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Miroslav Fiedler

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ON INVERTING PARTITIONED MATRICES

Miroslav Fiedler, Praha

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In terms of linear mappings, formulae for inverting partitioned matrices of a special form are given.

1. Introduction. We intend to investigate the form of the inverse matrix to a partitioned matrix whose combinatorial structure of non-zero blocks is of a certain simple kind (quasi-tridiagonal and quasi-triangular matrices are included). Similar questions have been studied by H. I. Meyer and B. J. Hollingsworth [2], S. Schechter [3], G. Swift [4] and others. The main result is theorem (3,3) which shows that – under certain restrictions – the inverse of such a matrix can be evaluated by reccurent formulas and that the problem of inverting such a matrix is equivalent with solving a smaller non-linear system of matrix equations.

2. Definitions and notation. Lemmas on e-simple graphs. In our considerations, we shall use the customary notions of the theory of graphs. The reader may find these notions in the book [1] by D. König. Moreover, we shall call a directed graph G e-simple if each edge of G is contained in at most one (directed) cycle of G. It is obvious that if a graph G has the property that for each pair $u, v (u \neq v)$ of its vertices there exists at most one (directed) path from u to v then G is e-simple. Especially, a finite symmetric (or, non-directed) connected graph without loops is e-simple if and only if it is a tree.

Further, we shall say that a finite-dimensional vector space X is partitioned if a set $X(u)$, $u \in U$, of its linear subspaces is given in such a manner that X is their direct sum:

$$X = \dot{\sum_{u \in U}} X(u) .$$

If A is a linear mapping in this partitioned vector space X, there exists for each pair $u, v \in U$ a (unique and linear) mapping $a(u, v)$ of $X(u)$ into $X(v)$ such that if for a vector $x \in X$ is $x = \sum_{u \in U} x(u)$, $x(u) \in X(u)$, then its image $xA = y = \sum_{v \in U} y(v)$, $y(v) \in X(v)$, where

$$y(v) = \sum_{u \in U} x(u)\, a(u, v) , \quad v \in U .$$

We shall write simply

$$A = [a(u, v)], \quad u, v \in U .$$

Analogously, we shall call a unitary vector space X partitioned if $X = \dot{\sum} X(u)$, $u \in U$, where $X(u)$ and $X(v)$ are orthogonal if $u \neq v$, $u, v \in U$. The scalar product of two vectors x, y of X will be denoted by $\langle x, y \rangle$.

If $A = [a(u, v)]$, $u, v \in U$, is a linear mapping in a vector space $X = \dot{\sum} X(u)$, $u \in U$, and if $V \subset U$, we shall denote by $A[V]$ the linear mapping $A[V] = [a(u, v)]$, $u, v \in V$, in the subspace $\dot{\sum} X(u)$, $u \in V$.

We shall associate with a linear mapping in a partitioned vector space $X = \dot{\sum} X(u)$, $u \in U$, a directed graph whose set of vertices is U and whose edges (u, v), $u, v \in U$, $u \neq v$, are those (ordered) pairs for which $a(u, v)$ is different from the zero-mapping of $X(u)$ into $X(v)$.

In section 3 we shall investigate mappings whose graph is e-simple and their inverse mappings. It is obvious that – with a fixed basis in each $X(u)$ – this is equivalent to a similar problem for partitioned matrices.

In the sequel, we shall need some further notations and six lemmas on e-simple graphs. We shall assume here that G is an e-simple graph whose set of vertices U is finite.

If $u \in U$, we shall denote by $U(u)$ the set of all vertices $w \in U$ such that either $w = u$, or $w \neq u$ and there exists a (directed) path from u to w as well as a path from w to u in G. If (u, v) is an edge of G, we shall denote by $U(u, v)$ the set of all vertices $w \in U$ such that either $w = u$, or $w \neq u$ and there exists a path from u to w not containing (u, v) as well as a path from w to u in G. The number $h(u, v) = \text{card}\, U(u, v)$ will be called weight of the edge (u, v). It is obvious that $U(u, v) \subset U(u)$. If (u, v), (u', v') are edges of G, we shall say that (u', v') is inferior to (u, v) if there exists a path from u to u' not containing (u, v) as well as a path from u' to u containing (u', v') (thus, $u' \neq u$). Further, if u is a vertex, (u', v') an edge of G, we shall say that (u', v') is inferior to u if there exists a path from u to u' in G as well as a path from u' to u containing (u', v').

(2,1) *If u, v are two different vertices of an e-simple graph G such that there exist both paths from u to v and from v to u in G, then these paths are unique.*

Proof. Assume that $P_i = (u, \ldots, u', w_i, \ldots, v)$, $i = 1, 2$, are two different paths from u to v whose parts $(u, \ldots, u')$ are common but $w_1 \neq w_2$. Since there exists a path from v to u and a path (possibly trivial if $u = u'$) from u to u', there exists a "joining" (with not necessarily distinct vertices) from v to u'. Consequently, there exists a path $P = (v, \ldots, w, u')$ from v to u'. But, if we denote by $v_i (i = 1, 2)$ the last vertex of P different from u' which P has in common with $P'_i = (u', w_i, \ldots, v) \subset P_i$, then (w, u') is contained in two different cycles $(u', w_i, \ldots, v_i, \ldots, w, u')$. This contradiction with the e-simplicity of G proves our assertion.

