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SEMI-GROUPS OF POSITIVE CONTRACTION OPERATORS

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The paper is concerned with the general problem of semi-groups of positive contraction operators in arbitrary Banach lattices. For discrete Banach lattices of l_p-type ($1 \leqq p < \infty$), the analogue of the Kolmogorov differential equations is considered.

1. Introduction. There is a voluminous literature dealing with a special class of strongly continuous semi-groups of positive contraction operators, namely stationary Markov processes. The usual setting for such a process is an L_1-type Banach lattice. Recently, however, some probabilists (see, for example, [6] and [8]) have found it convenient to study Markov processes in a Hilbert space setting, treating a special class of processes whose members were contraction operators in both the L_1 and the L_2 metrics. The present paper is concerned with the general problem of semi-groups of positive contraction operators in arbitrary Banach lattices.

Without assuming positivity, G. LUMER and R. S. PHILLIPS [11] have studied semi-groups of contraction operators, characterizing the generators of such semi-groups by means of the notion of a semi-inner-product, previously introduced by Lumer.

Definition 1.1. *A semi-inner-product (s. i. p.) associates with each ordered pair x, y of a real (complex) normed linear space $\mathfrak{X}$ a real (complex) number $[x, y]$ having the properties:*

$$(1.1)\qquad [x + y, z] = [x, z] + [y, z]\,, \quad [\lambda x, z] = \lambda[x, z]\,,$$
$$[x, x] = \|x\|^2\,, \quad |[x, z]| \leqq \|x\|\,\|z\|\,.$$

It is clear that such a s. i. p. is defined by choosing for each $y \in \mathfrak{X}$ a functional $Wy \in \mathfrak{X}^*$ such that $(y, Wy) = \|y\|^2$ and $\|Wy\| = \|y\|$. According to the Hahn-Banach theorem this can always be done in at least one way.

Definition 1.2. *An operator A with domain $\mathfrak{D}(A)$ is called dissipative if*

$$(1.2)\qquad \operatorname{re}[Ax, x] \leqq 0\,, \quad x \in \mathfrak{D}(A)\,,$$

and maximal dissipative if it is not the proper restriction of any other dissipative operator.

We state for future reference the following result on contraction semi-groups proved in [11]; for convenience we use the notation $\mathfrak{R}(A)$ to denote the range of A.

Theorem 1.1. *A necessary and sufficient condition for a linear operator A with dense domain to generate a strongly continuous semigroup of contraction operators is that A be dissipative with $\mathfrak{R}(I - A) = \mathfrak{X}$.*

The notion of positivity requires that we work within the structure of a partially ordered real vector space. As a matter of fact, we shall restrict our considerations to Banach lattices, defined in G. BIRKHOFF's treatise [1] as a complete normed real vector lattice for which the order relation and the norm are related by

$$(1.3) \qquad |x| \leqq |y| \quad \text{implies} \quad \|x\| \leqq \|y\| ;$$

here we have used the notation

$$(1.4) \qquad |x| = x^+ - x^- \quad \text{where} \quad x^+ = x \vee 0 \quad \text{and} \quad x^- = x \wedge 0 .$$

For such spaces we require two further properties of our s. i. p. (see lemma 2.1):

$$(1.5) \qquad \begin{array}{l} \text{i) If } x \geqq 0 \text{ then } [y, x] \geqq 0 \text{ for all } y \geqq 0 , \\ \text{ii) } [x, x^+] = \|x^+\|^2 . \end{array}$$

We now describe the essential property exhibited by generators of semi-groups of positive contraction operators.

Definition 1.3. *An operator A is called dispersive*[1]) *if*

$$(1.6) \qquad [Ax, x^+] \leqq 0 , \quad x \in \mathfrak{D}(A) .$$

In terms of this concept we can now state

Theorem 2.1. *A necessary and sufficient condition for a linear operator A with dense domain to generate a strongly continuous semi-group of positive contraction operators is that A be dispersive with $\mathfrak{R}(I - A) = \mathfrak{X}$.*

For discrete Banach lattices of the l_p-type $(1 \leqq p < \infty)$ we consider the analogue of the Kolmogorov differential equations solved by W. FELLER [2] for the case $p = 1$. To help formulate this problem it is convenient to introduce the following concepts.

Definition 1.4. *Let $\mathfrak{D}_0$ denote the set of all vectors having only a finite set of non-zero components. Then corresponding to the matrix (a_{ij}) we define the minimal operator A_0 with domain $\mathfrak{D}_0$ as*

$$(A_0 f)(i) = \textstyle\sum_j a_{ij} f(j) , \quad f \in \mathfrak{D}_0 ;$$

and the maximal operator A_1 with domain

$$\mathfrak{D}_1 = [f; f \in \mathfrak{X} , \quad g(i) = \textstyle\sum_j a_{ij} f(j) \ \text{converges absolutely for each } i \text{ and } g \in \mathfrak{X}],$$

$$(A_1 f)(i) = \textstyle\sum_j a_{ij} f(j) , \quad f \in \mathfrak{D}_1 .$$

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1) Bounded dispersive operators in l_2 spaces were previously considered by W. J. FIREY in a paper entitled "On ballistically closed regions", Applied Math. and Statistics Lab., Stanford University Technical Report No. *19*, 1954, 68 pages.

In order that A_0 make sense it is clear that the column vectors of (a_{ij}) must each belong to $\mathfrak{X}$. Employing a method of proof which combines ideas from the work of W. FELLER [3], T. KATO [7], and W. LEDERMANN and G. E. H. REUTER [10], we are able to establish

Theorem 3.1. *Let A_0 be a dispersive minimal matrix operator. Then there exists a strongly continuous semi-group of positive contraction operators $[F(t)]$ with generator A such that $A_0 \subset A \subset A_1$.*

It is shown that the semi-group $[F(t)]$ is minimal with respect to all semi-groups of contractions with generators $A' \supset A_0$ or $A' \subset A_1$. Actually $[F(t)]$ is even minimal with respect to all semigroups of positive contractions $[S(t) = (s_{ij}(t))]$ for which

$$\left.\frac{\mathrm{d}s_{ij}(t)}{\mathrm{d}t}\right|_0 = a_{ij}\,.$$

For the case $p = 1$, these results are well-known and are found in each of the above mentioned papers ([2], [3], [7], [10]). Moreover, W. B. JURKAT [5] has established the existence of a minimal solution to a generalized Kolmogorov equation in a much more general setting than ours; however, his development requires the *a priori* existence of some positivity preserving matrix solution to the given equations. What is novel in this part of the present work is the characterization of those matrices for which a solution exists in the form of a semi-group of positive contraction operators in the given (discrete) Banach lattice.

When $\mathfrak{X} = l_2$ and A_0 is symmetric as well as dispersive, we show that the generator A of $[F(t)]$ is the Friedrichs' self-adjoint extension of A_0. Another result (and a somewhat disturbing result) is that for $\mathfrak{X} = l_p$ $(1 < p < \infty)$ the only *honest process* (i. e., $\|S(t)\,x\| = \|x\|$ for all $x \geqq 0$ and all $t \geqq 0$) is the trivial semigroup $[S(t) \equiv I]$.

The previous theory can be used to shed some light on the existence of a generator A of a semi-group of *contraction* operators when it is required to be both an extension of a given *dissipative* minimal matrix operator A_0 and a restriction of the corresponding maximal matrix operator A_1.

Definition 1.5. *A minimal matrix operator A_0 with elements (a_{ij}) is said to be majorized by the matrix operator M_0 with elements (m_{ij}) if* (i) *M_0 is a dispersive minimal matrix operator, and* (ii) $0 \geqq m_{ii} \geqq \mathrm{re}\,[a_{ii}]$ *and* $|a_{ij}| \leqq m_{ij}$ *for all* $i \neq j$.

In terms of this concept we are able to prove

Theorem 4.1. *If A_0 is a dissipative minimal matrix operator which is majorizable, then there exists a dissipative generator A such that $A_0 \subset A \subset A_1$.*

Although this theorem is applicable in all discrete complex Banach spaces of the l_p-type $(1 \leqq p < \infty)$, it is only for the case $p = 1$ that all dissipative minimal matrix operators are majorizable (lemma 4.1). Hence it is only for $p = 1$ that we obtain a complete solution for the above posed problem.

