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A SEMIGROUP TREATMENT OF SOME THEOREMS ON NON-NEGATIVE MATRICES

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Dedicated to Profesor A. D. WALLACE
on the occasion of his sixtieth birthday.

The purpose of this paper is to give a systematic treatment of the fundamental properties of non-negative matrices from the standpoint of the elementary theory of semigroups.

Let A be an $n \times n$ matrix with non-negative entries. In large parts of investigations concerning non-negative matrices their properties depend only on the distributions of zeros and "non-zeros" in the matrix (regardless of the actual numerical values of positive entries). One of the main problems is to study the behaviour of the iterations $A, A^2, A^3, \ldots$.

In this paper we give some applications of the rather elementary parts of the theory of semigroups to this problem. The substance is the following idea. We introduce the semigroup S of "$n \times n$ – matrix units" (as defined below). To every matrix A we associate a subset of S denoted by C_A and called the support of A. By means of Lemma 1 (see below) the multiplicative semigroup of all non-negative matrices is homomorphically mapped onto the semigroup $\mathfrak{S}$ of all subsets of S (the multiplication in $\mathfrak{S}$ being the multiplication of complexes). $\mathfrak{S}$ contains only a finite number of different elements and the main problem reduces to the study of the cyclic subsemigroup $\{C_A, C_A^2, C_A^3, \ldots\}$ of $\mathfrak{S}$. This subsemigroup reflects all properties of A which depend only on the distribution of zeros and "non-zeros".

The treatment essentially differs from the classical methods described in [2]. It is in a rather loose connection with the papers [4], [5], [6] and the probabilistic methods used in the theory of finite Markov chains (see e.g. [3], [7]).

Though, possibly, our treatement is not the shortest one it seems to bee very natural and it enables a clear insight into the nature of non-negative matrices.

From the standpoint of the algebraic theory of semigroups the method and some results may be considered as a first step toward the description of subsemigroups of completely 0-simple semigroups. In contradistinction to the case of a completely simple semigroup without zero (which has been treated in [8]) the last problem seems – in general – to be rather difficult.

Let $N = \{1, 2, \ldots, n\}$. Consider the set S of symbols $\{e_{ij} \mid i, j \in N\}$ together with a zero element 0 adjoined. Define in S a multiplication by

$$e_{ij}e_{ml} = \begin{cases} 0 & \text{if } j \neq m, \\ e_{il} & \text{if } j = m, \end{cases}$$

and $e_{ij} . 0 = 0 . e_{ij} = 0 . 0 = 0$ (for any $i, j \in N$). The set S with this multiplication is the simplest case of a non-commutative completely 0-simple semigroup (i.e. a finite semigroup S which does not contain any two-sided ideal of S different from 0 and S). It is often called "the semigroup of $n \times n$-matrix units".

Definition. Let $A = (a_{ij})$ be a non-negative $n \times n$ matrix. By C_A we shall denote the subset of S containing all such elements $e_{ij} \in S$ for which $a_{ij} > 0$ together with the zero element 0.

The set C_A will be called the support of A.

Lemma 1. *If A, B are non-negative, we have $C_{A+B} = C_A \cup C_B$ and $C_{AB} = C_A C_B$.*

Proof. The first statement is evident. We prove the second. Let $A = (a_{ik})$, $B = (b_{jl})$, $AB = (c_{\varrho\sigma})$.

a) If $e_{ik} \in C_{AB}$, then $c_{ik} = \sum_j a_{ij}b_{jk} > 0$. There is therefore at least one j such that $a_{ij}b_{jk} > 0$, i.e. $e_{ij} \in C_A$, $e_{jk} \in C_B$, hence $e_{ij}e_{jk} = e_{ik} \in C_A C_B$. This implies $C_{AB} \subset C_A C_B$.

b) Let conversely $e_{ij} \in C_A$, $e_{kl} \in C_B$, i.e. $e_{ij}e_{kl} \in C_A C_B$. If $j \neq k$, then $e_{ij}e_{kl} = 0 \in C_{AB}$. If $j = k$, i.e. $e_{ij}e_{jl} \in C_A C_B$, then $c_{il} = \sum_\tau a_{i\tau}b_{\tau l} \geqq a_{ij}b_{jl} > 0$, hence $e_{il} \in C_{AB}$. Therefore $C_A C_B \subset C_{AB}$. This proves Lemma 1.

Corollary. *For any non-negative matrix A we have $C_{A^h} = C_A^h$ for every integer $h \geqq 1$. In particular, if A is idempotent, then C_A is a subsemigroup of S with $C_A^2 = C_A$.*

Lemma 2. *For any non-negative $n \times n$ matrix A we have*

$$(1) \qquad C_A^{n+1} \subset C_A \cup C_A^2 \cup \ldots \cup C_A^n .$$

Proof. The elements of C_A^{n+1} are products of $n + 1$ elements $\in S$ of the form $e_{i_1 i_2} . e_{j_1 j_2} \ldots e_{k_1 k_2}$. Such a product is 0 except the case when the subscripts follow in the following order

$$(2) \qquad (i_1, i_2)(i_2, i_3) \ldots (i_m, i_{m+1})(i_{m+1}, i_{m+2}) \ldots (i_n, i_{n+1})(i_{n+1}, i_{n+2}) .$$

Since the numbers $i_1, i_2, \dots, i_{n+1}$ cannot be all different, there exists a couple, say $m < l$, such that $i_m = i_l$. The sequence (2) is of the form

$$\dots (i_{m-1}, i_m)(i_m, i_{m+1}) \dots (i_{l-1} i_m)(i_m, i_{l+1}) \dots$$

and the corresponding product is the same if we delete $(i_m, i_{m+1}) \dots (i_{l-1}, i_m)$. The product contains then at most n factors, i.e. it is yet contained in $C_A \cup C_A^2 \cup \dots \cup C_A^n$. This proves our Lemma.

The relation (1) implies (in an obvious way) $C_A^{n+\tau} \subset C_A \cup C_A^2 \cup \dots \cup C_A^n$ for any integer $\tau \geqq 1$. Therefore $[C_A \cup \dots \cup C_A^n][C_A \cup \dots \cup C_A^n] \subset [C_A \cup \dots \cup C_A^n]$.

This implies:

Corollary. *For any non-negative $n \times n$ matrix A the set $C_A \cup C_A^2 \cup \dots \cup C_A^n$ is a subsemigroup of S.*

Notation. The multiplicative semigroup of all non-empty subsets of S will be denoted by $\mathfrak{S}$.

Consider now a non-negative $n \times n$ matrix A, the sequence of powers

(3) $$A, A^2, A^3, \dots ,$$

and the sequence of corresponding supports

(4) $$C_A, C_A^2, C_A^3, \dots$$

While all elements in (3) may be different each from the other, the sequence (4) contains in any case only a finite number of different elements $\in \mathfrak{S}$.

Let k be the least integer such that $C_A^k = C_A^{l_1}$ for some integer $l_1 > k$. Let l be the least integer l_1 satisfying this relation. Then the sequence (4) is of the form

(5) $$C_A, C_A^2, \dots, C_A^{k-1} \mid C_A^k, C_A^{k+1}, \dots, C_A^{l-1} \mid C_A^k, C_A^{k+1}, \dots, C_A^{l-1} \mid \dots$$

and it contains exactly $l - 1$ different elements $\in \mathfrak{S}$. It is well known from the elements of the theory of finite semigroups that $\mathfrak{G}_A = \{C_A^k, C_A^{k+1}, \dots, C_A^{l-1}\}$ is a subgroup of $\mathfrak{S}$ of order $d = l - k$.