(2,2) *Let (u, v) be an edge of G, $C = (u, w_1, \ldots, w_k, u)$, $k \geqq 1$, a cycle of G passing* *through u but not containing (u, v). Then, $(w_1, w_2), \ldots, (w_{k-1}, w_k)$, (w_k, u) are inferior to (u, v) but (u, w_1) is not.*

Proof. Obvious.

(2,3) *If (u, v) is an edge of G, $w_1, w_2 \in U(u, v)$, then all vertices of the path from w_1 to w_2 belong to $U(u, v)$. If $u \in U$, $w_1, w_2 \in U(u)$, then all vertices of the path $(w_1, \ldots, w_2)$ belong to $U(u)$.*

Proof. Obvious.

(2,4) *Let (u, v), (u', v') be edges of G such that (u', v') is inferior to (u, v). Then:*

1° *If $w \in U(u', v')$, then both paths $(u, \ldots, w)$ and $(w, \ldots, u)$ contain u';*

2° $U(u', v') \subset U(u, v)$;

3° $h(u', v') < h(u, v)$;

4° *the edge (w', u') in the path $(u, \ldots, w', u')$ is the only edge of the form (w_1, w_2) such that $w_1 \in U(u, v)$, $w_1 \notin U(u', v')$, $w_2 \in U(u', v')$.*

Proof. Let us denote $P_1 = (u, \ldots, u')$, $P_2 = (u', \ldots, u)$, $P_3 = (u', \ldots, w)$, $P_4 = (w, \ldots, u')$. To prove 1°, it is sufficient to show that P_1 and P_3 have the single vertex u' in common as well as P_2 and P_4 have only u' in common. Assume that for a vertex $z \neq u'$ is $z \in P_1$, $z \in P_3$. Then, $(u', \ldots, z) \subset P_2$ is a path not containing (u', v'); but, combining P_2 and $(u, \ldots, z) \subset P_1$ we obtain a joining from u' to z which contains u' just once (namely in the edge (u', v')), so that this joining contains a path from u' to z containing (u', v'). This contradiction with (2,1) proves the first part of 1°. Further, assume that P_2 and P_4 have a vertex $z' \neq u'$ in common. Then, $(u', \ldots, z') \subset P_2$ contains v'; but, combining P_3 with $(w, \ldots, z') \subset P_4$ we obtain a joining which does not contain (u', v'). This contradiction with (2,1) completes the proof of 1°. The part 2° follows easily from 1°: If $w \in U(u', v')$, then the path $P = (u, \ldots, w)$ exists and contains u'. But since $u' \neq u$ and $(u, \ldots, u')$ does not contain (u, v), P does not contain (u, v) as well. Since $(w, \ldots, u)$ exists, $w \in U(u, v)$. But $v' \in U(u, v)$ while $v' \notin U(u', v')$ so that $U(u', v') \subset U(u, v)$. The part 3° is an immediate consequence of 2°. To prove 4°, let us show first that $w' \notin U(u', v')$: otherwise there would exist a path $(u, \ldots, u', \ldots, w')$ according to 1° which contradicts the definition of w'. Let now (w_1, w_2) be an edge fulfilling the condition in 4°. Since $w_1 \notin U(u', v')$, there exists a path $P' = (u', v', \ldots, w_1)$. Assume that $w_2 \neq u'$. Then, combining P' with (w_1, w_2), we obtain a joining from u' to w_2 which contains u' just once, in the edge (u', v'). Thus, (u', v') is contained in $(u', \ldots, w_2)$ in contradiction with $w_2 \in U(u', v')$. Consequently, $w_2 = u'$. If we combine the path $(v', \ldots, u)$ not containing u' with $(u, \ldots, w, u')$, we obtain a joining v' to u' containing u' just once, in (w, u'). But $(v', \ldots, w_1) \subset P$ combined with (w_1, u') is a joining from v' to u' containing u' just once, in (w_1, u'). Consequently, $w_1 = w'$. The proof is complete.

(2,5) *The relation of inferiority of edges in G is transitive but neither reflexive nor symmetric.*

Proof. According to 3° of (2,4) it is sufficient to prove the transitivity. Let (u', v') be inferior to (u, v) and (u'', v'') inferior to (u', v'). If we combine the path $(u, \ldots, u')$ not containing (u, v) with the path $(u', \ldots, u'')$, we obtain according to 1° of (2,4) a path from u to u'' not containing (u, v). Similarly, the combination of the paths $(u'', v'', \ldots, u')$ and $(u', \ldots, u)$ yields a path from u'' to u containing (u'', v''). Thus, (u'', v'') is inferior to (u, v) and the proof is complete.

Remark. It can easily be proved that if an edge (u', v') of G is inferior to a vertex u and if (u'', v'') is inferior to (u', v'), then (u'', v'') is inferior to u. Further, an analogous argument as in (2,4) proves that if (u', v') is inferior to u then $U(u', v') \subset U(u)$ and the edge (w', u') in the path $(u, \ldots, w', u')$ is the only edge of the form (w_1, w_2) such that $w_1 \in U(u)$, $w_1 \notin U(u', v')$, $w_2 \in U(u', v')$.

