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LOCALLY CONNECTED TOPOLOGIES

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This paper deals with some locally connected topologies associated with a given topology.

The term "topology for a set P" will be used in the sense of Bourbaki [1]. Let τ be a topology for a set P. The symbols $\tau[M]$, $\mathfrak{O}(\tau)$ and $\mathfrak{F}(\tau)$ denote the τ-closure of a set $M \subset P$, the family of all τ-open sets and the family of all τ-closed sets, respectively. The pair (P, τ) is a topological space. Let τ_1 and τ_2 be topologies for a set P. The topology τ_2 is coarcer (or smaller) than τ_1, in symbols $\tau_2 \geqq \tau_1$, if and only if $\mathfrak{O}(\tau_2) \subset \subset \mathfrak{O}(\tau_1)$. If $\tau_2 \geqq \tau_1$, than τ_1 is said to be finer (or larger) than τ_2. Let T be a family of topologies for a set P. There exist topologies τ_1 and τ_2 such that

(a) if $\tau \in T$, then $\tau_1 \leqq \tau \leqq \tau_2$,

(b) if τ_1' and τ_2' are topologies satisfying the condition (a), then $\tau_1' \leqq \tau_1$ and $\tau_2' \geqq \tau_2$. The topology τ_2 (τ_1) is said to be the supremum or the least upper bound (the infimum or greatest lower bound) of the family T and it is denoted by sup T (inf T, respectively).

It is easy to show that

$$\mathfrak{O}(\sup T) = \bigcap\{\mathfrak{O}(\tau); \tau \in T\}$$

and that

$$\bigcup\{\mathfrak{O}(\tau); \tau \in T\}$$

is an open sub-base for $\mathfrak{O}(\inf T)$.

Let τ be a topology for a set P and let M be a subset of P. The relativization of τ to M is denoted by τ/M and defined by

$$\mathfrak{O}(\tau/M) = \mathfrak{O}(\tau) \cap M\,.$$

Of course, if $\mathfrak{A}$ is a family of sets and if M is a set, then $\mathfrak{A} \cap M$ denotes the family of all sets of the form $A \cap M$ where $A \in \mathfrak{A}$. The topology τ/M will be sometimes called the relative topology. We shall say that the topologies τ_1 and τ_2 agree on a set M if $\tau_1/M = \tau_2/M$.

Let T be the set of all topologies τ' for the set P satisfying the condition

$$\tau'/M = \tau/M .$$

The topology inf T will be denoted by τ_M. It is easy to show that a $U \subset P$ is τ_M-open if and only if $U \cap M$ is τ/M-open.

Now we shall recall the definition and some properties of locally connected topologies.

Let τ be a topology for a set P. The topology τ is said to be locally connected at a point $x \in P$ if for every neighborhood U of x the component of x in U is a neighborhood of x. The topology τ is locally connected if it is locally connected at every $x \in P$. τ is locally connected if and only if the components of open sets are open. Every open subspace of a locally connected space is locally connected. If $M_1, M_2 \subset P$ and the relative topologies τ/M_1 and τ/M_2 are locally connected, then the topology $\tau/M_1 \cup M_2$ is locally connected at every point of $M_1 \cap M_2$. Moreover if M_1 and M_2 are closed, then $\tau/M_1 \cup M_2$ is locally connected. If τ is connected, locally connected and complete metrizable (that is, if there exists a metric ϱ for (P, τ) such that (P, ϱ) is a complete metric space), then every two points of P may be joined by a connected locally connected compact subspace (that is, if $x, y \in P$ then there exists a locally connected continuum K with $x \in K$, $y \in K$); moreover, every two points may by joined by an arc.

In the present paper we shall investigate locally connected topologies associated with a given topology. Let τ be a topology for a set P. There exists a coarcest locally connected topology $s(\tau) \leqq \tau$ (section 1). This topology agrees with τ on locally connected subspaces of (P, τ). The finest topology possessing the preceding property is denoted by $m(\tau)$ and it is studied in the section 3. In the section 2 there is introduced a locally connected topology $c(\tau)$ satisfying $m(\tau) \leqq c(\tau) \leqq s(\tau)$.

1. THE COARCEST LOCALLY CONNECTED TOPOLOGY

Let τ be a topology for a set P. There exists a locally connected topology $s(\tau)$ for the set P such that $s(\tau) \leqq \tau$ and if τ_0 is a locally connected topology with $\tau_0 \leqq \tau$, then $\tau_0 \leqq s(\tau)$. In this section we shall study the topology $s(\tau)$.

The existence of $s(\tau)$ is guaranteed by the following proposition:

1.1. Lemma. *Let $\{\tau\}$ be a family of locally connected topologies for a set P. Then $\tau_0 = \sup \{\tau\}$ is a locally connected topology for a set P.*

Proof. We have to show that τ_0-components of non-void τ_0-open sets are τ_0-open. Let U be a non-void τ_0-open set and let C be a τ_0-component of U. To prove that C is τ_0-open, it is sufficient to show that C is τ-open for every τ in $\{\tau\}$. Selecting a point x in C we shall show that C is a τ-neighborhood of x (τ being an arbitrary topology belonging to the family $\{\tau\}$). The set U being open with respect to the locally connected topology τ, the τ-component C' of the point x in the set U is a τ-open set. But τ_0 is smaller that τ so that C' is contained in C. The proof is complete.