We have clearly $C_A^\alpha = C_A^{\alpha+\beta d}$ for every integer $\alpha \geqq k$ and every integer $\beta \geqq 0$.

The unit element of the group $\mathfrak{G}_A$ is C_A^ϱ with a suitably chosen ϱ satisfying $k \leqq \varrho \leqq$ $\leqq l - 1$. It is easy to show directly that $\varrho = \tau d$, where the integer τ is uniquely determined by the requirement $k \leqq \tau d \leqq l - 1 = k + d - 1$.

Moreover $\mathfrak{G}_A$ is a cyclic group, i.e. there is an integer t with $k \leqq t \leqq l - 1$ such that

$$\mathfrak{G}_A = \{C_A^t, C_A^{2t}, \dots, C_A^{dt}\} .$$

The number t is, in general, not uniquely determined but the set in the bracket is for any admissible t identical up to the order with the set $\{C_A^k, C_A^{k+1}, \dots, C_A^{l-1}\}$ and $C_A^{dt} = C_A^\varrho$.

Notation. Throughout all of the paper the integers $k = k(A)$, $d = d(A)$, $\varrho = \varrho(A)$ will always have the meaning just introduced. We shall suppose that the number t is fixed chosen. The subsemigroup of $\mathfrak{S}$ generated by C_A will be denoted by $\mathfrak{S}_A$.

Since $C_A^{\varrho} = C_A^{2\varrho}$, the set C_A^{ϱ} is a subsemigroup of S. We show that this is the unique subsemigroup of S among the elements $\in \mathfrak{G}_A$. Suppose for an indirect proof that $C_A^{\tau t}$, $1 \leqq \tau < d$ is a semigroup (subsemigroup of S), i.e. $C_A^{2\tau t} \subset C_A^{\tau t}$. This implies $C_A^{\tau t} \supset \supset C_A^{2\tau t} \supset C_A^{3\tau t} \supset \ldots \supset C_A^{d\tau t} = C_A^{\varrho}$, i.e. $C_A^{\varrho} \subset C_A^{\tau t}$. Therefore $C_A^{\varrho} . C_A^{\tau t} \subset C_A^{2\tau t}$. Since C_A^{ϱ} is the unit element of $\mathfrak{G}_A$ this says $C_A^{\tau t} \subset C_A^{2\tau t}$. Hence $C_A^{\tau t} = C_A^{2\tau t}$ in contradiction to the fact that $C_A^{\tau t}$ is not the unit element of $\mathfrak{G}_A$.

Remark. If $k > 1$, it may happen that one of the sets $C_A, C_A^2, \ldots, C_A^{k-1}$ is a semigroup. Let for instance $n = 2$ and $C_A = \{0, e_{12}\}$, then $C_A^2 = \{0\}$ and C_A is a semigroup, while $\mathfrak{G}_A = \{0\}$.

We summarise all these results as follows:

Lemma 3. *Let C_A be the support of a non-negative $n \times n$ matrix A. The sequence (4) contains a finite number of different elements $\in \mathfrak{S}$. These elements form (with respect to the multiplication of subsets) a subsemigroup $\mathfrak{S}_A$ of $\mathfrak{S}$. If the maximal group $\mathfrak{G}_A$ contained in $\mathfrak{S}_A$ has $d \geqq 1$ elements, then*

$$\mathfrak{S}_A = \{C_A, C_A^2, \ldots, C_A^{k-1}, C_A^k, \ldots C_A^{k+d-1}\}.$$

Hereby $k \geqq 1$ and $C_A^{k+d} = C_A^k$. The group $\mathfrak{G}_A = \{C_A^k, \ldots, C_A^{k+d-1}\}$ is cyclic and it contains a unique power C_A^{ϱ}, $k \leqq \varrho \leqq k + d - 1$, which itself considered as a subset of S is a semigroup. The set C_A^{ϱ} acts as the unit element of the group $\mathfrak{G}_A$.

II. IRREDUCIBLE MATRICES

A non − negative $n \times n$ matrix $A = (a_{ij})$ is called reducible if $N = \{1, 2, \ldots, n\}$ can be decomposed in two non − void disjoint subsets I, J such that $a_{ij} = 0$ for $i \in I$, $j \in J$. Otherwise it is called irreducible. If moreover $a_{ji} = 0$ for $j \in J$, $i \in I$, A is called completely reducible.

An equivalent definition is: A is said to be reducible if there is a permutation matrix P such that $P^{-1}AP$ is of the form

$$P^{-1}AP = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix},$$

where A_1 and A_2 are square matrices and 0 is a rectangular zero matrix. If moreover B is a zero matrix, then A is called completely reducible.

It is obvious what is the meaning of the words "a matrix A is completely reducible in u matrices" and "a matrix A is completely reducible into v irredubicle matrices".

It is well known that for a given A the number $v = v(A)$ of irreducible "matrix components" is uniquely determined. (See [2], p. 341.)

For convenience we shall adopt occasionally an analogous terminology for the subsets of S. A subset C of S is called reducible if $N = \{1, 2, \ldots, n\}$ can be decomposed in two non-empty disjoint sets I, J such that $\{e_{ij} \mid i \in I, j \in J\} \subset S - C$. If moreover $\{e_{ji} \mid j \in J, i \in I\} \subset S - C$, then C is called completely reducible.

If A is completely reducible into u matrices, then N can be decomposed into u non-empty disjoint sets $N = J_1 \cup J_2 \cup \ldots \cup J_u$ such that for any $\alpha \neq \beta$ we have $\{e_{ij} \mid i \in J_\alpha, j \in J_\beta\} \subset S - C_A$. Denoting $S_\alpha = \{e_{ij} \mid i \in J_\alpha, j \in J_\alpha\}$ we also have $C_A \subset \{0\} \cup S_1 \cup S_2 \cup \ldots \cup S_u$.

Conversely, if N can be decomposed into u non-empty disjoint sets $N = J_1 \cup J_2 \cup \cup \ldots \cup J_u$ and $C_A \subset \{0\} \cup S_1 \cup S_2 \cup \ldots \cup S_u$, then A is completely reducible in (at least) u matrices. Clearly: If $C_A = \{0\} \cup S_1 \cup S_2 \cup \ldots \cup S_u$, then A is completely reducible into u positive (and hence irreducible) matrices.

For further purposes we remark: If P is a permutation matrix, then $C_P \,.\, S = S \,.\, C_P = S$. Also an irreducible matrix cannot contain a zero row or column. Hence for such a matrix we have $C_A \,.\, S = S \,.\, C_A = S$. More generally $C_A^h \,.\, S = S \,.\, C_A^h = S$ for any integer $h \geqq 1$.

Theorem 1. *A non-negative $n \times n$ matrix A is irreducible if and only if*

$$C_A \cup C_A^2 \cup \ldots \cup C_A^n = S\,. \tag{6}$$

Proof. a) Suppose that A is reducible and $a_{ij} = 0$ for $i \in I$, $j \in J$ $(I \cap J = \emptyset, I \cup J = N)$, so that $e_{ij} \notin C_A$. Denote $A^2 = (b_{rs})$. For $i \in I$, $j \in J$ we then have $b_{ij} = \sum_{m \in I} a_{im} a_{mj} + \sum_{m \in J} a_{im} a_{mj} = 0$. Hence $e_{ij} \notin C_A^2$. Analogously $e_{ij} \notin C_A^h$ for any integer $h \geqq 1$. Therefore $C_A \cup C_A^2 \cup \ldots \cup C_A^n$ cannot be equal to S.

b) Suppose conversely that A is irreducible. We have to show that (6) holds.