For the sake of completeness, we shall conclude this section by proving the following characterization of e-simple graphs:

(2,6) *A graph is e-simple if and only if each its strong component (i.e. a maximal subgraph with the property that each vertex of it can be joined with any other by a path) is a union of cycles, each two of which have at most one vertex in common and such that this union contains no other cycles.*

Proof. Let a graph G be e-simple and let E be the set of all edges of G contained in at least one cycle. If $e \in E$, let $C(e)$ denote the corresponding (unique) cycle. Then, $\bigcup_{e \in U} C(e)$ has the strong components of G as its components. From (2,1) it follows that each two of the cycles $C(e)$ either concide or have at most one vertex in common. The first part of the theorem follows then immediately.

Let a graph G have the mentioned property. Then, each edge of G is evidently contained in at most one cycle and G is e-simple. The proof is complete.

3. Results. In this section, we shall prove the main theorems.

(3,1) *Let $C = [c(u, v)]$, $u, v \in U$, be a linear (auxiliary) mapping in a partititoned vector space $X = \dot{\sum} X(u)$, $u \in U$, whose graph G is e-simple. Let $c(u, u)$ be regular for all $u \in U$. Let us define a linear mapping $B = [b(u, v)]$ in X in the following manner:*

(1) $$b(u, u) = c(u, u) \quad \text{for} \quad u \in U\,; \quad \text{if} \quad u \neq v \quad \text{then} \quad b(u, v) =$$
$$= \sum c(u, w_1)\, c^{-1}(w_1, w_1)\, c(w_1, w_2)\, c^{-1}(w_2, w_2) \ldots c^{-1}(w_k, w_k)\, c(w_k, v)$$

where the sum is extended over all paths $(u, w_1, w_2, \ldots, w_k, v)$ in G from u to v.

Further, let us define for each edge $(u, v) \in G$ a linear mapping $m(u; v)$ in $X(u)$ in the following way:

(2) $$m(u; v) = c(u, u) - c(u, v)\, c^{-1}(v, v)\, c(v, w_1) \ldots c^{-1}(w_s, w_s)\, c(w_s, u)\,,$$

if there exists a (unique) cycle $(u, v, w_1, \ldots, w_s, u)$ in G containing the edge (u, v); $m(u; v) = c(u, u)$ otherwise.

Then:

1° *If one edge (u, v) in a cycle of G has the property that the corresponding mapping $m(u; v)$ is regular then each other edge of this cycle has this property as well.*

2° *The mapping B is regular if and only if $m(u; v)$ is regular for each edge $(u, v) \in G$.*

3° *If B is regular then its inverse mapping $B^{-1} = A = [a(u, v)]$ is determined by:*

$$a(u, u) = c^{-1}(u, u) + \sum_{w;(u,w)\in G} [m^{-1}(u; w) - c^{-1}(u, u)], \quad u \in U; \tag{3}$$

$$a(u, v) = - m^{-1}(u; v)\, c(u, v)\, c^{-1}(v, v) \quad \text{if} \quad u \neq v, \quad (u, v) \in G;$$

$$a(u, v) = 0 \text{ if } u \neq v \text{ and there is no edge } (u, v) \text{ in } G.$$

4° *G is the graph of A.*

Proof. Let first u, v, w be elements from U, $u \neq v$, and let $b(w; u, v)$ denote a sum analogous to (1) but extended over those paths $(u, \ldots, v)$ only which do not contain w(if $w = u$ or $w = v$ then of course $b(w; u, v) = 0$). We shall prove that, if u and w are in a cycle Z of G,

$$b(u, v) = b(u, w)\, c^{-1}(w, w)\, b(w, v) + \tag{4}$$
$$+ [c(u, u) - b(u, w)\, c^{-1}(w, w)\, b(w, u)]\, c^{-1}(u, u)\, b(w; u, v).$$

To prove this, notice that (4) is true if $w = u$ or $w = v$. If $u \neq w \neq v$, it follows from (2,1) that every path in G from u to v (if there is any) has with Z either the only vertex w or a segment $(w, \ldots, u')$ of Z in common. Thus, we obtain all paths from u to v which pass through w if we combine the path $(u, \ldots, w)$ with all possible paths from w to v, not passing through u. That means according to (1) that

$$b(u, v) = b(w; u, v) + b(u, w)\, c^{-1}(w, w)\, b(u; w, v). \tag{5}$$

Analogously,

$$b(w, v) = b(u; w, v) + b(w, u)\, c^{-1}(u, u)\, b(w; u, v).$$

By elimination of $b(u; w, v)$ we obtain (4).

Further, it is easy to see that if $u \neq v$ then

$$b(u, v) = \sum c(u, w)\, c^{-1}(w, w)\, b(u; w, v) \tag{6}$$

where we define $b(u; v, v) = b(v, v)$ and the sum is extended over all vertices $w \in U$ for which $(u, w) \in G$.