From 1.1 we have at once

1.2. Let τ be a topology for a set P. Denote by T the family of all locally connected topologies τ' for P for which $\tau' \leqq \tau$. Put

1.2.1. $$s(\tau) = \sup T .$$

By 1.1 the topology $s(\tau)$ is locally connected. The topology $s(\tau)$ has the following property:

1.2.2. $s(\tau) \leqq \tau$ and if τ_0 is a locally connected topology and $\tau_0 \leqq \tau$, then $\tau_0 \leqq$ $\leqq s(\tau)$.

The property 1.2.2 defines the topology $s(\tau)$. The topology $s(\tau)$ will be called the coarcest locally connected topology associated with τ.

Clearly, the topology τ is locally connected if and only if $\tau = s(\tau)$.

1.3. Note. The infimum of two locally connected topologies may fail to be a locally connected topology. For example, consider the following subsets of the Euclidean plane:

$$P_1 = \{(x, y);\ x > 0\} ,$$
$$P_2 = \{(x, y);\ x < 0\} ,$$
$$P_3 = \{(0, y);\ y \text{ rational}\} .$$

Denoting by τ the natural topology we have at once that the topologies $\tau_1 = \tau_{P_1 \cup P_3}$ and $\tau_2 = \tau_{P_2 \cup P_3}$ are locally connected. On the other hand we have

$$\inf(\tau_1, \tau_2) = \tau_{P_3} ,$$

and consequently, the infimum of τ_1 and τ_2 is not a locally connected topology.

1.4. Proposition. *Let τ be a topology for a set P and let $R \subset P$. If the relative topology τ/R is locally connected, then*

1.4.1. $$\tau/R = s(\tau)/R = s(\tau/R) .$$

Thus $s(\tau)$ and τ agree on locally connected subspaces of (P, τ).

Proof. The topology τ_R is locally connected provided that τ/R is locally connected. By definition of $s(\tau)$ we have $\tau_R \leqq s(\tau)$. It follows

$$\tau_R/R \leqq s(\tau)/R \leqq \tau/R ,$$

and consequentely, $\tau/R = s(\tau)/R$. The equality $\tau/R = s(\tau/R)$ is obvious.

1.5. Corollary. *In order that a topology τ for a set P be locally connected, it is necessary and sufficient that if $x \in P$ and $M \subset P$, then $x \in \tau[M]$ if and only if there exists a set $R \subset P$ such that the relative topology τ/R is locally connected and*

1.5.1. $$x \in (\tau/R)[M \cap R] .$$

Proof. The necessity is obvious; for the sufficiency we need only show that $\tau \leqq \leqq s(\tau)$. Let us suppose that $x \in P$ belongs to the τ-closure of a subset M of P. There exists a set $R \subset P$ such that τ/R is locally connected and 1.5.1 holds. According to 1.4 we have

$$x \in (s(\tau)/R)[M],$$

and consequently, x belongs to $s(\tau)[M]$.

Now we shall characterize the topology $s(\tau)$ as the infimum of a transfinite sequence of topologies.

1.6. *Let τ be a topology for a set P. The family $\mathfrak{B}$ of all τ-components of all τ-open sets is a base for some topology which will be denoted by τ^*. Of course, $\tau^* \leqq \tau$. In addition, if $\mathfrak{A}$ is an open base for (P, τ) then the family of all τ-components of sets from $\mathfrak{A}$ is an open base for (P, τ^*).*

Proof. We have to prove that for each B_1 and B_2 in $\mathfrak{B}$ and for each x in $B_1 \cap B_2$, there exists a B in $\mathfrak{B}$ such that

$$(*) \qquad x \in B \subset B_1 \cap B_2 .$$

There exist τ-open sets U_1 and U_2 such that B_i $(i = 1, 2)$ is the τ-component of the point x in U_i. Let B be the τ-component of the point x in $U_1 \cap U_2$. Evidently the relations $(*)$ hold. The proof of the remaining assertions may be left to the reader.

From the definition of τ^* we have at once

1.7. A set $U \subset P$ is τ^*-open if and only if the following condition is satisfied:

1.7.1. If $M \subset P$ and if the topology τ/M is locally connected at a point $x \in M \cap U$, then $U \cap M$ is a neighborhood of x in $(M, \tau/M)$.

It is easy to deduce the following

1.8. Proposition. If $\tau_1 \geqq \tau_2$, then $\tau_1^* \geqq \tau_2^*$. A topology τ is locally connected if and only if $\tau = \tau^*$.

