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REMARKS ON SPACES OF LARGE CARDINAL NUMBER

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It is proved that a completely regular space of sufficiently large cardinal number $F(n)$ must contain an arbitrarily large (n) discrete subspace.

1. This paper shows that for a completely regular space X to have a discrete subspace of power n, it suffices that the power of X exceed the sum of all the numbers exp exp exp m, $m < n$ (where exp p denotes 2^p). The method involves a subspace, in any space of more than exp m points, which contains more than m points but has a covering by open sets each containing at most m points. An additional consequence: a hereditarily Lindelöf space contains at most exp $\aleph_0$ points. P. S. ALEKSANDROV and P. S. URYSOHN [1] proved this in the compact case.

The number $F(n)$, the successor of the sum of all exp exp exp m, $m < n$, is too large if $n = \aleph_0$; for every other infinite cardinal n, I do not know whether $F(n)$ can be replaced by a smaller number. Product spaces D^m (D a space of two points) show that if $m < n$ then $F(n) > 2^m$. Note that a linearly ordered space of power greater than 2^m must contain a discrete subspace of power $> m$; this is essentially due to Urysohn (see [1]), though it is implicit in earlier work of F. HAUSDORFF [2; VI, 8].

2. Consider any completely regular space X. Fix an embedding of X in a Tychonoff cube; thus the points x of X are represented by functions on some index set J to the interval $I = [0,1]$.

We define by transfinite induction a set of functions on subsets of J to I, called *sorting functions*; the sorting functions introduced at the α-th step will be said to have *length* α. All sorting functions will be restrictions of limits of functions in X; those which are restrictions of just one $x \in X$ will be called *complete*.

We may begin with the empty function, which we suppose is not complete; in fact, let us assume X is infinite. Inductively, for each incomplete sorting function ξ of length α, $\xi : S \to I$, select an index $j \in J - S$ on which some two extensions of ξ, that are restrictions of functions in X, differ. Define the *immediate extensions* of ξ to be all such extensions of ξ over $S \cup \{j\}$. The sorting functions of length $\alpha + 1$ are defined as the immediate extensions of sorting functions of length α. For a limit ordinal β, a function (considered as a set of ordered pairs) is a sorting function of length β provided it is a union of sorting functions of all lengths $\alpha < \beta$. This completes the definition.

Evidently each x in X has one or more restrictions that are complete sorting functions. The number of sorting functions whose length is an ordinal of power at most m (an infinite cardinal) is at most 2^m. Hence the number of sorting functions of length less than $n(n > \aleph_0)$ is at most the sum of all 2^m, $m < n$.

If the power of X exceeds 2^m there must be a sorting function η whose length λ is the first ordinal of power greater than m, for there are at most 2^m shorter complete sorting functions. For $m \geqq 2^{\aleph_0}$, the same conclusion follows from the weaker hypothesis that the character of X exceeds m.

From the sorting function η of length λ we can determine points $x_\alpha(\alpha < \lambda)$ such that the restriction of η of length α is a restriction of x_α, but the restriction of η of length $\alpha + 1$ is not. The x_α form a subspace S of X having more than m points. The open sets $U_j = \{x \in S : x(j) \neq \eta(j)\}$, as j runs through the domain of η, cover S; and each contains at most m points. Taking account of limit cardinals, we find

Lemma. *If the power of X exceeds the sum of all 2^m for $m < n$ (or, for non-limit cardinals $n > 2^{\aleph_0}$, if X merely has character at least n) then X contains a subspace that has power less than n locally but not globally.*

3. Restating the lemma affirmatively, we have the bound on the size of hereditarily Lindelöf spaces:

Theorem 1. *If every family of open sets in X has the same union as some subfamily of power at most m, then X contains at most 2^m points, and if $m \geqq 2^{\aleph_0}$, X can even be embedded in a product of m intervals.*

Theorem 2. *If X has power at least $F(n)$, then X has a discrete subspace of at least n points.*

To prove Theorem 2, apply the lemma (to the cardinal successor of exp exp p when n is the successor of p; with suitable modification for the other case). Then build up a discrete subspace, cushioning each point as it is added by a neighborhood of small power, and always avoiding the closure of the set of points already added. As long as only r points have been added, the power of the closure is at most exp exp r.

References

[1] *P. S. Alexandroff* et *P. S. Urysohn:* Mémoire sur les espaces topologiques compacts. Verh. Akad. Wetensch. Amsterdam *14* (1929), 1–96.
[2] *F. Hausdorff:* Grundzüge der Mengenlehre. Leipzig, 1914.

Резюме

ЗАМЕЧАНИЕ О ПРОСТРАНТСВАХ БОЛЬШОЙ МОЩНОСТИ

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Для каждого кардинального числа n существует такое наименьшее число $G(n)$, что любое вполне регулярное пространство X, мощность которого превосходит $G(n)$, содержит дискретное подпространство Y, имеющее мощность n.

$G(n)$ не превосходит суммы всех чисел $2^{2^{2^m}}$, $m < n$, но для $\aleph_0$ это – не наилучшая оценка; является ли она найлучшей для кардинальных чисел $> \aleph_0$, мне не известно.