Let $F_1 = \{e_{1i_1}, e_{1i_2}, \ldots e_{1i_r}\}$ be the „first row" of C_A. Hereby $r \geqq 1$. Suppose $r < n$. We shall show that $F_1 C_A$ (i.e. „the first row" of C_A^2) contains at least one non-zero element not contained in F_1. Suppose for an indirect proof that $F_1 C_A \subset F_1 \cup \{0\}$. This means: for every $e_{\varrho\sigma} \in C_A$ we have

$$\{e_{1i_1}, e_{1i_2}, \ldots, e_{1i_r}\}\, e_{\varrho\sigma} \subset \{e_{1i_1}, e_{1i_2}, \ldots, e_{1i_r}\} \cup \{0\}\,.$$

Hence, if $\varrho \in \{i_1, i_2, \ldots, i_r\}$, then σ is necessarily $\in \{i_1, i_2, \ldots, i_r\}$ and therefore C_A does not contain the elements $e_{\varrho\sigma}$, where $\varrho \in \{i_1, \ldots, i_r\}$ and $\sigma \in N - \{i_1, \ldots, i_r\}$. But this is equivalent to the statement that A is reducible, contrary to the assumption.

We have proved that $F_1 \cup F_1 C_A$ contains at least $r + 1$ non-zero elements. (Hereby $r + 1 \leqq n$). The same argument implies that $(F_1 \cup F_1 C_A) \cup (F_1 \cup F_1 C_A)\, C_A = F_1 \cup F_1 C_A \cup F_1 C_A^2$ contains at least $\min(n, r + 2)$ non-zero elements. Repeating this argument $n - 1$ times we obtain that $F_1 \cup F_1 C_A \cup \ldots \cup F_1 C_A^{n-1}$ (i.e. "the first row" of $C_A \cup C_A^2 \cup \ldots \cup C_A^n$) contains at least $\min(n, r + n - 1)$ non-zero elements. Since $r \geqq 1$ the last number is equal to n and since this argument can be applied to any „row" of C_A (and 0 is ex definitione contained in C_A and S) our Theorem is proved.

Our next goal is to prove Theorem 2 which gives (for our purposes) a more convenient criterium for the irreducibility of A.

Consider an irreducible non-negative matrix A and the semigroup $\mathfrak{S}_A$. Since $\mathfrak{S}_A$ contains all powers of C_A, we have with respect to Theorem 1

$$C_A \cup C_A^2 \cup \ldots \cup C_A^k \cup \ldots \cup C_A^{k+d-1} = S .$$

Note that $\{e_{11}, e_{22}, \ldots, e_{nn}\} \subset C_A^\varrho$. For, if $e_{ii} \in C_A^u$, $1 \leqq u \leqq k + d - 1$, then $e_{ii} \in C_A^{\lambda u}$ for every integer $\lambda \geqq 1$ and since some power of C_A^u is the idempotent $\in \mathfrak{G}_A$ (i.e. C_A^ϱ), we have $e_{ii} \in C_A^\varrho$.

The set $C_A^k \cup C_A^{k+1} \cup \ldots \cup C_A^{k+d-1}$ is a two-sided ideal of S. For

$$S[C_A^k \cup \ldots \cup C_A^{k+d-1}] = [C_A \cup C_A^2 \cup \ldots \cup C_A^{k+d-1}] [C_A^k \cup \ldots \cup C_A^{k+d-1}] =$$
$$= [C_A^k \cup \ldots \cup C_A^{k+d-1}] [C_A \cup C_A^2 \cup \ldots \cup C_A^{k+d-1}] \subset [C_A^k \cup \ldots \cup C_A^{k+d-1}] .$$

Now S is a 0-simple semigroup, hence it contains only the trivial two-sided ideals (i.e. 0 and S itself). Since $C_A^\varrho \neq \{0\}$, we necessarily have

$$C_A^k \cup C_A^{k+1} \cup \ldots \cup C_A^{k+d-1} = S .$$

Since the summands on the left hand side are exactly the elements $\in \mathfrak{G}_A$, this relation can be rewritten in the form

$$C_A^t \cup C_A^{2t} \cup \ldots \cup C_A^{dt} = S . \tag{7}$$

Notation. For brevity we shall write throughout the rest of the paper $C_A^t = D_A$. (Note again that taking an other admissible t we only influence the order of the sets $D_A, D_A^2, \ldots, D_A^d$).

Our result can be formulated as follows:

Theorem 2. *A non-negative matrix A is irreducible if and only if*

$$D_A \cup D_A^2 \cup \ldots \cup D_A^d = S . \tag{8}$$

We mention also that Theorem 1 and the relation (7) imply the following

Corollary. *If A is irreducible, then A^t is also irreducible.*

The next theorem locates the non-zero idempotents $\in S$.

Theorem 3. *For a non-negative irreducible $n \times n$ matrix A write in the sense of the foregoing theorem*

$$S = D_A \cup D_A^2 \cup \ldots \cup D_A^d .$$

Denote $E = \{e_{11}, e_{22}, \ldots, e_{nn}\}$. Then

a) $E \subset D_A^d$.

b) $E \cap D_A^\tau = \emptyset$ *for* $\tau = 1, 2, \ldots, d - 1$.

Proof. 1) For any idempotent $e_{ii} \in S$ there is certainly a $\tau_i (1 \leqq \tau_i \leqq d)$ such that $e_{ii} \in D^{\tau_i}$. This implies $e_{ii} = e_{ii}^d \in D_A^{\tau_i d} = D_A^d$. Hence $E \subset D_A^d$.

2) We next show that D_A^d is the unique summand in (8) containing the whole set E. If $d = 1$, there is nothing to prove. Suppose therefore $d > 1$. Let $\tau (1 \leqq \tau \leqq d)$ be an integer such that $E \subset D_A^\tau$ holds. If $e_{ij} \in D_A^\tau$, then $e_{ij} = e_{ij} e_{jj} \subset D_A^\tau E \subset$ $\subset D_A^\tau D_A^\tau = D_{\tau A}^2$, hence $D_A^\tau \subset D_A^{2\tau}$. This implies

$$D_A^\tau \subset D_A^{2\tau} \subset D_A^{3\tau} \subset \ldots \subset D_A^{(d-1)\tau} \subset D_A^{d\tau} = D_A^d \subset D_A^{(d+1)\tau} = D_A^\tau .$$

Hence $D_A^\tau = D_A^d$, q.e.d.

3) Thirdly we prove: Let s_i be the least integer such that $e_{ii} \in D_A^{s_i}$. Then s_i / d and $e_{ii} \in D_A^\tau$ if and only if s_i / τ.

To prove this suppose first $s_i \not| \, d$. Then we may write $d = \alpha s_i + \beta$, where α is an integer and $0 < \beta < s_i$. Clearly $e_{ii} \in D_A^{(\alpha+1)s_i}$. Hence $e_{ii} \in D_A^{\alpha s_i + \beta + s_i - \beta} = D_A^d D_A^{s_i - \beta} =$ $= D_A^{s_i - \beta}$. Since $0 < s_i - \beta < s_i$, this is a contradiction to the definition of s_i.

Suppose further that $e_{ii} \in D_A^\tau$ $(1 \leqq \tau \leqq d)$ and $s_i \not| \, \tau$. We then may write $\tau =$ $= \alpha s_i + \beta$ with $0 < \beta < s_i$. Further, since s_i / d, we have $e_{ii} \in D_A^{d - \alpha s_i}$. Hence $e_{ii} \in$ $\in D_A^{\alpha s_i + \beta} D_A^{d - \alpha s_i} = D_A^{d+\beta} = D_A^\beta$, which is again a contradiction to the definition of s_i.