Let now (u_1, u_2) be an edge of G and let $m(u_1; u_2)$ be singular. Then, according to the definition of $m(u_1; u_2)$, (u_1, u_2) is contained in a cycle $Z = (u_1, u_2, \ldots, u_k, u_1)$, $k \geqq 2$, and there exists a non-zero vector $x_1 \in X(u_1)$ such that

$$x_1 m(u_1; u_2) = 0 . \tag{7}$$

Let us denote by x_j $(j = 2, \ldots, k)$ the vector

$$x_j = x_1\, b(u_1, u_j)\, c^{-1}(u_j, u_j) . \tag{8}$$

If $v \in U$, $v \neq u_1$, (4) yields that

$$x_1\, b(u_1, v) = x_1\, b(u_1, u_j)\, c^{-1}(u_j, u_j)\, b(u_j, v) \tag{9}$$

since $c(u_1, u_1) - b(u_1, u_j)\, c^{-1}(u_j, u_j)\, b(u_j, u_1) = m(u_1; u_2)$. If $v = u_1$, (9) is fulfilled as well. Thus,

$$x_1\, b(u_1, v) = x_j\, b(u_j, v) \quad \text{for each} \quad v \in U . \tag{10}$$

It follows that x_j are non-zero vectors for $j = 1, 2, \ldots, k$ and (if we put $u_{k+1} = u_1$)

$$x_j\, m(u_j; u_{j+1}) = x_1\, b(u_1, u_j)\, c^{-1}(u_j, u_j) \left[c(u_j, u_j) - \right.$$
$$\left. - \, b(u_j, u_1)\, c^{-1}(u_1, u_1)\, b(u_1, u_j)\right] = 0 .$$

This proves 1°. Moreover, in this case the non-zero vector $y = x_1 - x_2$ has the property that $yB = 0$ according to (10) for $j = 2$. Consequently, B is then singular. This proves "only if" in 2°.

Let now $m(u; v)$ be regular for each edge $(u, v) \in G$. Then, the mapping A defined in (3) exists. Thus, to prove both 3° and the remaining part of 2° it is sufficient to show that $\sum_{w \in U} a(u, w)\, b(w, v)$ is the identity mapping $e(u)$ in $X(u)$ if $v = u$ and zero if $v \neq u$.

But

$$\sum_{w \in U} a(u, w)\, b(w, u) = a(u, u)\, b(u, u) + \sum_{w;(u,w) \in G} a(u, w)\, b(w, u) =$$
$$= \{c^{-1}(u, u) + \sum_{w;(u,w) \in G} [m^{-1}(u; w) - c^{-1}(u, u)]\}\, c(u, u) -$$
$$- \sum_{w;(u,w) \in G} m^{-1}(u; w)\, c(u, w)\, c^{-1}(w, w)\, b(w, u) = e(u) +$$
$$+ \sum_{w;(u,w) \in G} \{m^{-1}(u; w) [c(u, u) - c(u, w)\, c^{-1}(w, w)\, b(w, u)] - e(u)\} = e(u)$$

since

$$c(u, u) - c(u, w)\, c^{-1}(w, w)\, b(w, u) = m(u; w) .$$

If $v \neq u$ then

$$\sum_{w \in U} a(u, w)\, b(w, v) = a(u, u)\, b(u, v) + \sum_{w;(u,w) \in G} a(u, w)\, b(w, v) =$$
$$= c^{-1}(u, u)\, b(u, v) + \{ \sum_{w;(u,w) \in G} [m^{-1}(u; w) - c^{-1}(u, u)]\}\, b(u, v) -$$
$$- \sum_{w;(u,w) \in G} m^{-1}(u; w)\, c(u, w)\, c^{-1}(w, w)\, b(w, v) .$$

Let us denote by M, N resp. the sets of those $w \in U$ for which $(u, w) \in G$ and (u, w) is resp. is not contained in a cycle of G. Then, by (4) and (5),

$$\sum_{w \in U} a(u, w)\, b(w, v) = c^{-1}(u, u)\, b(u, v) +$$
$$+ \sum_{w \in M} \{m^{-1}(u; w)\,[b(u, v) - c(u, w)\, c^{-1}(w, w)\, b(w, v)] - c^{-1}(u, u)\, b(u, v)\} -$$
$$- \sum_{w \in N} c^{-1}(u, u)\, c(u, w)\, c^{-1}(w, w)\, b(w, v) = c^{-1}(u, u)\, b(u, v) +$$
$$+ \sum_{w \in M} \{c^{-1}(u, u)\, b(w; u, v) - c^{-1}(u, u)\, b(u, v)\} +$$
$$+ \sum_{w \in N} c^{-1}(u, u)\, c(u, w)\, c^{-1}(w, w)\, b(w, v) = c^{-1}(u, u)\, b(u, v) +$$
$$- \sum_{w \in M \cup N} c^{-1}(u, u)\, c(u, w)\, c^{-1}(w, w)\, b(u; w, v) = 0 \quad \text{according to (6).}$$

It remains to prove 4°. But this is an easy consequence of (3). The proof is complete.

(3,2) *Let* $C = [c(u, v)]$, $u, v \in U$, *be a symmetric linear mapping in a partitioned unitary vector space* $X = \dot{\sum} X(u)$, $u \in U$, *whose graph* G *is e-simple.*[4]) *Let* $c(u, u)$ *be regular for all* $u \in U$. *The mapping* B *defined by* (1) *(where the sum consists of at most one term) is positive definite if and only if each* $c(u, u)$ *is positive definite as well as the mappings* $m(u; v) = c(u, u) - c(u, v)\, c^{-1}(v, v)\, c(v, u)$ *are positive definite for all edges* $(u, v) \in G$.

