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ON r-DIMENSIONAL INTEGRALS IN $(r + 1)$-SPACE

(Preliminary Communication)

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Some theorems are announced concerning the relations between the integrals over a closed oriented parametric r-surface S in $(r + 1)$-space and between "non-parametric" integrals over the boundaries of certain sets determined by means of the order of a point with respect to S.

We shall assume throughout that M^r is a fixed compact oriented r-manifold and f is a continuous mapping of M^r into E_{r+1}, the Euclidean $(r + 1)$-space (so that the pair (f, M^r) represents a closed parametric r-surface). We shall suppose that the $(r + 1)$-dimensional Lebesgue measure of $f(M^r)$ is equal to zero. Given $z = [z_1, \ldots, z_{r+1}] \in E_{r+1}$ and $1 \leqq i \leqq r + 1$ we put $P_i(z) = [z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{r+1}]$. Further we denote by $f_i = P_i(f)$ the superposition of f and P_i. Following J. H. MICHAEL (Proc. London Math. Soc. 1957) we write $u(z)$ for the order of $z \in E_{r+1} - f(M^r)$ with respect to (f, M^r), $\mathscr{R}$ for the collection of all those subsets A of M^r for which $f_i(\text{fr } A)$, the image under f_i of the boundary (with respect to M^r) of A, is contained in a finite number of $(r - 1)$-dimensional hyperplanes of the form $\{x = [x_1, \ldots, x_r],\ x_j = \text{const}\}$; finally, we define on the ring $\mathscr{R}$ the additive set function $\mu_{f_i} = \mu^i$ by

$$\mu^i(A) = \int_{E_r - f_i(\text{fr } A)} \mathrm{d}(f_i, A, x)\, \mathrm{d}x\,,$$

$\mathrm{d}(f_i, A, x)$ denoting the degree of f_i on A at x. In the case that the variation

$$\|f_i\| = \sup_{A \in \mathscr{R}} |\mu^i(A)|$$

is finite, μ^i can be extended to a finite signed measure defined on a σ-algebra of subsets of M^r containing the inverses under f of all Borel subsets of $f(M^r)$ and one can introduce a signed Borel measure λ^i over $f(M^r)$ by $\lambda^i(B) = \mu^i(f^{-1}(B))$.

Let $\mathscr{D}$ be the system of all infinitely differentiable functions with compact support on E_{r+1}. Given a Lebesgue measurable set $U \subset E_{r+1}$ we can investigate the distribution

$$T_i = -\frac{\partial}{\partial z_i}\, \chi_U(z)$$

defined by

$$T_i(\varphi) = \int_U \frac{\partial \varphi(z)}{\partial z_i} \mathrm{d}z, \ \varphi \in \mathscr{D} .$$

If the number

$$\|U\|_i = \sup_{\varphi} T_i(\varphi) , \quad \varphi \in \mathscr{D} , \quad \max_z |\varphi(z)| \leqq 1$$

is finite, then there exists a uniquely determined finite signed Borel measure P_i^U over the boundary $\tilde{U}$ of U such that

$$T_i(\varphi) = \int_{\tilde{U}} \varphi \, \mathrm{d}\mathsf{P}_i^U, \ \varphi \in \mathscr{D} .$$

Put

$$G(n) = \{z; z \in E_{r+1} - f(M^r), \ u(z) = n\} , \quad U(n) = \bigcup_{k=n}^{\infty} G(k) .$$

Theorem. $\sum_{n=-\infty}^{\infty} \|U(n)\|_i \leqq \|f_i\|, \ \sum_{n=-\infty}^{\infty} \|G(n)\|_i \leqq 2\|f_i\|.$

Thus $\|f_i\| < \infty$ implies that $\|U(n)\|_i + \|G(n)\|_i < \infty$ for every n and one can discuss the relations between the integrals with respect to λ^i and those with respect to $\mathsf{P}_i^{U(n)}$, $\mathsf{P}_i^{G(n)}$.

Theorem. *Let $\|f_i\| < \infty$ and let ψ be a λ^i-integrable function on $f(M^r)$. Then, for every n, ψ is integrable with respect to $\mathsf{P}_i^{U(n)}$ and $\mathsf{P}_i^{G(n)}$ over the boundary of $U(n)$ and $G(n)$ respectively and one has the relation*

$$(-1)^{i-1} \int_{f(M^r)} \psi \, \mathrm{d}\lambda^i = \sum_{n=-\infty}^{\infty} \int_{\widetilde{U(n)}} \psi \, \mathrm{d}\mathsf{P}_i^{U(n)} .$$

The series

$$\sum_{n=1}^{\infty} n \left(\int_{\widetilde{G(n)}} \psi \, \mathrm{d}\mathsf{P}_i^{G(n)} - \int_{\widetilde{G(-n)}} \psi \, \mathrm{d}\mathsf{P}_i^{G(-n)} \right)$$

(which need not be convergent) is Cesàro summable to

$$(-1)^{i-1} \int_{f(M^r)} \psi \, \mathrm{d}\lambda^i .$$

Hence a general theorem of the Gauss-Green type can be deduced. (For $r = 1$ related results were obtained by the author in Чех. мат. жур. 1957.) At this juncture, mention should be made of the work of J. Cecconi, L. Cesari, E. De Giorgi, H. Federer, W. H. Fleming, K. Krickeberg, J. Mařík, J. H. Michael, Chr. Y.

PAUC, L. H. TURNER, H. WHITNEY, L. C. YOUNG and others. The space of this note being limited we cannot give here the corresponding bibliographical comments; these, together with the proofs of the above theorems, will be given in a forthcoming paper of the author.

Резюме

ОБ r-МЕРНЫХ ИНТЕГРАЛАХ В $(r + 1)$-МЕРНОМ ПРОСТРАНСТВЕ

(*Предварительное сообщение*)

ИОСЕФ КРАЛ (Josef Král), Прага

Исследуются связи между r-мерными интегралами по параметрически представленном замкнутом r-мерном ориентированном многообразии и между „непараметрическими" интегралами по границам определенных множеств, которые определяются при помощи порядка точки относительно данного многообразия. Соответствующие определения и теоремы здесь, из-за недостатка места, формулировать невозможно.