4) Suppose now that s_i, s_j are the least integers for which $e_{ii} \in D_A^{s_i}$, $e_{jj} \in D_A^{s_j}$ respectively holds. We shall show that $s_i = s_j$.

Consider the relations $e_{jj} = e_{ji} e_{ii} e_{ij}$ and $e_{ii} = e_{ij} e_{jj} e_{ji}$. With respect to (8) there are integers $\alpha, \beta (1 \leqq \alpha \leqq d,\ 1 \leqq \beta \leqq d)$ such that $e_{ji} \in D_A^\alpha$ and $e_{ij} \in D^\beta$. Hence

$$e_{jj} \in D_A^{\alpha + s_i + \beta} \quad \text{and} \quad e_{ii} \in D_A^{\beta + s_j + \alpha} .$$

Therefore $s_j / \alpha + \beta + s_i$ and $s_i / \alpha + \beta + s_j$. Now $e_{jj} = e_{ji} e_{ij} \in D_A^{\alpha+\beta}$ implies $s_j / \alpha + \beta$, therefore s_j / s_i. Analogously s_i / s_j. This proves $s_i = s_j$.

The relation $s_i = s_j$ implies $E \subset D^{s_i} = D^{s_j}$. But by 2) the unique summand in (8) having this property is D_A^d. Hence $D_A^{s_i} = D_A^{s_j} = D_A^d$ and $s_i = d$ for every $i =$ $= 1, 2, \ldots, n$. This proves Theorem 3.

Corollary. *For an irreducible matrix A the number d is the least integer s for which $E \subset D^s$ holds.*

The next theorem is of a decisive importance for all the paper.

Theorem 4. *The sets $D_A, D_A^2, \ldots, D_A^d$ are pairwise quasidisjoint (i.e. the intersection of any two of them is the zero element 0).*

Proof. Suppose for an indirect proof that there is a couple (i, j), $1 \leqq i < j \leqq d$, such that $D_A^i \cap D_A^j \neq \{0\}$. Consider the set

$$T = \bigcup_{\alpha < \beta} [D_A^\alpha \cap D_A^\beta] , \quad \alpha = 1, 2, \ldots, d-1 \,;\ \beta = 1, 2, \ldots, d .$$

By supposition $T \neq \{0\}$. For any κ $(1 \leqq \kappa \leqq d)$ we have

$$D_A^\kappa [D_A^\alpha \cap D_A^\beta] = [D_A^\alpha \cap D_A^\beta] \, D_A^\kappa \subset D_A^{\alpha+\kappa} \cap D_A^{\beta+\kappa} .$$

Hereby $D_A^{\alpha+\kappa} \neq D_A^{\beta+\kappa}$, since $(\beta + \kappa) - (\alpha + \kappa) = \beta - \alpha$ is not divisible by d. Therefore $D^\kappa T = TD^\kappa \subset T$ and $ST = TS \subset T$. This says that T is a two-sided ideal of S. Since S is a 0-simple semigroup and $T \neq \{0\}$, we necessarily have $T = S$, i.e.

$$\bigcup_{\alpha<\beta} [D_A^\alpha \cap D_A^\beta] = D_A \cup D_A^2 \cup \ldots \cup D_A^d .$$

The set on the left hand side of this relation is contained in $D_A \cup D_A^2 \cup \ldots \cup D_A^{d-1}$. Hence

$$D_A^d \subset D_A \cup D_A^2 \cup \ldots \cup D_A^{d-1} .$$

But this is impossible since (by Theorem 3) D_A^d contains E, while $D_A \cup \ldots \cup D_A^{d-1}$ does not contain any non-zero idempotent $\in S$ at all. This proves Theorem 4.

Theorem 5. *For an irreducible non-negative $n \times n$ matrix A the number d satisfies the relation $1 \leqq d \leqq n$.*

Proof. By Theorem 1 $C_A \cup C_A^2 \cup \ldots \cup C_A^n = S$. If $k > 1$, multiply this relation by C_A^{k-1}. Since $C_A^{k-1}S = S$, we get

$$C_A^k \cup C_A^{k+1} \cup \ldots \cup C_A^{k+n-1} = S .$$

All summands on the left hand side are contained in $\mathfrak{G}_A$. Comparing with the relation (see Theorem 2)

$$C_A^k \cup C_A^{k+1} \cup \ldots \cup C_A^{k+d-1} = S ,$$

in which no summand can be deleted (since all are quasidisjoint), we obtain that $d \leqq n$, q.e.d.

A further characterization of the number d will be given in Theorem 7 below. But before we now give some informations concerning the "small powers" of C_A.

Theorem 6. *For a non-negative irreducible matrix A we have:*

a) *The sets $C_A, C_A^2, \ldots, C_A^d$ are quasidisjoint. More generally: Any consecutive d members $C_A^v, C_A^{v+1}, \ldots, C_A^{v+d-1}$ (for a $v \geqq 1$) are quasidisjoint.*

b) *For any $v \geqq 1$ we have*

$$C_A^v \cup C_A^{d+v} \cup C_A^{2d+v} \cup \ldots = C_A^{td+v} . \tag{9}$$

Proof. a) Since $E \subset C_A^{dt}$ and $C_A^v = C_A^v E$, we have $C_A^v \subset C_A^v C_A^{dt} = C_A^{dt+v}$, whence $C_A^{v+1} \subset C_A^{dt+v+1}, \ldots, C_A^{v+d-1} \subset C_A^{dt+v+d-1}$. Since $\{C_A^{dt+v}, \ldots, C_A^{dt+v+d-1}\}$ are exactly all elements $\in \mathfrak{G}_A$, and these are quasidisjoint, our statement is evident.

b) The relation $C_A \subset C_A^{dt+1}$ implies $C_A^{d+1} \subset C_A^{dt+d+1} = C_A^{dt+1}$, analogously $C_A^{2d+1} \subset C_A^{dt+1}$, etc., whence

$$C_A \cup C_A^{d+1} \cup C_A^{2d+1} \cup \ldots \subset C_A^{td+1} .$$

Since the converse inclusion is obvious, we have

$$C_A \cup C_A^{d+1} \cup C_A^{2d+1} \cup \ldots = C_A^{td+1},$$

whence (9) immediately follows.

Theorem 7. *For a non-negative irreducible matrix A the number d is the greatest common divisor of all natural numbers α such that $E \cap C_A^\alpha \neq \emptyset$.*

Proof. If a non-zero idempotent $e_{ii} \in S$ is contained in $C_A^{v_i}$, then by (9) $e_{ii} \in C_A^{td+v_i}$. Since $C_A^{td+v_i} \in \mathfrak{G}_A$, we have $C_A^{td+v_i} = D_A^{u_i} = C_A^{tu_i}$ with some integer $u_i \geqq 1$. This implies (by Theorem 3b) d/u_i and since $tu_i \equiv td + v_i \pmod d$, we have d/v_i. Hence a non-zero idepotent $\in S$ can be contained only in some powers of the form C_A^{ud} with $1 \leqq u \leqq t - 1$ and it is certainly contained in all the following powers C_A^{dt}, $C_A^{(t+1)d}$, $C_A^{(t+2)d}, \ldots$ The greatest common divisor of the numbers $\{ud\}$ and td, $(t + 1)d$, $(t + 2)d, \ldots$ is clearly d.