Proof. It is easy to prove this if U has one or two elements. In the last mentioned case $U = \{1, 2\}$ it follows from the relation

$$\langle xB, x\rangle =$$
$$= \langle [x(1) + x(2)\, c(2, 1)\, c^{-1}(1, 1)]\, c(1, 1),\ [x(1) + x(2)\, c(2, 1)\, c^{-1}(1, 1)]\rangle +$$
$$+ \langle x(2)\,[c(2, 2) - c(2, 1)\, c^{-1}(1, 1)\, c(1, 2)],\ x(2)\rangle$$

where $x = x(1) + x(2)$, $x(i) \in X(i)$, $i = 1, 2$.

From this and the fact that if $B = [b(u, v)]$, $u, v \in U$, is positive definite in $X = \dot{\sum} X(u)$, $u \in U$, then all mappings $B[V] = [b(u, v)]$, $u, v \in V \subset U$, are positive definite as well, follows the necessity of the condition. We shall prove the sufficiency by induction with respect to the number n of vertices in G. If $n = 1$ or if there is no edge in G, our assertion is true. Thus, let $n \geqq 2$ and let there exist an edge in G, contained in a component K of G. Since K is a tree (with at least two vertices), there exists an end-vertex u in K such that (u, v) is the unique edge in G from u. Consequently,

(11) $$b(u, z) = c(u, v)\, c^{-1}(v, v)\, b(v, z) \quad \text{for each} \quad z \in U,\ z \neq u\,.$$

If $U_1 = U - \{u\}$, $B_1 = B[U_1]$ then the graph G_1 of B_1 is e-simple, too, and has $n - 1$ vertices. Since B_1 is generated in the same manner as B, the induction hypo-

[4]) This means that each component of G is a tree.

thesis yields that B_1 is positive definite. But if $x \in X$, $x = x(u) + x_1$, $x(u) \in X(u)$, $x_1 \in \dot{\sum} X(v)$ for $v \in U_1$, it follows from (11) that

$$\langle xB, x\rangle = \langle x(u)\,[c(u, u) - c(u, v)\, c^{-1}(v, v)\, c(v, u)],\, x(u)\rangle +$$
$$+ \langle [x_1 + x(u)\, c(u, v)\, c^{-1}(v, v)]\, B_1,\, [x_1 + x(u)\, c(u, v)\, c^{-1}(v, v)]\rangle\,.$$

Let $x \neq 0$. If $x(u) \neq 0$, the first member of the right-hand side is positive according to our assumption, the second being non-negative. If $x(u) = 0$, $x_1 \neq 0$ and $\langle xB, x\rangle = \langle x_1 B_1, x_1\rangle > 0$ as well. Thus, B is positive definite and the proof is complete.

(3,3) Theorem. *Let $A = [a(u, v)]$, $u, v \in U$, be a linear mapping in a partitioned vector space $X = \dot{\sum} X(u)$, $u \in U$, whose graph G is e-simple. Let there exist a solution of the following system (12) with unknown mappings $c(u, u)$ in $X(u)$ for each $u \in U$ and $c(u, v)$ of $X(u)$ into $X(v)$ for those pairs u, v for which (u, v) is an edge of G:*

$$\text{(12)} \qquad a(u, u) = c^{-1}(u, u) + \sum_{w;(u,w)\in G} [m^{-1}(u; w) - c^{-1}(u, u)],\quad u \in U\,,$$
$$a(u, v) = -\, m^{-1}(u; v)\, c(u, v)\, c^{-1}(v, v) \quad \text{if} \quad u \neq v,\ (u, v) \in G\,,$$

where we denote by $m(u; v)$ etc. the mapping

$$m(u; v) = c(u, u) - c(u, v)\, c^{-1}(v, v)\, c(v, w_1) \ldots c^{-1}(w_s, w_s)\, c(w_s, u)$$

if there exists a (unique) cycle $(u, v, w_1, \ldots, w_s, u)$ in G containing the edge (u, v); $m(u; v) = c(u, u)$ otherwise.

Then, the inverse mapping $A^{-1} = B = [b(u, v)]$ exists and is determined by

$$\text{(1')} \qquad b(u, u) = c(u, u) \quad \text{for} \quad u \in U\,; \quad \text{if} \quad u \neq v \quad \text{then} \quad b(u, v) =$$
$$= \sum c(u, w_1)\, c^{-1}(w_1, w_1)\, c(w_1, w_2)\, c^{-1}(w_2, w_2) \ldots c^{-1}(w_k, w_k)\, c(w_k, v)$$

where the sum is extended over all paths $(u, w_1, w_2, \ldots, w_k, v)$ in G from u to v.

Moreover, the solution of (12) exists if and only if the following algorithm is available:

Associate with each edge $(u, v) \in G$ a linear mapping $r(u; v)$ in $X(u)$ as follows supposing that all of them are regular:

$r(u; v) = a(u, u)$ if the weight $h(u, v) = 0$; if $h(u, v) > 0$ put by induction with respect to the weights

$$\text{(13)} \qquad r(u; v) = a(u, u) + \sum (-1)^k\, a(u, w_1)\, r^{-1}(w_1; w_2)\, a(w_1, w_2)\,.$$
$$.\, r^{-1}(w_2; w_3)\, a(w_2, w_3) \ldots r^{-1}(w_k; u)\, a(w_k, u)$$

where the sum is extended over all cycles $(u, w_1, w_2, \ldots, w_k, u)$ in G which contain u but not (u, v).

