$ is the kernel of S.*

1.7. Theorem. *If S is a semigroup of matrices with a completely simple kernel K, then $K = m(S)$.*

Proof. Since $m(S)$ is an ideal, we need only show that $m(S) \subseteq K$. Let $x \in m(S)$. Then $\varrho(x) = \varrho(x^2)$ and by 1.1 $x \in H(e, M_n)$ for some $e = e^2 \in M_n$. Let y be an element of K. Then $xyx \in K$ and $xyx = exyxe \in eKe$. Since $K \subseteq m(S)$ we know that $\varrho(e) = \varrho(xyx)$. These facts imply that e and xyx have the same range and null space; hence by [1, 6(b), p. 57] e and xyx are in the same $\mathscr{H}$-class of M_n. Since this $\mathscr{H}$-class contains an idempotent, it is a group [1, p. 59]. Now since K is a union of groups and $xyx \in K$ we must have $e \in K$. Whence $x = ex \in K$, q.e.d.

1.8. Corollary. *If S is a compact semigroup of real or complex matrices, then $m(S)$ is the kernel of S.*

Proof. This follows from the well-known fact that a compact topological semigroup always has a completely simple kernel (see e.g. [12]).

2. Example. Define a product on Euclidean 4-space as follows:

$$(a, b, c, d)(x, y, z, w) = (a + bx + cw, by, bz, d).$$

This semigroup may be faithfully represented as a semigroup of real matrices by letting (x, y, z, w) correspond to

$$\begin{pmatrix} 1 & 0 & 0 \\ x & y & z \\ w & 0 & 0 \end{pmatrix}.$$

Associativity is then automatic. We now take K to be the subsemigroup consisting of all (x, y, z, w) with

$$x > 0,\ y > 0,\ z \geqq 0 \quad \text{and} \quad w \geqq 0.$$

Let $e = (0, 1, 0, 0)$ and set $S = K \cup \{e\}$. It is routine to verify that S is a semigroup and that K is the kernel of S. Clearly $m(S) = S$ and hence $m(S) \neq K$.

To see that S settles the question of Clifford and Preston mentioned in the introduction, let L be the principal left ideal of K generated by any element $p = (a, b, c, d)$ of K with $d > 0$. One may easily show that $ep \notin L$ and hence L is not a left ideal of S.

The following lemma is straightforward and we therefore omit its proof.

2.1. Lemma. *Let Q be a completely o-simple semigroup, and let S be a subsemigroup of $Q \setminus \{0\}$ such that every $\mathscr{H}$-class of Q meeting S is a group. Then the union T of all $\mathscr{H}$-classes of Q meeting S is a completely simple semigroup contained in $Q \setminus \{0\}$.*

We observe that if S is a simple subsemigroup of M_n, then $m(S) = S$ since $m(S)$ is an ideal of S. Hence $\varrho(x) = \varrho(x^2)$ for all x in S, and by 1.1, x lies in a subgroup $H(x, M_n)$ of M_n.

2.2. Theorem. *If S is a simple subsemigroup of M_n, then*

$$T = \bigcup \{H(x, M_n) : x \in S\}$$

is a completely simple subsemigroup of M_n.

Proof. As remarked above $m(S) = S$, and therefore every element of S has the same rank, say k. In the notation of 1.3, let $Q = J_k / J_{k-1}$. Identifying the elements of T with those of Q which correspond under the quotient mapping, one easily sees that T is the union of all $\mathscr{H}$-classes of Q meeting S. The theorem now follows from the lemma of 2.1.

2.3. Corollary. *If S is a simple semigroup of matrices, the following are equivalent:*

(i) *S is completely simple.*
(ii) *S contains a minimal left (right) ideal.*
(iii) *$a \in Sa \cup aS$ for some a in S.*
(iv) *S contains an idempotent.*

Proof. Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). To show that (iii) $\Rightarrow$ (iv), we first note that by 2.2 we may assume that S is contained in a completely simple semigroup of matrices. Using the Rees representation it is easily seen that $a = ba$ in a completely simple semigroup implies that $b^2 = b$. Similarly $a = ba$ implies $b^2 = b$. In either case S contains an idempotent. Now any idempotent in S must be primitive since every element of S has the same rank; so clearly (iv) implies (i).

Remark. SCHWARZ [10, p. 229] shows that if S is a simple subsemigroup of a completely o-simple semigroup T then $P_{\alpha\beta} = S \cap H_{\alpha\beta}$, where $H_{\alpha\beta}$ is an $\mathscr{H}$-class of T, is also simple and contains an idempotent if and only if S does. If S is completely simple then $P_{\alpha\beta}$ are isomorphic. Is this true if S is simple but not completely simple?

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Резюме

ЗАМЕТКИ О ЯДРЕ МАТРИЧНЫХ ПОЛУГРУПП

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Пусть S — полугруппа матриц конечного порядка. Пусть $m(S)$-множество всех элементов $\in S$, имеющих минимальный ранг. Если S содержит минимальный двусторонний идеал K (так называемое ядро S), то $K \subseteq m(S)$. На примере показано, что K необязательно равно $m(S)$.

Назовем S псевдо-инверзной полугруппой, если для всякого $a \in S$ существует натуральное число $n = n(a)$ так, что a^n содержится в некоторой подгруппе, лежащей в S. В работе доказывается, что для псевдо-инверзной полугруппы матриц S имеет место $m(S) = K$ и K-вполне простая полугруппа. Наоборот, если S-любая полугруппа матриц, для которой ядро K вполне простое, то $K = m(S)$.

Цитированый пример дает отрицательный ответ на один открытый вопрос из теории полугрупп. А именно, доказывается: Если K-ядро полугрупп S, то левый идеал из K не является необходимо левым идеалем из S.