2. General theory. The principal result of this section is theorem 2.1 which characterizes the generators of strongly continuous semi-groups of positive contraction operators. Before proceeding to the proof of this theorem, we shall verify the fact that there exists a s. i. p. with the properties (1.5) in a Banach lattice. Since $x^+ \wedge (-x^-) = 0$ for any $x \in \mathfrak{X}$, it is clear that it suffices to prove

Lemma 2.1. *Given $x \geqq 0$, there exists an $F \in \mathfrak{X}^*$ satisfying* a) *F is positive,* b) *$Fx = \|x\|^2 = \|F\|^2$. and* c) *$Fy = 0$ for every y such that $x \wedge |y| = 0$.*

Proof. Setting $N = [y; x \wedge |y| = 0]$; it can be shown that N is a closed linear subspace and that if $|z| \leqq |y|$ for $y \in N$, then $z \in N$. Moreover $\|x - y\| \geqq \|x\|$ for all $y \in N$. In fact, according to [1; p. 220]

$$|x - y| = x \vee y - x \wedge y$$

and since $x \vee y \geqq x$ and $x \wedge y \leqq x \wedge |y| = 0$, we see that $|x - y| \geqq x$ and hence the assertion follows from (1.3). By the Hahn-Banach theorem there exists an $F \in \mathfrak{X}^*$ such that $\|F\| = \|x\|$, $Fx = \|x\|^2$, and $F(N) = 0$. Next we decompose F into its positive and negative parts (cf. [1; p. 245 and p. 248]): $F = F^+ - F^-$ where for $y \geqq 0$, $F^+y = \sup [Fz; 0 \leqq z \leqq y]$. It is clear from the above stated properties of N that $F^+(N) = 0$. Further for arbitrary $z \in \mathfrak{X}$, we have

$$|F^+z| = |F^+z^+ + F^+z^-| \leqq \max(|F^+z^+|, |F^+z^-|) \leqq \|F\| \max(\|z^+\|, \|z^-\|) \leqq \\ \leqq \|F\| \, \|z\|$$

so that $\|F^+\| \leqq \|F\|$. Finally for the given x

$$Fx \leqq F^+x \leqq \|F^+\| \, \|x\| \leqq \|F\| \, \|x\| = \|x\|^2 = Fx$$

and consequently $F^+x = Fx = \|x\|^2$ and $\|F^+\| = \|F\|$. It follows that F^+ satisfies the assertion of the lemma.

The following lemma is essential to the proof of theorem 2.1:

Lemma 2.2. *If T is a linear positive operator contractive on positive elements, that is $\|Tx\| \leqq \|x\|$ if $x \geqq 0$, then T is a contraction operator.*

Proof. Since $|z + y| \leqq |z| + |y|$, we see that

$$|Tx| = |Tx^+ + Tx^-| \leqq |Tx^+| + |Tx^-| = T(x^+ - x^-) = T|x|$$

and hence by (1.3)

$$\|Tx\| \leqq \|T|x|\| \leqq \||x|\| = \|x\| \, .$$

Theorem 2.1. *A necessary and sufficient condition for a linear operator A with dense domain to generate a strongly continuous semi-group of positive contraction operators is that A be dispersive with $\mathfrak{R}(I - A) = \mathfrak{X}$.*

Proof. If A generates a semi-group of positive contraction operators $[S(t)]$, then $\mathfrak{R}(I - A) = \mathfrak{X}$ by the Hille-Yosida theorem [4; theorem 12.3.1]; and further

$$(2.1) \qquad [x, x^+] = \|x^+\|^2 \geqq \|S(t)\,x^+\| \, \|x^+\| \geqq [S(t)\,x^+, x^+] \\ \geqq [S(t)\,x^+, x^+] + [S(t)\,x^-, x^+] = [S(t)\,x, x^+]$$

so that for $x \in \mathfrak{D}(A)$

$$[Ax, x^+] = \frac{d}{dt} [S(t)\, x, x^+] \Big|_0 \leqq 0 ,$$

which proves that A is dispersive.

In order to prove the converse assertion, let us suppose for the moment that $\mathfrak{R}(\lambda I - A) = \mathfrak{X}$ for some $\lambda > 0$. Then for fixed $f > 0$ in $\mathfrak{X}$ there is an $x \in \mathfrak{D}(A)$ such that $\lambda x - Ax = f$. Making use of the dispersive property of A we see that

$$\lambda \|x^-\|^2 = \lambda[-x, (-x)^+] \leqq \lambda[-x, (-x)^+] - [A(-x), (-x)^+] = \\ = [-f, (-x)^+] \leqq 0$$

consequently $x \geqq 0$ and

$$\lambda \|x\|^2 = \lambda[x, x^+] \leqq \lambda[x, x^+] - [Ax, x^+] = [f, x^+] \leqq \|f\| \, \|x\| .$$

Thus

$$\lambda \|x\| \leqq \|f\| . \tag{2.2}$$

Since 0 is a non-negative element, the relations (2.2) implies that $(\lambda I - A)$ is one-to-one. Hence (2.2) together with lemma 2.2 implies that

$$\lambda R(\lambda; A) \equiv \lambda(\lambda I - A)^{-1}$$

is a positive contraction operator. Now according to [4; corollary 2 to theorem 5.8.4]

$$R(\mu; A) = R(\lambda; A) [I - (\mu - \lambda) R(\lambda; A)]^{-1}$$

holds for $|\mu - \lambda| < 1/\lambda$. In particular then, $\mathfrak{R}(\mu I - A) = \mathfrak{X}$ for $|\mu - \lambda| < 1/\lambda$ and the dispersive property shows as above that $\mu R(\mu; A)$ is a positive contraction operator in this range. This permits us to extend the result by analytic continuation to all $\mu > 0$ once it is known that $\mathfrak{R}(\lambda I - A) = \mathfrak{X}$ for some $\lambda > 0$. However this is precisely what is assumed in the hypothesis to the theorem. The Hille-Yosida theorem [4; theorem 12.3.1] therefore applies and establishes the fact that A is the generator of a strongly continuous semi-group of contraction operators $[S(t)]$. It is evident from the proof of the Hille-Yosida theorem that

$$S(t)\, x = \lim_{\lambda \to \infty} \exp(-\lambda t) \sum_{m=0}^{\infty} \frac{(\lambda t)^n}{n!} [\lambda\, R(\lambda; A)]^n\, x \tag{2.3}$$

and it follows from this expression that $S(t)$ is a positive operator if $\lambda\, R(\lambda; A)$ is positive.

Combining theorems 1.1 and 2.1, we obtain

Corollary. *If $\mathfrak{X}$ is a Banach lattice and A is a dispersive semi-group generator, then A is also dissipative.*

We do not know whether an arbitrary dispersive operator is dissipative. However, as the following lemma shows this is the case for the familiar Banach lattices:

Lemma 2.3. *If $\mathfrak{X}$ is a Banach lattice with s. i. p. satisfying the condition*

$$[y, x] = \alpha[y, x^+] - \beta[y, (-x)^+] , \quad y \in \mathfrak{X} \tag{2.4}$$

for some $\alpha, \beta \geqq 0$ (depending on x), then each dispersive operator on $\mathfrak{X}$ is also dissipative.

Proof. For $x \in \mathfrak{D}(A)$, the relation (2.4) implies that

$$[Ax, x] = \alpha[Ax, x^+] - \beta[Ax, (-x)^+] ;$$

and since A is dispersive, we have $[Ax, x^+] \leqq 0$ and $[A(-x), (-x)^+] \leqq 0$ from which $[Ax, x] \leqq 0$ follows.

3. Generalized Kolmogorov differential equations. In this section we study the analogue of the Kolmogorov differential equations for a general class of discrete Banach lattices. More specifically we suppose that $\mathfrak{X}$ is a function space, that is a class of real-valued functions $[f(i); i \in \mathfrak{I}]$ on an abstract set $\mathfrak{I}$, satisfying the usual algebraic relations and in addition

(3.1) (i) The set $\mathfrak{D}_0$ of all functions with only a finite set of non-zero components belongs to $\mathfrak{X}$;

(ii) $f \leqq g$ is taken to mean that $f(i) \leqq g(i)$ for all $i \in \mathfrak{I}$;

(iii) Any monotone increasing directed system of positive elements $[f_\mu]$ which is bounded in norm is a Cauchy sequence and converges to $\bigvee f_\pi$.

As a consequence $\mathfrak{D}_0$ is dense in $\mathfrak{X}$. In fact, for $f \in \mathfrak{X}$ let π denote any finite subset of $\mathfrak{T}$, order the π's by inclusion, and set $f_\pi(i) = f(i)$ for $i \in \pi$ and $= 0$ otherwise. Then for $\pi_1 \leqq \pi_2$, $|f_{\pi_1}| \leqq |f_{\pi_2}| \leqq |f|$ and $|f - f_\pi| = |f| - |f_\pi|$; hence

$$\|f_\pi - f\| \leqq \||f_\pi| - |f|\|$$

which converges to zero by property (iii) above. It also follows that if $f \in \mathfrak{X}$ and $|g| \leqq |f|$, then $g \in \mathfrak{X}$. It is clear that the l_p spaces $(1 \leqq p < \infty)$ over sets of any cardinality are examples of such spaces, as are product spaces such as $l_p \times l_q$ $(1 \leqq p, q < \infty)$.