Consider the relation (8) and define $D_A^0 = D_A^d$. We close this section with the following.

Lemma 5. *If A is irreducible and $e_{ij} \in D_A^\tau$ $(1 \leqq \tau \leqq d)$, then $e_{ji} \in D_A^{d-\tau}$.*

Proof. With respect to (8) there is an s $(0 \leqq s \leqq d - 1)$ such that $e_{ji} \in D_A^s$. We have

$$e_{ii} = e_{ij} e_{ji} \in D_A^\tau D_A^s = D_A^{\tau+s}.$$

Since $e_{ii} \in D_A^d$, we have $\tau + s = d$, q.e.d.

Corollary. *If A is irreducible and $e_{ij} \in D_A^d$, we also have $e_{ji} \in D_A^d$.*

III. THE POWERS OF AN IRREDUCIBLE MATRIX

We shall now study the powers of an irreducible non-negative matrix A.

Let $u \geqq 1$ be any integer. Consider the sequence

$$A^u, A^{2u}, A^{3u}, \ldots$$

The set of the corresponding supports

$$C_A^u, C_A^{2u}, C_A^{3u}, \ldots, C_A^{(l-1)u} \tag{10}$$

is clearly a subsemigroup of the semigroup

$$\mathfrak{S}_A = \{C_A, C_A^2, \ldots, C_A^{k-1}, C_A^k, \ldots, C_A^{k+d-1}\}.$$

Hence the maximal group contained in (10) is a subgroup of

$$\mathfrak{G}_A = \{C_A^k, \ldots, C_A^{k+d-1}\} = \{D_A, \ldots, D_A^d\}. \tag{11}$$

If $\alpha u \geqq k$, then $\mathfrak{G}_{A^u} = \{C_A^{\alpha u}, C_A^{(\alpha+1)u}, C_A^{(\alpha+2)u}, \ldots\}$. Two sets $C_A^{\beta u}$, $C_A^{\gamma u}$ are identical if and only if $\beta u \equiv \gamma u \pmod{d}$, i.e. $\beta \equiv \gamma\ [\mathrm{mod}\ d/(d, u)]$. Denote $u_1 = (d, u)$, $d_1 = d/u_1$. Then

$$\mathfrak{G}_{A^u} = \{C_A^{\alpha u}, C_A^{(\alpha+1)u}, \ldots, C_A^{(\alpha+d_1-1)u}\}.$$

This is a subgroup of $\mathfrak{G}_A$ of order d_1, so that we may write

$$\mathfrak{G}_{A^u} = \{D_A^{u_1}, D_A^{2u_1}, \ldots, D_A^{d_1 u_1}\}. \tag{12}$$

A formally other expression for the group $\mathfrak{G}_{A^u}$ is obtained as follows. Consider the subgroup of $\mathfrak{G}_A$ generated by D_A^u, i.e. the subgroup

$$\{D_A^u, D_A^{2u}, \ldots, D_A^{du}\}. \tag{13}$$

Here $D_A^{\alpha u} = D_A^{\beta u}$, i.e. $C_A^{t\alpha u} = C_A^{t\beta u}$, if and only if $\alpha t u \equiv \beta t u \pmod{d}$, i.e. $\alpha \equiv \beta \pmod{d_1}$. Hence (13) contains exactly d_1 different elements

$$\{D_A^u, D_A^{2u}, \ldots, D_A^{d_1 u}\}. \tag{14}$$

This is a subgroup of order d_1 of $\mathfrak{G}_A$, hence it is identical with (12).

Summarily we have proved:

Lemma 6. *If the maximal group $\mathfrak{G}_A$ is given by (11) and $u_1 = (d, u)$, then $\mathfrak{G}_{A^u}$ is of order $d_1 = d/u_1$ and $\mathfrak{G}_{A^u}$ is given by (12) or (14).*

Lemma 7. *If A is irreducible and some power $A^v (v > 1)$ is reducible, then A^v is completely reducible into irreducible matrices.*

Proof. a) We first prove: If $A^v (v > 1)$ is reducible and N can be decomposed in two non-void disjoint subsets $N = I \cup J$ such that $e_{ij} \notin C_A^v$ for $i \in I$, $j \in J$, we then also have $e_{ji} \notin C_A^v$ for $j \in J$, $i \in I$. (Hence A^v is completely reducible.)

To prove this note first that $e_{ij} \notin C_A^v$ (for $i \in I$, $j \in J$) implies $e_{ij} \notin C_A^{v(dt+1)} = C_A^{dt+v}$. Since (by Theorem 6) $C_A^v \subset C_A^{dt+v}$, it is sufficient to prove that $e_{ji} \notin C_A^{dt+v}$. Now $C_A^{dt+v} = D_A^\tau$ with some τ, $1 \leqq \tau \leqq d$. Hence it is sufficient to prove that if $e_{ij} \notin$ $\notin D_A^\tau (i \in I, j \in J)$, we also have $e_{ji} \notin D_A^\tau (j \in J, i \in I)$. Suppose for an indirect proof that $e_{ji} \in D_A^\tau$. By Lemma 5 we then have $e_{ij} \in D_A^{d-\tau} = D_A^{d-\tau} D_A^{(\tau-1)d} = D_A^{\tau(d-1)}$ (hereby $D_A^0 = D_A^d$). On the other hand $e_{ij} \notin D_A^\tau (i \in I, j \in J)$ implies $e_{ij} \notin D_A^{\tau(d-1)}$. This contradiction proves our statement.

b) Suppose now that A is irreducible and $A^v (v > 1)$ is reducible. By a) A^v is completely reducible and there is a permutation matrix P such that

$$PA^vP^{-1} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

If A_1, A_2 are irreducible, Lemma 7 holds. Suppose therefore that e.g. A_1 is reducible. Then there is a permutation matrix Q such that

$$QA^vQ^{-1} = \begin{pmatrix} A_1' & 0 & 0 \\ B_1 & A_1'' & 0 \\ 0 & 0 & A_2 \end{pmatrix}$$

(Since A is irreducible none of the diagonal block matrices can be a zero matrix). Now the last matrix can be considered as a reducible matrix of the form $\begin{pmatrix} A_1' & 0 \\ B & M \end{pmatrix}$, where $B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$, $M = \begin{pmatrix} A_1'' & 0 \\ 0 & A_2 \end{pmatrix}$. Since QAQ^{-1} is irreducible and $(QAQ^{-1})^v$ is reducible, and of the form $\begin{pmatrix} A_1' & 0 \\ B & M \end{pmatrix}$, it follows by the statement proved sub a) that B is necessarily a rectangular zero matrix. Hence B_1 is a zero matrix and we have $QA^vQ^{-1} = \operatorname{diag}(A_1', A_1'', A_2)$.

This proceeding can be repeated until all diagonal square matrices are irreducible. This proves Lemma 7.

Theorem 8. *Let A be a non-negative irreducible matrix. Denote $u_1 = (d, u)$. Then A^u is completely reducible into u_1 irreducible matrices.*

Proof. a) We first show that A^{tu} is complety reducible into u_1 irreducible matrices Denote

$$Z_u = D_A^{u_1} \cup D_A^{2u_1} \cup \ldots \cup D_A^{d_1u_1}.$$

The set Z_u, which is equal to $D_A^u \cup D_A^{2u} \cup \ldots \cup D_A^{d_1u}$ is the support of the matrix $B = A^{ut} + A^{2ut} + \ldots + A^{d_1ut}$.

Consider the decomposition of S into quasidisjoint summands of the following form

$$S = Z_u \cup D_AZ_u \cup D_A^2Z_u \cup \ldots \cup D_A^{u_1-1}Z_u.$$

Define also $Z_u = D_A^0Z_u = D_A^{u_1}Z_u$.