If $u \in U$, let a mapping $r(u)$ in $X(u)$ be defined by an analogous sum (14) as in (13) but extended over all cycles in G containing u without exception. Suppose that these mappings are all regular as well.

Then, the solution of (12) *is*

(15) $$c(u, u) = r^{-1}(u)\,, \quad c(u, v) = -\, r^{-1}(u; v)\, a(u, v)\, r^{-1}(v) \quad \textit{for} \quad (u, v) \in G\,.$$

Finally, this solution exists if and only if all mappings $A[U(u)]$ and $A[U(u, v)]$ with $u \in U$ and $(u, v) \in G$ are regular. Especially, if A is symmetric positive definite (in a unitary space), this condition is fulfilled.

Proof. If there exists a solution of (12) then $c(u, u)$ is regular for each $u \in U$ as well as $m(u; v)$ for each edge $(u, v) \in G$. Thus, it is possible to define the mappings A and B in (3,1) given by (3) and (1). According to this theorem, these mappings are inverse to each other. But since our original mapping A coincides with the mapping A from (3), B is of the form (1) or (1′) as asserted.

To prove the second part, observe that the possibility of inductive definition of the mappings $r(u; v)$ in (13) follows from (2,2) and 3° of (2,4). Thus, let the algorithm (13) and (14) be available and let $c(u, u)$ and $c(u, v)$ be defined by (15). Let us show first that the corresponding mappings $m(u; v)$ (for $(u, v) \in G$) fulfil

(16) $$m(u; v) = r^{-1}(u; v)\,.$$

This is true if there is no cycle in G containing (u, v) since then $m(u; v) = c(u, u) = r^{-1}(u) = r^{-1}(u; v)$. If $(u, v, w_1, \ldots, w_s, u)$ is such a (unique) cycle, then according to (2′), (13) and (14)

$$c(u, u) - m(u; v) = (-1)^s\, r^{-1}(u; v)\, a(u, v)\, r^{-1}(v; w_1)\, a(v, w_1) \ldots$$
$$\ldots r^{-1}(w_s; u)\, a(w_s, u)\, r^{-1}(u) = -\, r^{-1}(u; v)\,[r(u) - r(u; v)]\, r^{-1}(u) =$$
$$= -\, r^{-1}(u; v) + r^{-1}(u) = c(u, u) - r^{-1}(u; v)$$

which completes the proof of (16).

Now, it is easy to see (using (13) and (14)) that the mappings $c(u, u)$ and $c(u, v)$ fulfil the system (12). Conversely, if there exists a solution $c(u, u)$ (for $u \in U$) and $c(u, v)$ (for $(u, v) \in G$) of (12) then all mappings $c(u, u)$ as well as the mappings $m(u; v)$ defined by (2′) are regular. If we define $r(u; v)$ as $m^{-1}(u; v)$ and $r(u)$ as $c^{-1}(u, u)$, then it follows easily from $r^{-1}(u; v)\, a(u, v) = -\, c(u, v)\, c^{-1}(v, v)$ (according to the second relation in (12)) and from both parts of (12) that (13) and (14) are fulfilled. Thus, the algorithm is available.

To prove the last assertion of the theorem, let us prove the following two statements:

A. Let (u, v) be an edge of G and let all mappings $r(w_1; w_2)$ exist and be regular for which $(w_1, w_2) \in G$ is inferior to (u, v). Then:

1° If $yA[U(u, v)] \in X(u)$ for a vector $y = \sum y(z)$, $y(z) \in X(z)$, then

(17) $$yA[U(u, v)] = y(u)\, r(u; v)$$

and

(18) $$y(w')\, a(w', w) + y(w)\, r(w; w'') = 0$$

whenever $w \in U(u, v)$, $w \neq u$, where $(u, \ldots, w', w)$, $(w, w'', \ldots, u)$ are the (unique) paths in G from u to w (not containing (u, v)) and from w to u.

2° If $y(u)$ is a vector from $X(u)$ and if vectors $y(w)$ $(u \neq w \in U(u, v))$ are defined inductively by (18) with respect to the length of the path $(u, \ldots, w)$, then (17) is fulfilled for $y = \sum y(z)$, $z \in U(u, v)$.

B. Let $u \in U$ and let all mappings $r(w_1; w_2)$ exist and be regular for which $(w_1, w_2) \in G$ is inferior to u. Then:

1° If $yA[U(u)] \in X(u)$ for a vector $y = \sum y(z)$, $y(z) \in X(z)$ then

$$yA[U(u)] = y(u)\, r(u) \tag{19}$$

and (18) holds where $(u, \ldots, w', w)$, $(w, w'', \ldots, u)$ are paths in G.

2° If $y(u)$ is a vector in $X(u)$ and if vectors $y(w)$ $(w \neq u,\ w \in U(u))$ are defined by (18) inductively with respect to the length of $(u, \ldots, w)$, then (19) is fulfilled for $y = \sum y(z)$.