Any operator A with domain containing $\mathfrak{D}_0$ can be represented on $\mathfrak{D}_0$ as a matrix operator: $(Af)(i) = \sum_j a_{ij} f(j)$, $f \in \mathfrak{D}_0$.

Lemma 3.1. *If A is a dispersive operator with $\mathfrak{D}(A) \supset \mathfrak{D}_0$, then $a_{ii} \leqq 0$ and $a_{ij} \geqq 0$ for $i \neq j$.*

Proof. Suppose x_j is defined as $x_j(i) = 0$ for $i \neq j$ and $x_j(j) = 1$. Then it is clear that $[f, x_j] = \|x_j\|^2 f(j)$. Hence $[Ax_j, x_j] \leqq 0$ implies $a_{jj} \leqq 0$. Likewise setting $x = \varepsilon x_i - x_j$, $i \neq j$ and $\varepsilon > 0$, the relation

$$[Ax, x^+] = \varepsilon\|x_i\|^2 (\varepsilon a_{ii} - a_{ij}) \leqq 0$$

for all $\varepsilon > 0$, implies $a_{ij} \geqq 0$

Remark 1. If $\mathfrak{X} = l_1(w)$ with norm $\|f\| = \sum w_i |f(i)|$ (here the w_i are positive weight factors), the notion of a dispersive minimal matrix operator and a Kolmogorov matrix operator coincide. In fact for a fixed finite subset π of $\mathfrak{I}$, suppose $i \in \pi$ and define $x(i) = 1$, $x(j) = \varepsilon > 0$ for $j \in \pi$, $j \neq i$, and $x(j) = 0$ otherwise.

Then

$$0 \geqq [Ax, x] = \|x\|[\sum_{k\in\pi} w_k(a_{ki} + \varepsilon \sum_{\substack{j\in\pi \\ j\neq i}} a_{kj})]$$

for all $\varepsilon > 0$ and π implies

$$\sum_{k\in I} w_k a_{ki} \leqq 0 \tag{3.2}$$

which is the Kolmogoroff condition when combined with $a_{ii} \leqq 0$ and $a_{ij} \geqq 0$ for $i \neq j$. It is easy to see that this condition also suffices to make the minimal matrix operator dispersive.

Remark 2. Let $\mathfrak{X} = l_p(w)$ with norm $\|f\| = [\sum w_i |f(i)|^p]^{1/p}$. Then if A is a dissipative minimal matrix operator such that $a_{ii} \leqq 0$ and $a_{ij} \geqq 0$ for $i \neq j$, then A is necessarily dispersive. In fact given $x \in \mathfrak{D}_0$ and setting $y(i) = w(i)\, x(i)^{p-1}/\|x^+\|^{p-2}$ for $x(i) > 0$ and $= 0$ otherwise, we see that

$$[Ax, x^+] = \sum_{x(i)>0} (\sum_j a_{ij} x(j))\, y(i) = [Ax^+, x^+] + \sum_{\substack{x(j)<0 \\ x(i)>0}} a_{ij} x(j)\, y(i) \leqq [Ax^+, x^+] \leqq 0\,,$$

since $a_{ij} \geqq 0$ if $i \neq j$ and $x(j)\, y(i) < 0$ for $x(i) > 0$ and $x(j) < 0$.

We include for completeness the following generalization of a lemma due to G. E. H. Reuter [15; lemma 1.1] (cf. W. Feller [3; theorem 3.1]):

Lemma 3.2. *In order that a family of linear bounded operators* $[R_\lambda;\, \lambda > 0]$ *be resolvent operators for the generator of a semi-group of (positive) contraction operators it is necessary and sufficient that*

(i) $R_\lambda - R_\mu = (\mu - \lambda)\, R_\mu R_\lambda\,, \quad \lambda, \mu > 0\,,$

(ii) λR_λ *is a (positive) contraction operator for each* $\lambda > 0$,

(iii) $\lim_{\lambda\to\infty} \lambda R_\lambda x = x\,, \quad x \in \mathfrak{X}\,.$

Proof. The necessity is clear from well-known properties of the resolvents of generators of semi-groups of (positive) contraction operators (see [4; theorems 5.8.1, 11.7.1, 11.7.2, and lemma 12.2.1]). On the other hand, operators R_λ satisfying the above properties must be one-to-one. For if $R_\lambda x = 0$, then by (i) $R_\mu x = 0$ for all $\mu > 0$ and (iii) implies that $x = 0$. According to [4; theorem 5.8.3] the R_λ's are resolvent operators for some closed linear operator, say A. Since $\mathfrak{D}(A) = \mathfrak{R}[R_\lambda]$ it follows from (iii) that $\mathfrak{D}(A)$ is dense. Hence (ii) together with the Hille-Yosida theorem ([4; theorem 12.3.1]) implies that A generates a strongly continuous semi-group of (positive) contraction operators.

Corollary. *The lemma remains valid if condition* (iii) *is replaced by*

$$R_\lambda(\lambda I - A_0)\, x = x\,, \quad x \in \mathfrak{D}(A_0)\,, \tag{iv}$$

for some $\lambda > 0$ *where* $\mathfrak{D}(A_0)$ *is dense in* $\mathfrak{X}$. *In this case the generator* A *is an extension of* A_0.

Proof. It suffices to show that (iv) implies (iii). However, for $x \in \mathfrak{D}(A_0)$, we see from (ii) and (iv) that $\|\lambda R_\lambda x - x\| = \|R_\lambda A_0 x\| = O(1/\lambda)$. Thus (iii) holds for all x in $\mathfrak{D}(A_0)$ and since this set is dense, condition (ii) allows us to assert (i) for all x in $\mathfrak{X}$.

We now establish the existence of a semi-group solution to our generalized Kolmogorov equations and in deference to Feller we denote this solution by $[F(t)]$. The minimal properties of this solution will be verified afterwards.

Theorem 3.1. *Let A_0 be a dispersive minimal matrix operator. Then there exists a strongly continuous semi-group of positive contraction operators $[F(t)]$ with generator A such that $A_0 \subset A \subset A_1$.*

Proof. Let π denote a generic finite subset of $\mathfrak{J}$. The class of π's, ordered by inclusion, forms a directed set. Corresponding to each π we define the matrix operator $C_\pi = (c_{ij}^\pi)$ where $c_{ij}^\pi = a_{ij}$ if $i, j \in \pi$ and $i \neq j$, and $c_{ij}^\pi = 0$ otherwise; then $c_{ij}^\pi \geqq 0$ for all i, j. Since C_π has only a finite set of non-zero elements it is well defined with $\mathfrak{D}(C_\pi) = \mathfrak{X}$. Next we define $B = (b_{ij})$ where $b_{ij} = a_{ii}$ for $i = j$ and $b_{ij} = 0$ otherwise; then $b_{ij} \leqq 0$ for all i, j. As to its domain, we set

$$\mathfrak{D}(B) = [f; f \text{and} \{a_{ii} f(i)\} \in \mathfrak{X}] .$$

We now approximate the desired operator by

$$A_\pi = B + C_\pi \quad \text{with} \quad \mathfrak{D}(A_\pi) = \mathfrak{D}(B) . \tag{3.3}$$

Finally we decompose $\mathfrak{X}$ into $\mathfrak{X}_\pi$ and $\mathfrak{X}'_\pi$ where

$$\begin{aligned} \mathfrak{X}_\pi &\equiv [f; f(i) = 0 \quad \text{if} \quad i \notin \pi] , \\ \mathfrak{X}'_\pi &\equiv [f; f(i) = 0 \quad \text{if} \quad i \in \pi] . \end{aligned} \tag{3.4}$$

It is clear that A_π leaves $\mathfrak{X}_\pi$ and $\mathfrak{X}'_\pi$ invariant and that A_π restricted to $\mathfrak{X}_\pi$ (in symbols $A_\pi/\mathfrak{X}_\pi$) is the same as $A_0/\mathfrak{X}_\pi$ as concerns the dispersive relation. Hence $A_\pi/\mathfrak{X}_\pi$ is dispersive and since $I/\mathfrak{X}_\pi - (A_\pi/\mathfrak{X}_\pi)$ is one-to-one (by 2.2)) and $\mathfrak{X}_\pi$ is finite dimensional we have $\mathfrak{R}[(I/\mathfrak{X}_\pi) - (A_\pi/\mathfrak{X}_\pi)] = \mathfrak{X}_\pi$. On the other hand $A_\pi/\mathfrak{X}'_\pi$ is diagonal with non-positive elements and hence dispersive and it is readily verified that $\mathfrak{R}[(I/\mathfrak{X}'_\pi) - - (A_\pi/\mathfrak{X}'_\pi)] = \mathfrak{X}'_\pi$. Again by (2.2) we see that for $\lambda > 0$, $\lambda R(\lambda; A_\pi)$ exists and is a positive contraction operator when restricted to either $\mathfrak{X}_\pi$ or $\mathfrak{X}'_\pi$; consequently it is positive and of norm $\leqq 2$ on $\mathfrak{X}$ itself.