For $\kappa = 1, 2, \ldots, u_1$ let $J_\kappa = \{i_1^{(\kappa)}, i_2^{(\kappa)}, \ldots, i_{\sigma_\kappa}^{(\kappa)}\}$ be the set of all indices such that

$$\{e_{1i_1^{(\kappa)}}, e_{1i_2^{(\kappa)}}, \ldots, e_{1i_{\sigma_\kappa}^{(\kappa)}}\} \in D_A^\kappa Z_u.$$

We have $J_1 \cup J_2 \cup \ldots \cup J_{u_1} = N$ and $J_\kappa \cap J_\lambda = \emptyset$ for $\kappa \neq \lambda$. Moreover $J_\kappa \neq \emptyset$ for every $\kappa = 1, 2, \ldots, u_1$. For, if there were $J_{\kappa_0} = \emptyset$ for some $\kappa_0 (1 \leqq \kappa_0 \leqq u_1)$ the "first row" of $D_A^{\kappa_0}Z_u$ would consist of zeros, and consequently the same would be true for $D_A^{\kappa_0+h}Z_u$ for every integer $h \geqq 1$. But this is impossible, since then $D_A^{\kappa_0}Z_u \cup \cup D_A^{\kappa_0+1}Z_u \cup \ldots \cup D_A^{\kappa_0+u_1-1}Z_u$ cannot be equal to S, a contradiction with the irreducibility of A.

We next prove that

α) for every κ we have $S_\kappa = \{e_{il} \mid i \in J_\kappa; \, l \in J_\kappa\} \subset Z_u$,

β) while if $i \in J_\kappa$, $l \in J_\lambda$, $\kappa \neq \lambda$, we have $e_{il} \notin Z_u$.

α) Since $e_{1i} \in D_A^\kappa Z_u = \bigcup_{\varrho=0}^{d_1-1} D_A^{\varrho u_1 + \kappa}$, we have by Lemma 5

$$e_{i1} \in \bigcup_{\varrho=0}^{d_1-1} D_A^{d_1 u_1 - \varrho u_1 - \kappa} = D_A^{d_1 u_1 - \kappa} \bigcup_{\varrho=0}^{d_1-1} D_A^{d_1 u_1 - \varrho u_1} = D_A^{d_1 u_1 - \kappa} Z_u ,$$

hence

$$e_{il} = e_{i1} e_{1l} \in D_A^{d_1 u_1 - \kappa} Z_u D_A^\kappa Z_u = D_A^{d_1 u_1} Z_u^2 = Z_u .$$

β) Since $e_{1i} \in D_A^\kappa Z_u$, we have $e_{i1} \in D_A^{d_1 u_1 - \kappa} Z_u$ and

$$e_{il} = e_{i1} e_{1l} \in D_A^{d_1 u_1 - \kappa} Z_u D_A^\lambda Z_u = D_A^{d_1 u_1 - \kappa + \lambda} Z_u$$

and this last set is different from Z_u since $d_1 u_1 - \kappa + \lambda = d - \kappa + \lambda \not\equiv 0 (\text{mod } u_1)$.

Now since $N = J_1 \cup J_2 \cup \ldots \cup J_{u_1}$, we have

$$Z_u = \{0\} \cup S_1 \cup S_2 \cup \ldots \cup S_{u_1} , \tag{15}$$

where $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$.

The relation (15) shows that B is completely reducible into u_1 positive (and hence irreducible) matrices. This implies that A^{ut} is completely reducible into u_1 irreducible matrices. For, if A^{ut} were (completely) reducible into $u_2 > u_1$ matrices, B would be completely reducible into u_2 matrices, a contradiction with (15).

b) Consider now the matrix A^u. A^u is either irreducible or by Lemma 7 completely reducible into irreducible matrices, i.e. there is a permutation matrix P such that

$$P^{-1} A^u P = \text{diag}(B_1, B_2, \ldots, B_\sigma) \tag{16}$$

with B_i irreducible and $\sigma \geqq 1$. By a) A^{tu} is completely reducible into $u_1 = (u, d)$ irreducible matrices. Since (16) implies

$$P^{-1} A^{tu} P = \text{diag}(B_1^t, B_2^t, \ldots, B_\sigma^t) ,$$

we clearly have $\sigma \leqq u_1$.

On the other hand recall that $C_A^u \subset C_A^{u+td}$. Since $C_A^{u+td} \in \mathfrak{G}_A$, we have $C_A^{u+dt} = C_A^{wt}$ for some w $(1 \leqq w \leqq d)$. Hence $u + dt \equiv wt (\text{mod } d)$. Now $(d, u + dt) = (d, wt)$, and since $(d, t) = 1$, we have $(d, w) = (d, u)$. By a) the matrix A^{wt} is completely reducible into $(w, d) = (u, d) = u_1$ irreducible matrices. Since the support of A^u is a subset of the support of A^{wt}, we conclude that A^u is completely reducible in at least u_1 irreducible matrices. Therefore $\sigma \geqq u_1$. The equality $\sigma = u_1$ completes the proof of our theorem.

An immediate consequence of Theorem 8 is

Theorem 9. *If A is a non-negative irreducible matrix, then A^u is irreducible if and only if $(u, d) = 1$.*

In the course of the proof of Theorem 8 we also proved the following

Corollary. *With the same notations as above the matrix $A^{ut} + A^{2ut} + \dots + A^{d_1 ut}$ is completely reducible into u_1 positive matrices.*

Remark. It should be noted that the matrix A^{ut} itself is (completely) reducible into not necessarily positive matrices. This is shown on the following simple example. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad A^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here we have $t = 1$, $d = 4$. Choose $u = 2$. Then $d_1 = (2, 4) = 2$ and

$$A^2 + A^4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

is completely reducible into two positive matrices while A^2 is reducible only in two non-negative matrices.

Taking $u = d$ we also have the following

Corollary. *Under the same suppositions as above the matrix A^{td} is completely reducible into d positive matrices.*

IV. DECOMPOSITION INTO PRIMITIVE MATRICES

Definition. A non-negative matrix A is said to be primitive if there is an integer w such that $C_A^w = S$.

A positive matrix is primitive. A reducible matrix cannot be primitive.

Theorem 2 implies immediately:

Theorem 10. *A non-negative irreducible matrix is primitive if and only if $d = d(A) = 1$.*

The sequence (5) has then the form

$$C_A, C_A^2, \ldots, C_A^{k-1}, C_A^k = C_A^{k+1} = \ldots$$

Theorem 11. *If A is primitive, then every power A^u is primitive. Conversely, if A^w for some $w \geqq 1$ is primitive, then A is primitive.*

Proof. a) $C_A^v = S$ implies $C_A^{uv} = C_{A^u}^v = S$. b) By supposition there is an integer ϱ such that $(C_{A^w})^\varrho = S$. This implies $(C_A)^{w\varrho} = S$, which says that A is primitive.