To prove **A**, we shall proceed by induction over the weight $h(u, v)$. If $h(u, v) = 0$, both parts are fulfilled in a trivial way. Thus, let $h(u, v) > 0$ and suppose that the assertion is true for all edges $(u', v') \in G$, for which $h(u', v') < h(u, v)$. Let $yA[U(u, v)] \in X(u)$ and let $w \neq u$, $w \in U(u, v)$. If $(u, \ldots, w', w)$, $(w, w'', \ldots, u)$ are the corresponding paths in G, then (w, w'') is inferior to (u, v). From 4° of (2,4) it follows that, if $\tilde{y}$ denotes the vector $\tilde{y} = \sum_{z \in U(w, w'')} y(z)$, then

$$\tilde{y}A[U(w, w'')] = \sum_{z, z' \in U(w, w'')} y(z)\, a(z, z') = \sum_{z' \in U(w, w'')} \sum_{z \in U(u, v)} y(z)\, a(z, z') - $$
$$- y(w')\, a(w', w) = - y(w')\, a(w', w) \in X(w)\,.$$

Since $h(w, w'') < h(u, v)$ according to 3° of (2,4) and since the induction can be applied according to (2,5), the induction hypothesis yields that

$$\tilde{y}A[U(w, w'')] = y(w)\, r(w; w'')\,.$$

Consequently,

$$y(w')\, a(w', w) + y(w)\, r(w; w'') = 0$$

as asserted in (18).

Further, let $C_1, C_2, \ldots, C_k$ be all cycles in G passing through u and not containing (u, v),

$$C_i = (u, v_i, w_i, \ldots, z_i, u), \quad i = 1, 2, \ldots, k\,.$$

Then,

$$yA[U(u, v)] = \sum_{z \in U(u, v)} y(z)\, a(z, u) = y(u)\, a(u, u) +$$
$$+ \sum_{i=1}^{k} y(z_i)\, a(z_i, u) = y(u)\, [a(u, u) +$$
$$+ \sum_{i=1}^{k} (-1)^{s_i - 1} a(u, v_i)\, r^{-1}(v_i; w_i)\, a(v_i, w_i) \ldots r^{-1}(z_i; u)\, a(z_i, u)]$$

according to 1° of (2,4) where s_i denotes the number of distinct vertices in C_i. Thus,

$$yA[U(u, v)] = y(u)\, r(u; v)$$

by (13).

This proves not only (17) but also the fact that the definition of y in 2° which is possible according to (2,1) implies that

$$\sum_{z \in U(u,v)} y(z)\, a(z, u) = y(u)\, r(u; v)\,.$$

It remains to prove that from this definition it follows that

$$\sum_{z \in U(u,v)} y(z)\, a(z, w) = 0 \quad \text{if} \quad w \in U(u, v),\ w \neq u\,.$$

Let thus $w \neq u$ and let $(u, \ldots, w', w)$ (not containing (u, v)) and $(w, w'', \ldots, u)$ be the corresponding paths in G. Then, in $U(w, w'')$, the analogous vector $\tilde{y} = \sum_{z \in U(w,w'')} y(z)$ is defined according to 1° of (2,4) in the same manner by $y(w)$ as y by $y(u)$. Consequently,

$$\sum_{z \in U(w,w'')} y(z)\, a(z, w) = y(w)\, r(w; w'')\,,$$

so that by 4° of (2,4)

$$\sum_{z \in U(u,v)} y(z)\, a(z, w) = y(w')\, a(w', w) + \sum_{z \in U(w,w'')} y(z)\, a(z, w) =$$
$$= y(w')\, a(w', w) + y(w)\, r(w; w'') = 0$$

by the definition of $y(w)$. The proof of **A** is complete.

To prove **B**, let $yA[U(u)] \in X(u)$, $y = \sum_{z \in U(u)} y(z)$, $y(z) \in X(z)$. If $w \in U(u)$, $w \neq u$, and $\tilde{y} = \sum_{z \in U(w,w'')} y(z)$ where $(w, w'', \ldots, u)$ is the path from w to u, then

$$\tilde{y}A[U(w, w'')] = \sum_{z,z' \in U(w,w'')} y(z)\, a(z, z') = \sum_{z' \in U(w,w'')} \sum_{z \in U(u)} y(z)\, a(z, z') -$$
$$- y(w')\, a(w', w) = -\, y(w')\, a(w', w) \in X(w)$$

according to the remark in section 2. Consequently, it follows from 1° of **A** that $\tilde{y}A[U(w, w'')] = y(w)\, r(w; w'')$ so that (18) in 1° of **B** holds. To prove both (19) and the converse part 2° of B, it is sufficient to use quite analogous methods as in the proof of A.

We are now able to conclude the proof of the theorem. Suppose first that all mappings $A[U(u, v)]$ and $A[U(u)]$ exist and are regular. Then, an easy induction over the weights $h(u, v)$ shows that all mappings $r(u; v)$ exist and are regular: If (u, v) is an edge in G such that $h(u, v) = 0$, then $r(u; v)$ exists and is regular. Let $h(u, v) > 0$ and suppose that $r(u', v')$ all exist and are regular if $h(u', v') < h(u, v)$. Then, $r(u; v)$ exists by (13). Let $r(u; v)$ be singular, $y(u)\, r(u; v) = 0$ where $y(u) \neq 0$, $y(u) \in X(u)$. Then $yA[U(u, v)] = 0$ as well if $y \neq 0$ is defined as in 2° of assertion **A** according to (17) in 2° of **A**, which is a contradiction.