For a given $f \geqq 0$ in $\mathfrak{D}_0$, we consider only those π which contain the support of f. In this case $x_\pi = R(\lambda; A_\pi) f \in \mathfrak{X}_\pi$ and $\lambda\|x_\pi\| \leqq \|f\|$. For $\pi_1 \leqq \pi_2$, it is clear that $C_{\pi_1} \leqq C_{\pi_2}$ so that

$$R(\lambda; A_{\pi_2}) - R(\lambda; A_{\pi_1}) = R(\lambda; A_{\pi_2}) (C_{\pi_2} - C_{\pi_1}) R(\lambda; A_{\pi_1}) \geqq 0 .$$

Thus $0 \leqq x_{\pi_1} \leqq x_{\pi_2}$ and we may conclude from (3.1) that $\{x_\pi\}$ for a Cauchy sequence with $\lim_\pi x_\pi \equiv x = \bigvee x_\pi$ and $\lambda\|x\| \leqq \|f\|$. Since $\mathfrak{D}_0$ is dense in $\mathfrak{X}$, we see that

$$\lambda R_\lambda \equiv \text{strong limit}_\pi \, \lambda \, R(\lambda; A_\pi)$$

exists, that it is positive and contracting on positive elements, and hence by lemma 2.2 that it is a positive contraction operator. Further the strong limit of resolvent operators satisfies the first resolvent equation and thus condition (i) of lemma 3.2. Finally for each $x \in \mathfrak{D}_0 \subset \mathfrak{D}(A_\pi)$ we have

$$R(\lambda; A_\pi)(\lambda I - A_\pi)x = x$$

and

$$\lim_\pi A_\pi x = A_0 x .$$

Passing to the limit we then obtain $R_\lambda(\lambda I - A_0)x = x$. It follows from lemma 3.2 that R_λ is the resolvent of a generator A of a semi-group $[F(t)]$ of positive contraction operators and that $A \supset A_0$.

It remains to show that $A \subset A_1$. It clearly suffices to consider only elements in $\mathfrak{D}(A)$ of the form $x = R(\lambda; A)f$ for $f \geqq 0$. In the notation of the previous paragraph $x = \lim_\pi x_\pi$ where $(\lambda I - A_\pi)x_\pi = f$; in particular

$$(\lambda - a_{ii})x_\pi(i) = f(i) + \sum_{\substack{j \in \pi \\ j \neq i}} a_{ij} x_\pi(j), \quad i \in \pi .$$

The sum on the right consists of non-negative terms each of which is monotonic non-decreasing in π. The monotonicity which was proved only for positive f in $\mathfrak{D}_0$ holds for all $f \geqq 0$ by continuity. Since the equality is termwise convergent, it follows by Fatou's lemma that the equation holds in the limit; that is

$$(\lambda - a_{ii})x(i) = f(i) + \sum_{j \neq i} a_{ij} x(j), \quad i \in \mathfrak{J} .$$

Transposing the infinite sum to the left hand member we see that $\sum_j a_{ij} x(j)$ is absolutely convergent for each $i \in \mathfrak{J}$ and that

$$(Ax)(i) = (\lambda x - f)(i) = \sum_j a_{ij} x(j), \quad i \in \mathfrak{J} .$$

This concludes the proof of theorem 3.1.

Remark. For any $f \geqq 0$ and $x_\pi = R(\lambda; A_\pi)f \in \mathfrak{D}(A_\pi) = \mathfrak{D}(B)$, it is clear that

$$\lambda x_\pi - Bx_\pi = f + C_\pi x_\pi$$

so that

$$x_\pi = R(\lambda; B)f + R(\lambda; B)C_\pi x_\pi = \sum_{k=0}^{n} R(\lambda; B)[C_\pi R(\lambda; B)]^k f + [R(\lambda; B)C_\pi]^{n+1} x_\pi .$$

Hence

$$0 \leqq \sum_{k=0}^{\infty} R(\lambda; B)[C_\pi R(\lambda; B)]^k f \leqq x_\pi$$

and it follows that the infinite series converges in norm for $f \geqq 0$ and hence for arbitrary $f \in \mathfrak{X}$. In particular then $[R(\lambda; B)C_\pi]^n R(\lambda; B)f \to 0$ and consequenctly $[R(\lambda; B)C_\pi]^n z \to 0$ for all $z \in \mathfrak{D}(B)$. Therefore

$$R(\lambda; A_\pi)f = \sum_{k=0}^{\infty} R(\lambda; B)[C_\pi R(\lambda; B)]^k f . \tag{3.5}$$

We now consider the minimal properties of the process $[F(t)]$.

Theorem 3.2. *Let A_0 be a dispersive minimal matrix operator, let A_1 be the corresponding maximal matrix operator, and let A be the generator of the process $[F(t)]$ constructed in theorem 3.1. Suppose that A' is the generator of a semi-group of positive contraction operators $[S(t)]$ and either $A' \subset A_1$ or $A' \supset A_0$. Then $F(t) \leqq S(t)$ for all $t \geqq 0$.*

Proof. In order to prove that $F(t) \leqq S(t)$ for all $t \geqq 0$, it suffices to show that $R(\lambda; A) \leqq R(\lambda; A')$ for all $\lambda > 0$. For in this case $[R(\lambda; A)]^n \leqq [R(\lambda; A')]^n$ for all $\lambda > 0$ and integers $n \geqq 0$ and it follows from (2.3) that $F(t) \leqq S(t)$. Suppose first that $A' \supset A_0$ and let $f \geqq 0$ belong to $\mathfrak{D}_0$. Then in the notation of the proof of theorem 3.1, we have $R(\lambda; A_\pi) f \in \mathfrak{D}_0$ and since $A' - A_\pi = A_0 - A_\pi$ on $\mathfrak{D}_0$ (and hence has only non-negative matrix elements as an operator on $\mathfrak{D}_0$), the second resolvent equation yields

$$R(\lambda; A') f - R(\lambda; A_\pi) f = R(\lambda; A')(A' - A_\pi) R(\lambda; A_\pi) f \geqq 0 .$$

Now $\mathfrak{D}_0^+$ is dense in $\mathfrak{X}^+$ so that $R(\lambda; A') f \geqq R(\lambda; A_\pi) f$ for all $f \geqq 0$, and passing to the limit with π we obtain $R(\lambda; A') f \geqq R(\lambda; A) f$, which was to be proved.

Next suppose that $A' \subset A_1$ and take $f \geqq 0$. Setting $x' = R(\lambda; A') f$ and $x_\pi = R(\lambda; A_\pi) f$, we see that

$$(3.6) \qquad (\lambda - a_{ii})\, x'(i) = f(i) + \sum_{j \neq i} a_{ij}\, x'(j) ;$$

$$(\lambda - a_{ii})\, x_\pi(i) \begin{cases} = f(i) + \sum_{\substack{j \in \pi \\ j \neq i}} a_{ij}\, x_\pi(j), & i \in \pi, \\ = f(i), & i \notin \pi . \end{cases}$$

For $i \notin \pi$ it is clear from these relations that $x'(i) \geqq x_\pi(i) \geqq 0$. On the other hand

$$[\lambda(I/\mathfrak{X}_\pi) - (A_\pi/\mathfrak{X}_\pi)]\, \{x'(j) - x(j);\ \ j \in \pi\} = \{ \sum_{j\,\text{non} \in \pi} a_{ij}\, x'(j) ;\ \ i \in \pi\}$$

has a unique (positive) solution because of the dispersive property of $A_0/\mathfrak{X}_\pi = A_\pi/\mathfrak{X}_\pi$; thus $x'(i) \geqq x_\pi(i)$ for all $i \in \pi$. Consequently $x' \geqq x_\pi$ and passing to the limit with π we conclude that $R(\lambda; A') f \geqq R(\lambda; A) f$.