Let A be irreducible and card $\mathfrak{G}_A = d$. Suppose that $A^2, A^3, \ldots, A^n$ are all irreducible. By Theorem 9 we necessarily have $(1, d) = (2, d) = \ldots = (n, d) = 1$. Since (by Theorem 5) $d \leqq n$, this implies $d = 1$. Hence

Theorem 12. *If $A, A^2, \ldots, A^n$ are all irreducible, then A is primitive.*

Theorem 13. *Let A be non-negative irreducible matrix with card $\mathfrak{G}_A = d$. Then A^d is completely reducible into d primitive matrices and d is the least integer u for which A^u is reducible into primitive matrices.*

Proof. a) By Theorem 8 the matrix A^d is completely reducible into d irreducible matrices, i.e. there is a permutation matrix P such that

$$PA^dP^{-1} = \operatorname{diag}(A_1, \ldots, A_d)$$

with irreducible $A_1, \ldots, A_d$. This relation implies

$$PA^{dt}P^{-1} = \operatorname{diag}(A_1^t, \ldots, A_d^t)\,.$$

Now by the Corollary at the end of section III A^{td} is completely reducible into d positive matrices. Hence $A_1^t, \ldots, A_d^t$ are positive and therefore $A_1, \ldots, A_d$ are primitive. (We use hereby that fact that the decomposition and diagonalization of A^{td} into positive matrices is up to the order of summands uniquely determined).

b) Let now $u < d$. Denote $u_1 = (u, d) < d$. A^u can be decomposed into u_1 irreducible matrices, i.e. there is a permutation matrix Q such that

$$Q^{-1}A^uQ = \operatorname{diag}(B_1, \ldots, B_{u_1})\,. \tag{17}$$

If all $B_1, \ldots, B_{u_1}$ were primitive there would exist a number w_0 such that for all $w > w_0$ the matrices $B_1^w, \ldots, B_{u_1}^w$ would be positive. Choose h so that $hdt > w_0$. Then (17) implies

$$Q^{-1}A^{uhdt}Q = \operatorname{diag}(B_1^{hdt}, \ldots, B_{u_1}^{hdt})\,.$$

The matrix to the left is of the form $\operatorname{diag}(A_1, \ldots, A_d)$ with positive $A_1, \ldots, A_d$, while to the right we have only u_1 positive matrices. This contradiction proves that $B_1, \ldots, B_{u_1}$ cannot be all primitive, which completes the proof of our theorem.

For a primitive matrix A there is an integer w such that $C_A^w = S$. In this section we find estimations for the number w.

The results formulated in Theorems 14 and 16 are known. (See [1], [5].)

We begin with the following

Lemma 8. *If A is irreducible and A has at least one non-zero in the main diagonal, then A is primitive.*

Proof. Since C_A contains a non-zero idempotent $\in S$, so does C_A^v for every $v > 1$. The group $\mathfrak{G}_A = \{D_A, \ldots, D_A^d\}$ contains necessarily a unique element since otherwise we would have a contradiction with Theorem 3. Since $d = 1$, A is primitive.

Lemma 9. *Suppose that A is a non-negative irreducible $n \times n$ matrix $(n > 1)$ containing $r > 0$ non-zero elements in the main diagonal, i.e. $\{e_{\beta_1\beta_1}, \ldots, e_{\beta_r\beta_r}\} \subset$ $\subset C_A$. Denote $B = \{\beta_1, \ldots, \beta_r\}$. Then*

a) *To every $j \in N$ there is a $\beta \in B$ and an $s = s(j)$ such that $e_{j\beta} \in C_A^s$. Hereby: If $j \in B$, we may choose $s = 1$. If $B \neq N$ and $j \in N - B$, we may choose $s = s(j) \leqq$ $\leqq n - r$.*

b) *For any $l \in N$ and any $\beta \in B$ we have $e_{\beta l} \in C_A^{n-1}$.*

Proof. a) If $j \in B$, then $e_{jj} \in C_A$ and our statement is true with $s = 1$. We may restrict ourselves to the case $B \neq N$ and $j \in N - B$.

Suppose for an indirect proof that C_A^v, $v > n - r$, is the least power of C_A for which $e_{j\beta} \in C_A^v$ holds (for some $\beta \in B$). Then there exist v different integers $j, \alpha_1, \alpha_2, \ldots$, $\ldots, \alpha_{v-1}$ all $\in N - B$ such that $e_{j\beta} = e_{j\alpha_1} \cdot e_{\alpha_1\alpha_2} \ldots e_{\alpha_{v-1}\beta}$. Since $v \geqq n - r + 1$ the set N would contain at least $(n - r + 1) + r = n + 1$ elements, which is a contradiction.

b) Let l, β be fixed. The irreducibility implies the existence of a $\lambda = \lambda(l, \beta) \leqq n$ such that $e_{\beta l} \in C_A^\lambda$. We have therefore $e_{\beta l} = e_{\beta\alpha_1} \cdot e_{\alpha_1\alpha_2} \ldots e_{\alpha_{\lambda-2}\alpha_{\lambda-1}} \cdot e_{\alpha_{\lambda-1},l}$ with all factors in C_A. Choose λ as small as possible. If $l = \beta$, the idempotent $e_{\beta\beta}$ is clearly contained in C_A^{n-1}. Suppose therefore $\beta \neq l$. Then $\beta, \alpha_1, \alpha_2, \ldots, \alpha_{\lambda-1}, l$ are all different, hence $\lambda + 1 \leqq n$, so that $\lambda = \lambda(l, \beta) \leqq n - 1$. If $\lambda = n - 1$, our statement is proved. If $\lambda < n - 1$, we many insert at the beginning $e_{\beta\beta}^{n-1-\lambda}$ so that $e_{\beta l} = e_{\beta\beta}^{n-1-\lambda} \cdot$ $\cdot\, e_{\beta l} \in C_A^{n-1-\lambda} \cdot C_A^\lambda = C_A^{n-1}$. This proves our Lemma.

Theorem 14. *If A is a non-negative irreducible $n \times n$ matrix $(n > 1)$ with $r > 0$ non-zero entries along the main diagonal, then $C_A^{2n-r-1} = S$.*

Proof. Let j and l be fixed chosen. If $B = N$, i.e. $r = n$, we have by Lemma 9b $e_{\beta l} \in C_A^{n-1}$ for any $\beta, l \in N$, so that $C_A^{n-1} = S$. Suppose therefore $B \neq N$. With the same notations as in Lemma 9, there is a $\beta \in B$ and a $s = s(j)$ such that $e_{j\beta} \in C_A^s$. Further by Lemma 9b $\{e_{\beta 1}, e_{\beta 2}, \ldots, e_{\beta n}\} \subset C_A^{n-1}$. Hence

$$e_{jl} \in \{e_{j1}, e_{j2}, \ldots, e_{jn}\} = e_{j\beta}\{e_{\beta 1}, e_{\beta 2}, \ldots, e_{\beta n}\} \subset C_A^{s+n-1}.$$

If $s = s(j) = n - r$, we have $e_{jl} \in C_A^{2n-r-1}$. If $s = s(j) < n - r$, multiply both sides by C_A^{n-r-s}. Since A^{n-r-s} contains in each column a non-zero element, we have

$$e_{jl} \in \{e_{j1}, e_{j2}, \ldots, e_{jn}\} C_A^{n-r-s} \subset C_A^{s+n-1+(n-r-s)} = C_A^{2n-r-1}.$$

This proves our Theorem.

Remark. It is known that this result is the best possible. (See [1].)

If all entries in the main diagonal of A are zeros, it is natural to find an exponent g such that the support of A^g contains non-zero idempotents $\in S$ and use Theorem 14. To this purpose we prove the following

Lemma 10. *Let A be a primitive $n \times n$ matrix with $n > 1$. Then there is a positive integer $g \leqq n - 1$ such that C_A^g contains at least g non-zero idempotents $\in S$.*

Proof. a) If A is primitive, there is at least one row in A that contains at least two non-zero elements. For, if each row of A contains a unique element different from zero, then there is either a zero column or there exists a permutation matrix B such that $C_A = C_B$. In both cases A cannot be primitive.