An analogous argument based on 2° of **B** shows that all mappings $r(u)$ exist and are regular as well.

Conversely, let all mappings $r(u; v)$ and $r(u)$ exist and be regular. Then, $yA[U(u, v)] = 0$, $y = \sum_{w \in U(u,v)} y(w)$, $y(w) \in X(w)$ implies that all vectors $y(w)$ are zero according to (17) and (18) in 1° of assertion **A**. Analogously, $A[U(u)]$ is then regular by assertion **B**. The proof is complete since the rest is obvious.

Remark. If A is symmetric (in a unitary space), then the mappings $r(u)$ and $r(u; v)$ are all symmetric, too.

(3,4) *Let $F = [f(u, v)]$, $u, v \in U$, be a symmetric positive definite mapping in a partitioned unitary vector space $X = \dot{\sum} X(u)$, $u \in U$, such that $f(u, u)$ is identity in $X(u)$ and the graph G of the inverse mapping is e-simple. Then:*

1° *$f(u, v) = 0$ if $u \neq v$ and there is no path in G from u to v,*

$$f(u, v) = f(u, w_1) f(w_1, w_2) \dots f(w_k, v) \quad \text{if} \quad u \neq v$$

and $(u, w_1, \dots, w_k, v)$ is such a (unique) path,

2° *the Euclidean norms of $f(u, v)$*

$$n(u, v) = \sup \{\|x(u) f(u, v)\|;\ x(u) \in X(u),\ \|x(u)\| \leqq 1\}$$

(where $\|x\|^2 = \langle x, x \rangle$) have then the property that

$$n(u, v) < 1$$

if $u \neq v$, $u, v \in U$.

Proof. The first part follows immediately from the preceding theorem. To prove 2°, let (u, v) be an edge in G. Since F is positive definite, it follows from (3,2) that $e(u) - f(u, v) f(v, u)$ is positive definite where $e(u)$ is the identity mapping in $X(u)$.

Thus, if $x(u) \in X(u)$, $x(u) \neq 0$, $\|x(u)\| \leqq 1$,

$$0 < \langle x(u) [e(u) - f(u, v) f(v, u)],\ x(u) \rangle = \langle x(u), x(u) \rangle - \\ - \langle x(u) f(u, v),\ x(u) f(u, v) \rangle \leqq 1 - \|x(u) f(u, v)\|^2 .$$

Consequently, $\|x(u) f(u, v)\| < 1$ and, since X is finite-dimensional, $n(u, v) < 1$.

The general case in 2° follows then from 1° and the submultiplicative property of the Euclidean norm.

4. Concluding remarks. Let us show first that the task of inversion partitioned matrices of the type mentioned occurs at solving important problems of the numerical praxis. Thus, if we solve the Dirichlet problem for a simply connected region in the plane by finite difference methods using the most simple square lattice, then this lattice (without the boundary points) can be identified with the graph of the (not partitioned) matrix of the corresponding linear system. This graph is in general not e-simple. But, if we group the unknowns properly, e.g. so that we collect to one group those unknowns which form a connected segment of each "horizontal" lattice line, we obtain a partitioned matrix of the system whose graph is e-simple.

The relations in Th. (3,3) may have an importance at numerical inversion since they show that there exists a one-to-one correspondence between the mappings $a(u, u)$ and $a(u, v)$ – for $(u, v) \in G$ – on one side and $c(u, u)$ and $c(u, v)$ on the other side, provided that these mappings exist. This correspondence is according to (12) of a very simple kind in one direction. That may help to improve an approximate solution by iterative methods. The direct solution given by (13)–(15) contains perhaps too much inversions than to be of greater practical value. On the other side, the knowledge of elements $c(u, u)$ and $c(u, v)$ enables us to evaluate all elements of A by simple formulas (1'). If A is positive definite, this procedure is numerically stable in the following sense:

Let us put[2])

$$[c(u', u')]^{-1/2}\, c(u', v')\, [c(v', v')]^{-1/2} = f(u', v')$$

if $(u', v') \in G$; then,

$$b(u, v) = [c(u, u)]^{1/2} f(u, w_1) f(w_1, w_2) \ldots f(w_k, v)\, [c(v, v)]^{1/2}$$

where $(u, w_1, w_2, \ldots, w_k, v)$ is the path from u to v in G. But (3,4) shows then that the Euclidean norms of $f(u, w_1), f(w_1, w_2), \ldots, f(w_k, v)$ are all smaller than one.

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Резюме

ОБ ОБРАЩЕНИИ КЛЕТОЧНЫХ МАТРИЦ

МИРОСЛАВ ФИДЛЕР (Miroslav Fiedler), Прага

Доказываются теоремы о свойствах матрицы, обратной к клеточной матрице, клеточный граф которой таков, что всякое его ребро содержится не более чем в одном цикле. При ограничениях, касающихся невырожденности некоторых главных миноров этой матрицы, приведены рекуррентные формулы для нахождения обратной матрицы при помощи конечного числа шагов. При этом обращаются только матрицы меньшего порядка.

[2]) Since $c(u', u')$ is positive definite, a positive definite mapping $[c(u', u')]^{-\frac{1}{2}}$ exists.