The $[F(t)]$ process is minimal with respect to an even larger class of semi-groups which can be associated with the matrix (a_{ij}) by means of the following result due to W. F. Jurkat [5]: Let $[(p_{ij}(t))]$ denote a semi-group of positive matrices satisfying the condition $p_{ij}(t) \to \delta_{ij}$ as $t \to 0^+$; then

$$a_{ii} \equiv \lim_{t \to 0+} \frac{p_{ii}(t) - 1}{t} \leqq 0$$

exists but may be infinite, and

$$a_{ij} \equiv \lim_{t \to 0+} p_{ij}(t)/t \geqq 0$$

exists and is finite for all $i \neq j$. In particular this applies to any strongly continuous semi-group of positive contraction operators.

Lemma 3.3. *Let $[S(t) = (s_{ij}(t))]$ be a strongly continuous semi-group of positive contraction operators and set $a_{ij} = s'_{ij}(0)$. If the column vectors of the matrix (a_{ij}) belong to $\mathfrak{X}$, then the minimal matrix operator A_0 associated with (a_{ij}) is dispersive.*

Proof. Let $y \in \mathfrak{D}_0$ and suppose that the support of y is contained in the finite subset π of $\mathfrak{J}$. Then the s. i. p. functional associated with y as in lemma 2.1 vanishes for all z with $z(i) = 0$ for all i in π. Consequently $[S(t)\, y, y^+]$ depends only on the $[s_{ij}(t);\ i, j \in \pi]$ portion of $S(t)$ so that its derivative at $t = 0$ exists and depends only on the $[a_{ij};\ i, j \in \pi]$ portion of A_0. Applying the inequality (2.1) we obtain

$$\frac{d}{dt}[S(t)\, y, y^+]|_0 = [A_0 y, y^+] \leqq 0,$$

which was to be proved.

It should be emphasized that the above lemma does not require the infinitesimal generator A' of $[S(t)]$ to be an extension of A_0, nor, for that matter, a restriction of the maximal matrix operator A_1. Never-the-less we have the following result:

Theorem 3.3. *Suppose $[S(t)]$ is a strongly continuous semi-group of positive contraction operators with the column vectors of $(a_{ij} \equiv s'_{ij}(0))$ in $\mathfrak{X}$ and let $[F(t)]$ be the process associated with (a_{ij}) as in theorem 3.1. Then $S(t) \geqq F(t)$ for all $t \geqq 0$.*

Proof. Let A' denote the infinitesimal generator of $[S(t)]$ and suppose that $x \geqq 0$ belongs to $\mathfrak{D}(A')$. Then

$$(A'x)(i) = \lim_{t \to 0+} \{t^{-1}(s_{ii}(t) - 1)\, x(i) + \sum_{j \neq i} t^{-1} s_{ij}(t)\, x(j)\},$$

so that by Fatou's lemma we have

$$(A'x)(i) \geqq a_{ii}\, x(i) + \sum_{j \neq i} a_{ij}\, x(j)\,. \tag{3.7}$$

Now let $f \geqq 0$ be given and set $x = R(\lambda; A')f$ and $x_\pi = R(\lambda; A_\pi)f$, where again we use the notation of theorem 3.1. Then $\lambda x - A'x = f$ implies

$$(\lambda - a_{ii})\, x(i) \geqq f(i) + \sum_{j \neq i} a_{ij} x(j)\,.$$

Comparing this with the corresponding relation for x_π namely (3.6), we obtain precisely as in the proof of theorem 3.2 the fact that $R(\lambda; A') \geqq R(\lambda; A)$, where A is the generator for the $[F(t)]$ process. As in the proof of theorem 3.2, this implies the assertion of the theorem.

Remark 1. It is interesting to note that when $[S(t)]$ is a strongly continuous semi-group of positive contraction operators with generator A' and when $A' \supset A_0$ or $A' \subset A_1$, where as before A_0 and A_1 are minimal and maximal matrix operators associated with (a_{ij}), then $s'_{ij}(0) = a_{ij}$. This is obvious when $A' \supset A_0$ for in this case $x_i = \{x_i(j) = \delta_{ij}\} \in \mathfrak{D}_0 \subset \mathfrak{D}(A')$ and $s'_{ij}(0) = (A'x_j)(i) = (A_0 x_j)(i) = a_{ij}$.

On the other hand when $A' \subset A_1$ then theorem 3.2 applies and we see that $S(t) \geqq$ $\geqq F(t)$. Thus if we set $\alpha_{ij} = s'_{ij}(0)$, then it follows from this that

$$\alpha_{ij} \geqq a_{ij} \tag{3.8}$$

and in particular that $\alpha_{ii} > -\infty$. Moreover for $x \geqq 0$ in $\mathfrak{D}(A') \subset \mathfrak{D}(A_1)$ we have

$$(A'x)(i) = \sum_j a_{ij}\, x(j)\,;$$

whereas by Fatou's lemma we have as in (3.7)

$$(A'x)(i) \geqq \sum_j \alpha_{ij}\, x(j)\,.$$

Consequently $\sum a_{ij}\, x(j) \geqq \sum \alpha_{ij}\, x(j)$ and combining this with (3.8) we see that $a_{ij} =$ $= \alpha_{ij}$ provided $x(j) \neq 0$. However for any $f \geqq 0$ $\lambda R(\lambda; A')f \geqq 0$ and converges to f as $\lambda \Rightarrow \infty$. Thus for each j there is an $x \geqq 0$ in $\mathfrak{D}(A')$ such that $x(j) > 0$, and therefore $a_{ij} = \alpha_{ij}$ for all i, j.

Remark 2. The preceding theorems can be extended so as not to require the column vectors of (a_{ij}) to lie in $\mathfrak{X}$. In this case the notion of a minimal matrix operator may not be meaningful. Never-the-less the operators $A_\pi/\mathfrak{X}_\pi$ are well defined and we can require that each of these operators be dispersive. We can then proceed to construct the process $[F(t)]$ as in the proof of theorem 3.1. The argument showing that R_λ = strong limit $R(\lambda; A_\pi)$ exists and satisfies the first resolvent equation for $\lambda > 0$ remains valid. The relation $R_\lambda(\lambda I - A_0)\, x = x, x \in \mathfrak{D}_0$, no longer makes sense. Instead we can prove that $\lim_{\lambda \to \infty} \lambda R_\lambda f = f$ for all $f \in \mathfrak{X}$, provided we further assume that $\mathfrak{X}$ is a uniformly monotone Banach lattice. As defined in [1, p. 248] this means that given $\varepsilon > 0$ there is a $\delta > 0$ such that for $f, g \geqq 0$ and $\|f\| = 1$, then $\|f + g\| \leqq \|f\| + \delta$ implies $\|g\| \leqq \varepsilon$. Now for $f > 0$,

$$\|\{\lambda R_\lambda f - \lambda\, R(\lambda; A_\pi) f\} + \lambda\, R(\lambda; A_\pi) f\| = \|\lambda R_\lambda f\| \leqq \|f\|$$

and since $\lambda R(\lambda; A_\pi) f \to f$, the uniform monotonicity of the norm implies that $\|\lambda R_\lambda f - \lambda\, R(\lambda; A_\pi) f\| \to 0$ and hence that $\lambda R_\lambda f \to f$.

Lemma 3.2 now shows that R_λ is the resolvent of a generator A of a semi-group of positive contraction operators. Finally one shows as in the proof of theorem 3.1 that $A \subset A_1$. The proof of theorem 3.2 shows that $[F(t)]$ is minimal over all semi-groups of positive contraction operators having generators $A' \subset A_1$. For an arbitrary semi-group of positive contraction operators $[S(t)]$ with $a_{ij} \equiv s'_{ij}(0)$ finite for all i, j, one proves as in lemma 3.2 that $A_\pi/\mathfrak{X}_\pi$ is dispersive and the proof of theorem 3.3 shows that $F(t) \leqq S(t)$ for all $t \geqq 0$.

Theorem 3.4. *Suppose $\mathfrak{X} = l_2(w)$ and A_0 is a symmetric dispersive minimal matrix operator. In this case the generator A of the minimal process $[F(t)]$ constructed in theorem* 3.1 *is the Friedrichs' self-adjoint extension of A_0.*

Proof. It will be recalled that $R(\lambda; A)$ is the strong limit of the approximating resolvents $R(\lambda; A_\pi)$ where A_π is defined as in (3.3). Now A_π is obviously self-adjoint and hence so is $R(\lambda; A_\pi)$ and $R(\lambda; A)$ for $\lambda > 0$, and finally so is A.