Without loss of generality suppose that the first row of A contains at least two elements different from zero. By the proof of Theorem 1 (part b) the "first row" of C_A^2 contains at least one element not contained in "the first row" of C_A. Analogously C_A^3 contains in the "first row" at least one element not contained in the "first row" of $C_A \cup C_A^2$, and so on. This implies that the "first row" of $C_A \cup \ldots \cup C_A^{n-1}$ contains all elements $e_{11}, e_{12}, \ldots, e_{1n}$. Hence $e_{11} \in C_A^g$ with $g \leqq n - 1$.

b) Let g be the least integer such that C_A^g contains a non-zero idempotent $\in S$. If $g = 1$, our statement is trivially true. If $e_{\beta_1\beta_1} \in C_A^g$, $g > 1$, we have

$$e_{\beta_1\beta_1} = e_{\beta_1\beta_2} e_{\beta_2\beta_3} \ldots e_{\beta_{g-1}\beta_g} e_{\beta_g\beta_1}$$

with all factors in C_A. Hereby, clearly, all integers $\beta_1, \beta_2, \ldots, \beta_g$ are different one from an other. But then the following $g - 1$ elements (arising by cyclic permutations)

$$\begin{aligned} e_{\beta_2\beta_2} &= e_{\beta_2\beta_3} e_{\beta_3\beta_4} \ldots e_{\beta_g\beta_1} e_{\beta_1\beta_2}, \\ &\vdots \\ e_{\beta_g\beta_g} &= e_{\beta_g\beta_1} e_{\beta_1\beta_2} \ldots e_{\beta_{g-1}\beta_g}, \end{aligned}$$

are also contained in C_A^g. This proves our Lemma.

Theorems 14 and 10 imply

Theorem 15. *Let A be a primitive $n \times n$ matrix. Let g be the least integer for which C_A^g contains a non-zero idempotent $\in S$. Then $C_A^{g(2n-g-1)} = S$.*

The exponent $g(2n - g - 1)$ takes its greatest value for $g = n - 1$ and this value is $n^2 - n$, so that we always have $C_A^{n^2-n} = S$. But this exponent is not the lowest possible. By modifying the argument used above we shall obtain in Theorem 16 the best possible exponent. We first give a reformulation of Lemma 9 necessary for this purpose.

Lemma 11. *Let g be the least integer such that C_A^g contains at least g non-zero idempotents $\in S$. Denote these idempotents by $e_{\beta_1\beta_1}, \ldots, e_{\beta_g\beta_g}$. Denote further $B = \{\beta_1, \ldots, \beta_g\}$. Then:*

a) *For every $j \in N$ there is a $\beta \in B$ and an $s = s(j)$ such that $e_{j\beta} \in C_A^s$. Hereby: If $j \in B$, we may choose $s = g$; if $B \neq N$ and $j \in N - B$, we may choose $s = s(j) \leqq n - g$.*

b) *For any $l \in N$ and any $\beta \in B$ we have $e_{\beta l} \in C_A^{g(n-1)}$.*

Proof. a) If $j \in B$, choose $\beta = j$. Then $e_{jj} \in C_A^g$, so that our statement holds. Suppose therefore $B \neq N$ and $j \in N - B$. The proof follows then in the same way as in Lemma 9 (part a) writing g instead of r.

b) The proof follows by Lemma 9 by considering the matrix A^g (instead of A) and writing g instead of r.

Theorem 16. *If A is a primitive $n \times n$ matrix, we always have $C_A^{(n-1)^2+1} = S$.*

Proof. a) If $j \in B$, we have by Lemma 11b

$$e_{jl} \in \{e_{j1}, e_{j2}, \ldots, e_{jn}\} \subset C_A^{g(n-1)}.$$

Since the matrix A^{n-g} is primitive it contains in each column at least one element different from zero so that

$$e_{jl} \in \{e_{j1}, \ldots, e_{jn}\} \,.\, C_A^{n-g} \quad (\text{for any } l \in N).$$

Therefore $e_{jl} \in C_A^{n-g+g(n-1)}$.

b) If $B \neq N$ and $j \in N - B$, then by Lemma 11 a there is a $\beta \in B$ such that $e_{j\beta} \in C_A^s$ where $s \leqq n - g$. Hence

$$e_{jl} \in \{e_{j1}, e_{j2}, \ldots, e_{jn}\} = e_{j\beta}\{e_{\beta 1}, \ldots, e_{\beta n}\} \subset C_A^{s+g(n-1)}.$$

If $s = n - g$, we have $e_{jl} \in C_A^{n-g+g(n-1)}$. If $s < n - g$, note again that A^{n-g-s} has in each column at least one element different from zero, so that $e_{jl} \in \{e_{j1}, \ldots, e_{jn}\} \,.\, C_A^{n-g-s}$. Therefore

$$e_{jl} \in C_A^{s+g(n-1)} \,.\, C_A^{n-g-s} = C_A^{n-g+g(n-1)}.$$

We have proved: for any $j, l \in N$ the relation $e_{jl} \in C_A^{n-g+g(n-1)}$ holds.

But now (by Lemma 10)

$$n - g + g(n-1) = n + g(n-2) \leqq n + (n-1)(n-2) = (n-1)^2 + 1.$$

This proves Theorem 16.

Remark. It is known that the result of Theorem 16 is the best possible (see e.g. [5]).

References

[1] *A. L. Dulmage-N. S. Mendelsohn:* The exponent of a primitive matrix. Canadian Math. Bull. *5* (1962), 241—244.
[2] *Ф. Р. Гантмахер:* Теория матриц, Москва 1953.
[3] *K. L. Chung:* Markov chains with stationary transition probabilities. Berlin 1960.
[4] *J. Mařík-Vl. Pták:* Norms, spectra and combinatorial properties of matrices. Czechoslovak Math. J. *10* (*85*) (1960), 181—196.
[5] *Vl. Pták:* Об одной комбинаторной теореме и ее применении к неотрицательным матрицам. Чех. мат. ж. *8* (*83*) (1958), 487—495.
[6] *Vl. Pták-I. Sedláček:* Об индексе импримитивности неотрицательных матриц. Чех. мат. ж. *8* (*83*) (1958), 496—501.
[7] *В. И. Романовский:* Дискретные цепы Маркова. Москва 1949.
[8] *Š. Schwarz:* Subsemigroups of simple semigroups. Czechoslovak Math. J. *13* (*88*) (1963), 226—239.

Резюме

ПОЛУГРУППОВАЯ ТРАКТОВКА ТЕОРИИ НЕОТРИЦАТЕЛЬНЫХ МАТРИЦ

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Пусть $N = \{1, 2, \ldots, n\}$ и S – множество символов $\{e_{ik} \mid i, k \in N\}$ вместе с присоединенным нулем 0. Введем в S умножение еtветсвенным образом. Тогда S является вполне простой полугруппой с нулем.

Назовем носителем неотрицательной $n \times n$ матрицы $A = (a_{ik})$ подмножество $C_A \subset S$ тех $e_{ik} \in S$, для которых $a_{ik} > 0$ вместе с нулем 0.

Пусть $\mathfrak{S}$ обозначает мультипликативную полугруппу всех подмножеств из S. Исследуя конечную циклическую полугруппу $\{C_A, C_A^2, C_A^3, \ldots\}$ элементов $\in \mathfrak{S}$, автор получил не только многие теоремы, касающиеся неотрицательных матриц, но нашел и новые характеристики таких понятий как, например, индекс импримитивности данной матрицы.