We next show that the Friedrichs' extension, which we denote by A', is dispersive. The Friedrichs' extension is defined as follows: Let

$$(3.9) \qquad \langle x, y\rangle = -(A_0 x, y) + (x, y), \quad x, y \in \mathfrak{D}_0 .$$

Condition (2.4) is satisfied in $l_2(w)$ so that A_0 is also dissipative, that is $(A_0 x, x) \leqq 0$ for all $x \in \mathfrak{D}_0$. As a consequence (3.9) defines a new inner product on $\mathfrak{D}_0$. If $\mathfrak{D}_1$ denotes the completion of $\mathfrak{D}_0$ with respect to this new metric, then it can be shown that $\mathfrak{D}_1 \subset l_2(w)$. In terms of these notions, the Friedrichs' extension is given by

$$A' \subset A_0^* \quad \text{and} \quad \mathfrak{D}(A') = \mathfrak{D}_1 \cap \mathfrak{D}(A_0^*) .$$

Now for $x \in \mathfrak{D}_0$, $(x, x) = (x^+, x^+) + (x^-, x^-)$ and

$$(A_0 x, x) = (A_0 x^+, x^+) + (A_0 x^+, x^-) + (A_0 x^-, x^+) + (A_0 x^-, x^-) .$$

Each term on the right in this last expression is non-positive; the first and last because of the dissipative property, and the middle two because $a_{ij} \geqq 0$ for $i \neq j$ so that

$$(A_0 x^+, x^-) = \sum_{\substack{x(i)<0 \\ x(j)>0}} w_i a_{ij}\, x(j)\, x(i) \leqq 0 , \quad (A_0 x^-, x^+) = \sum_{\substack{x(i)>0 \\ x(j)<0}} w_i a_{ij}\, x(j)\, x(i) \leqq 0 .$$

Therefore we can assert

$$(3.10) \qquad \langle x, x\rangle \geqq \langle x^+, x^+\rangle .$$

Suppose next that $x \in \mathfrak{D}(A')$. Then there exists a sequence $\{x_n\} \subset \mathfrak{D}_0$ which converges to x in the $\langle . \rangle$ norm. By (3.10) the sequence $\{x_n^+\}$ will be bounded in the $\langle . \rangle$ norm. Hence there is a subsequence, which we renumber as $\{x_n^+\}$, converging weakly in both the $\langle . \rangle$ and the $(.)$ metrics. It is clear that $\{x_n^+\}$ converges to x^+ in the $(.)$ metric since this was true of the original sequence. Moreover since

$$\langle y, x_n^+\rangle = -(A_0 y, x_n^+) + (y, x_n^+) \to \langle y, x^+\rangle , \quad y \in \mathfrak{D}_0 ,$$

and since $\mathfrak{D}_0$ is dense in $\mathfrak{D}_1$, we see that $\{x_n^+\}$ converges weakly to x^+ in the $\langle . \rangle$ metric. Further

$$\langle x_n, x_m^+\rangle - \langle x, x^+\rangle = \langle x_n - x, x_m^+\rangle + \langle x, x_m^+ - x^+\rangle ;$$

the first term on the right converges to 0 uniformly in m and the second term converges to 0 uniformly in n. Hence the double limit exists and in particular $\lim_{n,m} (A_0 x_n, x_m^+)$ exists. Now

$$(A' x, x^+) = \lim_m (A' x, x_m^+) = \lim_m (x, A_0 x_m^+)$$
$$= \lim_m \lim_n (x_n, A_0 x_m^+) = \lim_n (A_0 x_n, x_n^+) \leqq 0 .$$

It follows that A' is dispersive.

Once we know that A' is dispersive as well as dissipative and self-adjoint, theorem 2.1 implies that A' generates a semi-group of positive contraction operators. According to theorem 3.2

$$(3.11) \qquad R(\lambda; A') \geqq R(\lambda; A), \quad \lambda > 0,$$

since $A' \supset A_0$. On the other hand, M. KREIN [9] has shown that the Friedrichs' extension is minimal among all self-adjoint extensions of A_0 in the sense that

$$(3.12) \qquad (R(\lambda; A')f, f) \leqq (R(\lambda; A)f, f), \quad \lambda > 0, f \in l_2(w).$$

The relations (3.11) and (3.12) together imply

$$(3.13) \qquad (R(\lambda; A')f, f) = (R(\lambda; A)f, f), \quad f \geqq 0.$$

Replacing f by $f + g$ in (3.13) for $f, g \geqq 0$ and using the symmetry of the resolvent operators, we see that

$$(R(\lambda; A')f, g) = (R(\lambda; A)f, g) \quad \text{and from this we infer that}$$

$$R(\lambda; A')f = R(\lambda; A)f \quad \text{first for all } f \geqq 0 \text{ and then for all } f \in l_2(w).$$

This establishes the identity of A and A'.

In the theory of Markov processes on L_1-spaces the honest processes play a very important role. It is therefore somewhat surprising to find that there are no non-trivial honest processes in $l_p(w)$, $1 < p < \infty$.

Theorem 3.5. *For $\mathfrak{X} = l_p(w)$, $1 < p < \infty$, the only honest process is $[S(t) \equiv I]$.*

Proof. If $f, g \geqq 0$, then

$$\lim_{\varepsilon \to 0+} \varepsilon^{-1}[\|f + \varepsilon g\|^p - \|f\|^p] = p \sum w_i\, g(i)\, [f(i)]^{p-1},$$

as can be readily verified by using a termwise Taylor series expansion (two terms plus a remainder) of the expression on the left. Suppose that $[S(t)]$ is honest, that is suppose it consists only of positive contraction operators which are isometric on positive vectors. Then for $x_i = \{x_i(j) = \delta_{ij}\}$ and $\varepsilon > 0$, we have

$$\varepsilon^{-1}[\|S(t)(x_i + \varepsilon x_j)\|^p - \|S(t)\, x_i\|^p] = \varepsilon^{-1}[\|x_i + \varepsilon x_j\|^p - \|x_i\|^p],$$

and passing to the limit as $\varepsilon \to 0+$ we obtain

$$(3.14) \qquad \sum w_k\, s_{kj}(t)\, [s_{ki}(t)]^{p-1} = \sum w_k\, \delta_{kj} [\delta_{ki}]^{p-1} = 0$$

for $i \neq j$. Now $S(t) \geqq 0$ implies $s_{ij}(t) \geqq 0$. Further

$$s_{ii}(t + \tau) = \sum_k s_{ik}(t)\, s_{ki}(\tau) \geqq s_{ii}(t)\, s_{ii}(\tau),$$

and since $s_{ii}(t) \to 1$ as $t \to 0$, we may conclude that $s_{ii}(t) > 0$ for all $t \geqq 0$. Thus (3.14) implies $s_{ij}(t) = 0$ for all $i \neq j$. Finally since $\|S(t)\, x_i\| = \|x_i\|$ we conclude that $s_{ii}(t) \equiv 1$; in other words $S(t) = I$ for all $t \geqq 0$.

4. On the extension of dissipative matrix operators. The problem of extending a dissipative minimal matrix operator A_0 to a dissipative generator A (of a semi-group of contraction operators) so that A is at the same time a restriction of the corresponding maximal matrix operator A_1, is not in general solvable. However, by utilizing the previous dispersive theory we obtain a complete solution in $l_1(w)$ spaces and a partial solution in the case of some other discrete Banach spaces.

In the present section we deal with Banach spaces of the type $\mathfrak{Y} = \mathfrak{X} \times \mathfrak{X}$, where $\mathfrak{X}$ is a discrete Banach lattice satisfying the conditions (3.1). Thus a generic element of $\mathfrak{Y}$ is of the form $\{x_1, x_2\}$ with $x_1, x_2 \in \mathfrak{X}$ and for real a, b we have

$$(a + ib)\{x_1, x_2\} = \{ax_1 - bx_2, bx_1 + ax_2\}.$$

We employ the notation $|\{x_1, x_2\}|$ for the variation of $\{x_1, x_2\} \in \mathfrak{Y}$ where

$$|\{x_1, x_2\}|(i) \equiv [|x_1(i)|^2 + |x_2(i)|^2]^{\frac{1}{2}}. \tag{4.1}$$

From the fact that $\mathfrak{D}_0$ is dense in $\mathfrak{X}$, it is easily verified that $|\{x_1, x_2\}| \in \mathfrak{X}$. Finally we assume that

$$\|y\| = \||y|\| \tag{4.2}$$

as given in $\mathfrak{X}$. It is clear that the familiar complex $l_p(w)$ spaces are of this type.

The notion of majorizing as defined in Definition 1.5 plays the central role in this section. Not all dissipative operators are majorizable. For instance, for $\mathfrak{Y} = l_2$ (complex) of dimension 2 and

$$A_0 = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix},$$

it is easy to see that $(A_0 y, y) \leqq 0$ for all y. According to the second remark following lemma 3.1, in order that a majorizing operator M_0 be dispersive, it suffices that it satisfy conditions (i) and (ii) of Definition 1.5 and be dissipative. However, in the case of A_0 this requires that

$$\frac{1}{4} \geqq m_{11} m_{22} \geqq \left(\frac{m_{12} + m_{21}}{2}\right)^2 \geqq 1,$$

which is impossible. Never-the-less for $l_1(w)$ we have

Lemma 4.1. *For $\mathfrak{Y} = l_1(w)$ a minimal matrix operator A_0 is dissipative if and only if*

$$w_i \operatorname{re}[a_{ii}] + \sum_{j \neq i} w_j |a_{ji}| \leqq 0, \quad i \in \mathfrak{J}. \tag{4.3}$$

Such an operator is always majorizable by $M_0 = (m_{ij})$ where $m_{ii} = \operatorname{re}[a_{ii}]$ and $m_{ij} = |a_{ij}|$ for $i \neq j$.

Proof. For $y \in \mathfrak{D}_0$, the s. i. p. is defined as

$$[z, y] = \|y\| \sum_{y(i) \neq 0} w_i\, z(i)\, \overline{y(i)}/|y(i)|.$$

In particular, for a finite subset π of $\mathfrak{J}$ and for fixed $i \in \pi$, if we set $y(i) = 1$, $y(j) = \varepsilon(\operatorname{sgn} a_{ji})$ for $j \in \pi$, $j \neq i$, and $y(j) = 0$ otherwise, then

$$\operatorname{re}[A_0 y, y] = \|y\| \, [w_i \operatorname{re}[a_{ii}] + \sum_{\substack{k \neq i \\ k \in \pi}} w_k |a_{ki}| + O(\varepsilon)] \leqq 0.$$

Since this holds for all ε and π we see that (4.3) holds. Conversely if (4.3) holds and $y \in \mathfrak{D}_0$ with carrier π, then we have

$$\operatorname{re}[A_0 y, y] = \|y\| \operatorname{re}[\sum_{i \in \pi} w_i \overline{y(i)} |y(i)|^{-1} \sum_{j \in \pi} a_{ij} y(j)] \leqq$$
$$\leqq \|y\| [\sum_{i \in \pi} \{w_i \operatorname{re}[a_{ii}] + \sum_{\substack{k \neq i \\ k \in \pi}} w_k |a_{ki}|\} |y(i)|] \leqq 0 .$$

Setting $m_{ii} = \operatorname{re}[a_{ii}]$, $m_{ij} = |a_{ij}|$ for $i \neq j$, it is clear from the first remark following lemma 3.1 that M_0 is dispersive and hence that it majorizes A_0.

The principal result of the present section is

Theorem 4.1. *Let A_0 be a dissipative minimal matrix operator which is majorizable. Then there exists a dissipative generator A such that $A_0 \subset A \subset A_1$, where A_1 is the corresponding maximal matrix operator.*

Proof. Let $M_0 = (m_{ij})$ be a majorizing minimal matrix operator for A_0. Following the approach employed in the proof of theorem 3.1, we define the operators N and P_π on the discrete Banach lattice $\mathfrak{X}$ and B and C_π on $\mathfrak{Y} = \mathfrak{X} \times \mathfrak{X}$ (π being a finite subset of $\mathfrak{I}$) as follows:

$$(4.4)\qquad \begin{aligned}
&(Nx)(i) = m_{ii} x(i), \quad \mathfrak{D}(N) = [x; \{m_{ii} x(i)\} \in \mathfrak{X}] ;\\
&(P_\pi x)(i) \begin{cases} = \sum_{\substack{j \neq i \\ j \in \pi}} m_{ij} x(j), & i \in \pi, \\ = 0, & i \notin \pi, \ \mathfrak{D}(P_\pi) = \mathfrak{X} ; \end{cases}\\
&(By)(i) = a_{ii} y(i), \quad \mathfrak{D}(B) = [y; \{a_{ii} y(i)\} \in \mathfrak{Y}] ;\\
&(C_\pi y)(i) \begin{cases} = \sum_{\substack{j \neq i \\ j \in \pi}} a_{ij} y(j), & i \in \pi, \\ = 0, & i \notin \pi, \ \mathfrak{D}(C_\pi) = \mathfrak{Y} . \end{cases}
\end{aligned}$$

Setting $A_\pi = B + C_\pi$, $M_\pi = N + P_\pi$, where $\mathfrak{D}(A_\pi) = \mathfrak{D}(B)$ and $\mathfrak{D}(M_\pi) = \mathfrak{D}(N)$, and defining $\mathfrak{Y}_\pi$ and $\mathfrak{Y}'_\pi$ as in (3.4), it is readily verified that $A_\pi/\mathfrak{Y}_\pi$ and $A_\pi/\mathfrak{Y}'_\pi$ are dissipative and that the equations

$$(\lambda I - A_\pi) y_\pi = f, \quad (\lambda I - M_\pi) x_\pi = |f|, \quad f \in \mathfrak{Y},$$

have unique solutions for $\lambda > 0$. Since M_0 is dispersive, the results established for A_0 in the proof of theorem 3.1 apply. In particular the relation (3.5) holds and we have

$$(4.5)\qquad x_\pi = R(\lambda; M_\pi)|f| = \sum_{k=0}^{\infty} R(\lambda; N) [P_\pi R(\lambda; N)]^k |f|$$

and $\lim_n [R(\lambda; N) P_\pi]^n z = 0$ for all $z \in \mathfrak{D}(N)$. On the other hand, $(\lambda I - B) y_\pi = f + C_\pi y_\pi$ so that $y_\pi = R(\lambda; B) f + R(\lambda; B) C_\pi y_\pi$. Iterating this relation gives

$$y_\pi = \sum_{k=0}^{n-1} R(\lambda; B) [C_\pi R(\lambda; B)]^k f + [R(\lambda; B) C_\pi]^n y_\pi .$$

Now the elements of C_π are dominated in absolute value by those of P_π and the elements of $R(\lambda; B)$ are dominated in absolute value by those of $R(\lambda; N)$.

It follows that

$$|[R(\lambda; B)\, C_\pi]^n\, y_\pi| \leqq [R(\lambda; N)\, P_\pi]^n\, |y_\pi| \,.$$

Since $y_\pi \in \mathfrak{D}(B)$ implies $|y|_\pi \in \mathfrak{D}(N)$, we can assert that

$$\|[R(\lambda; B)\, C_\pi]^n\, y_\pi\| \leqq \|[R(\lambda; N)\, P_\pi]^n\, |y_\pi|\| \to 0$$

as $n \to \infty$. As a consequence

$$y_\pi = R(\lambda; A_\pi) f = \sum_{k=0}^{\infty} R(\lambda; B)\, [C_\pi R(\lambda; B)]^k f \,. \tag{4.6}$$

We now wish to show that $\{y_\pi\}$ defines a convergent system. To this end we note that for $\pi_1 \leqq \pi_2$ we have

$$\begin{aligned} R(\lambda; B)\, [C_{\pi_2} R(\lambda; B)]^k f - R(\lambda; B)\, [C_{\pi_1} R(\lambda; B)]^k f = \\ = \sum_{i=1}^{k} \{R(\lambda; B)\, [C_{\pi_2} R(\lambda; B)]^i\, [C_{\pi_1} R(\lambda; B)]^{k-i} f - \\ - R(\lambda; B)\, [C_{\pi_2} R(\lambda; B)]^{i-1}\, [C_{\pi_1} R(\lambda; B)]^{k-i+1} f\} = \\ = \sum_{i=1}^{k} R(\lambda; B)\, [C_{\pi_2} R(\lambda; B)]^{i-1}\, (C_{\pi_2} - C_{\pi_1})\, R(\lambda; B)\, [C_{\pi_1} R(\lambda; B)]^{k-i} f \,. \end{aligned}$$

It is readily verified that the i-th term of the left member is majorized componentwise by replacing all matrix elements by their absolute value majorants and by replacing f by $|f|$. Since $P_{\pi_1} \leqq P_{\pi_2}$, we find that

$$\begin{aligned} |y_{\pi_2} - y_{\pi_1}| \leqq \\ \leqq \sum_{k=0}^{\infty} \sum_{i=1}^{k} |R(\lambda; B)\, [C_{\pi_2} R(\lambda; B)]^{i-1}\, (C_{\pi_2} - C_{\pi_1})\, R(\lambda; B)\, [C_{\pi_1} R(\lambda; B)]^{k-i} f| \leqq \\ \leqq \sum_{k=0}^{\infty} \sum_{i=1}^{k} \{R(\lambda; N)\, [P_{\pi_2} R(\lambda; N)]^{i-1}\, (P_{\pi_2} - P_{\pi_1})\, R(\lambda; N)\, [P_{\pi_1} R(\lambda; N)]^{k-i}\, |f|\} = \\ = \sum_{k=0}^{\infty} \{R(\lambda; N)\, [P_{\pi_2} R(\lambda; N)]^k\, |f| - R(\lambda; N)\, [P_{\pi_1} R(\lambda; N)]^k\, |f|\} = x_{\pi_2} - x_{\pi_1} \,. \end{aligned}$$

Consequently $\|y_{\pi_2} - y_{\pi_1}\| \leqq \|x_{\pi_2} - x_{\pi_1}\|$. It was shown in the proof of theorem 3.1 that $\{x_\pi\}$ forms a Cauchy system and therefore the same is true of $\{y_\pi\}$. Thus $R_\lambda f \equiv \lim_\pi R(\lambda; A_\pi) f$ exists for all $f \in \mathfrak{Y}$. Moreover comparing (4.5) and (4.6) we see that

$$\lambda \|R_\lambda f\| \leqq \lambda \|R(\lambda; M)\, |f|\| \leqq \|f\| \,,$$

where M is the dispersive generator of the $[F(t)]$ process corresponding to M_0. It is further clear that R_λ satisfies the first resolvent equation for $\lambda > 0$ along with the approximating resolvent operators $R(\lambda; A_\pi)$. Finally for $y \in \mathfrak{D}_0$ we have $\lim_\pi (\lambda I - A_\pi)\, y = (\lambda I - A_0)\, y$ and hence

$$R_\lambda(\lambda I - A_0)\, y = \lim_\pi R(\lambda; A_\pi)\, (\lambda I - A_\pi)\, y = y \,.$$

By lemma 3.2 we conclude that R_λ is the resolvent of an operator A which is the dissipative generator of a semi-group of contraction operators and that $A \supset A_0$.

It remains to show that $A \subset A_1$. Again comparing (4.5) and (4.6), we see that $|y_\pi| \leqq x_\pi \leqq x = R(\lambda; M)|f|$. Consequently $|y| \leqq x$ and since $\sum m_{ij}\, x(j)$ converges (i. e., $M \subset M_1$), it follows that $\sum a_{ij}\, y(j)$ converges absolutely for each $i \in \mathfrak{J}$. Finally $(\lambda I - A_\pi)\, y_\pi = f$ implies that

$$\lambda\, y_\pi(i) - \sum_{j\in\pi} a_{ij}\, y_\pi(j) = f(i)\,, \quad i \in \pi\,,$$

and the dominated convergence theorem can be used to show that

$$\lambda\, y(i) - \sum_j a_{ij} y(j) = f(i)$$

for all $i \in \mathfrak{J}$. Since $(\lambda I - A)\, y = f$, this proves that

$$(Ay)(i) = \sum_j a_{ij}\, y(j) = (A_1 y)(i)\,.$$

Without the assumption that A_0 is majorizable, theorem 4.1 is no longer valid as the following example shows. Let $\mathfrak{Y} = l_2$ and consider the triangular matrix (a_{ij}): $a_{ij} = 0$ for $i > j$, $a_{ii} = -1$, and $a_{ij} = -2$ for $j > i$. It is readily verified that A_0 is dissipative; we need only note that for $y \in \mathfrak{D}_0$ we have

$$\operatorname{re}(A_0 y, y) = \operatorname{re}\left[\sum_i \{-y(i) - 2\sum_{j>i} y(j)\}\, \overline{y(i)}\right] = -\left|\sum y(i)\right|^2 \leqq 0\,.$$

Now the smallest closed extension of A_0, namely $\bar{A}_0$, exists (by [12; lemma 1.3.1]) and is actually maximal dissipative so that $\bar{A}_0$ generates a semi-group of contraction operators. In fact, because of the triangular property of (a_{ij}) the equation $(I - A_0)\, y = f$ has a solution $y \in \mathfrak{D}_0$ for each $f \in \mathfrak{D}_0$ given by $y(i) = \frac{1}{2}[f(i) - - f(i+1)]$, $i \in \mathfrak{J}$. Thus $\mathfrak{R}(I - A_0)$ is dense in $\mathfrak{Y}$ and since $\|(I - A_0)^{-1}\| \leqq 1$, it follows that $\bar{A}_0$ is a maximal dissipative generator. On the other hand for $f(j) = (-1)^j j^{-1}$, the equation $(I - \bar{A}_0)\, y = f$ has the solution $y(j) = (-1)^j (2j + + 1)[2j(j+1)]^{-1}$. Consequently $\sum_j a_{ij}\, y(j)$ is convergent but not absolutely convergent. Further all of the above properties except the convergence of $\sum_j a_{ij}\, y(j)$ are independent of the ordering of the integers $\mathfrak{J}$. Thus by a suitable reordering of $\mathfrak{J}$ we see that there exist y in $\mathfrak{D}(\bar{A}_0)$ such that $\sum a_{ij}\, y(j)$ is not even convergent. In this example there is only one dissipative generator A extending A_0, namely $\bar{A}_0$, and $\bar{A}_0$ is not a restriction of A_1, even if we modify Definition 1.4 so as to allow merely the convergence of $\sum_j a_{ij}\, y(j)$ (rather than its absolute convergence) to qualify y to be in $\mathfrak{D}(A_1)$.

In the case $\mathfrak{Y} = l_2$ it is known that any dissipative operator with dense domain has a maximal dissipative extension which generates a semi-group of contraction operators (see [12, theorem 1.1.1]). It is also known (see [13]) that if both the rows and columns of (a_{ij}) lie in l_2, then there exists a dissipative generator A such that $A_0 \subset A \subset A_1$. It is not known whether either of these results hold in the other l_p spaces $1 < p < \infty$.

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Резюме

ПОЛУГРУППЫ СЖИМАЮЩИХ ПОЛОЖИТЕЛЬНЫХ ОПЕРАТОРОВ

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В работе исследуются полугруппы сжимающих положительных операторов в структуре Банаха $\mathfrak{X}$ общего типа. В такой структуре всегда можно ввести полу-скалярное произведение $[x, y]$, обладающее свойствами (1.1) и (1.5).

Определение 1.3. Оператор A называется *дисперсионным*, если

$$[Ax, x^+] \leqq 0\,, \quad x \in \mathfrak{D}(A)\,.$$

Теорема 2.1. *Для того, чтобы линейный оператор со всюду плотной областью определения был производящим оператором сильно непрерывной полугруппы сжимающих положительных операторов, необходимо и достаточно, чтобы оператор A дыл дисперсионным и чтобы имело место равенство $\mathfrak{R}(I - A) = \mathfrak{X}$ ($\mathfrak{R}$ — область изменения).*

Пусть $\mathfrak{X}$ — банахова структура вещественных функций $[f(i); i \in \mathfrak{I}]$ на абстрактном множестве $\mathfrak{I}$ с обычными алгебраическими операциями, которая удовлетворяет соотношениям:

(i) Множество $\mathfrak{D}_0$ всех функций, имеющих лишь конечное число ненулевых составляющих, входит в $\mathfrak{X}$.

(ii) $f \leqq g$ означает $f(i) \leqq g(i)$ для всех $i \in \mathfrak{I}$.

(iii) Каждое монотонное направленное множество неотрицательных элементов $[f_\pi]$, являющееся ограниченным по норме, сходится к $\bigvee f_\pi$.

Каждой матрице (a_{ij}), столбцевые векторы которой входят в $\mathfrak{X}$, можно поставить в соответствие минимальный оператор A_0 с областью определения $\mathfrak{D}_0$, определенный при помощи соотношения

$$(A_0 f)(i) = \sum_j a_{ij} f(j), \quad f \in \mathfrak{D}_0 ,$$

а также максимальный оператор A_1 с областью определения
$\mathfrak{D}_1 = [f; f \in \mathfrak{X}, \quad g(i) = \sum_j a_{ij} f(j)$ сходится абсолютно для всякого i и $g \in \mathfrak{X}]$, определенный при помощи соотношения

$$(A_1 f)(i) = \sum_j a_{ij} f(j), \quad f \in \mathfrak{D}_1 .$$

Теорема 3.1. *Пусть A_0 — дисперсионный минимальный матричный оператор. Тогда существует сильно непрерывная полугруппа сжимающих положительных операторов $[F(t)]$ с производящим оператором A таким, что $A_0 \subset A \subset A_1$.*

В разделе 4 приводится аналогичная теорема о расширении диссипационного оператора A_0 при условии, что он надлежащим образом мажорируется дисперсионным оператором.