1.9. Definition. Let τ be a topology for a set P. Let us defince $\tau^0 = \tau$, $\tau^\alpha = (\tau^{\alpha-1})^*$ if $\alpha \geqq 1$ is an isolated ordinal and

$$\tau^\alpha = \inf \{\tau^\beta; \beta < \alpha\}$$

if α is a limit ordinal. Thus for every $\alpha \geqq 1$ we have

$$\tau^\alpha = \inf \{(\tau^\beta)^*; \beta < \alpha\} .$$

The *order of disconnectedness* of the topology τ is the least ordinal α for which τ^α is a locally connected topology.

Evidently, if α is the order of disconnectedness of a topology τ, then τ^α is finer than $s(\tau)$ since τ^α is locally connected. On the other hand, by 1.8 we have at once that τ^β is smaller than $s(\tau)$ for every ordinal β. Thus we have proved the following

1.10. Proposition. *If α is the order of disconnectedness of a topology τ, then $s(\tau) = \tau^\alpha$.*

1.11. Proposition. *If τ is a metrizable topology for a set P, then the topology τ^* is metrizable.*

Proof. Let us suppose that the topology τ is defined by a metric φ. Without loss of generality we may assume that $\varphi(x, y) \leqq 1$ for each x and y in P. We define a metric φ^* for the set P in the following way:

Let x and y belong to P. Put $\varphi^*(x, y) = 1$ if there exists no τ-connected set M containing both x and y. In the other case, let $\varphi^*(x, y)$ be the greatest lower bound of the set of diameters (in the metric φ) of all τ-connected sets containing both x and y.

It is easy to see that φ^* is a metric and

$$(*) \qquad \varphi^*(x, y) \geqq \varphi(x, y)$$

for each x and y in P. Let us denote by τ' the topology defined by φ^*. We shall prove that $\tau' = \tau^*$.

First we shall show that τ' is smaller than τ^*. Choose a point x in P and let ε be a positive real-number. Put

$$K = \left\{y;\, y \in P, \varphi(x, y) < \frac{\varepsilon}{2}\right\}.$$

Consider the τ-component C of the point x in K. C is τ^*-open and the diameter of C (with respect to the metric φ) is $\leqq \varepsilon$. It follows that $\varphi^*(x, y) \leqq \varepsilon$ for each y in C. Thus φ^* is τ^*-continuous, that is, $\tau' \geqq \tau^*$.

It remains to prove that τ^* is smaller than τ'. If $\mathfrak{B}$ is an open base for the topology τ, then the set $\mathfrak{B}'$ of all τ-components of all belonging to $\mathfrak{B}$ is an open base for the topology τ^*. Hence, to show $\tau^* \geqq \tau'$, it is sufficient to prove that for every x in P and for every positive real number ε there exists a positive real number δ such that the set

$$K_1 = \{y;\, \varphi^*(x, y) < \delta\}$$

is contained in the τ-component C of the point x in

$$K_2 = \{y;\, \varphi(x, y) < \varepsilon\}.$$

The set K_1 is τ-connected as the union of a family of τ-connected sets containg a common point x. Put $\delta = \varepsilon$. By $(*)$ we have $K_1 \subset K_2$. Consequentely, $K_1 \subset \mathsf{C}$, since C is the τ-component of the point x in K_2. The proof is complete.

As an immediate consequence of 1.11 and of the fact that the greatest lower bound of a countable family of metrizable topologies is a metrizable topology we have:

1.12. Proposition. *If the order of disconnectedness of a metrizable topology τ is countable, then $s(\tau)$ is a metrizable topology.*

1.13. Definition. Two topologies τ_1 and τ_2 for a set P are said to be homeomorphic at a point $x \in P$ if for every subset M of P

$$x \in \tau_1[M] \Leftrightarrow x \in \tau_2[M].$$

We have at once from the definition that τ_1 and τ_2 are homeomorphic at the point x if and only if a set U is a τ_1-neighborhood of x if and only if U is a τ_2-neighborhood of x.

1.14. Proposition. *Let τ be a topology for a set P. The topologies τ and τ^* are homeomorphic at a point $x \in P$ if and only if the topology τ is locally connected at the point x.*

Proof. Let us suppose that the topology τ is locally connected at the point x, *i. e.* that the family of all connected neighborhoods of the point x is a local base at x. If U is a τ^*-open set containing the point x, then there exists a τ-open set V such that the τ-component C of the point x in V is contained in U. The topology τ is locally connected at x and hence there exists a τ-connected neighborhood K of the point x such that $K \subset U$. Evidently K is contained in C. We have proved that every τ^*-neighborhood of the point x contains a τ-neighborhood of the point x. Since $\tau \geqq \tau^*$, τ and τ^* are homeomorphic at the point x.

Conversely, if the topology τ is not locally connected at the point x then there exists an open τ-neighborhood U of x such that the τ-component K of the point x in U is not an τ-neighborhood of x. If follows that τ and τ^* are not homeomorphic at the point x.

1.15. Example. For every positive integer k there exists a subspace M_k of the Euclidean plane having the order of disconnectedness k.

Construction. Let us denote by P_n $(n = 1, 2, \ldots)$ the line segment in the Euclidean plane joining the points $\left(\frac{1}{n}, -1\right)$ and $\left(\frac{1}{n}, 1\right)$. Let P_0 be the segment joining the points $(0, -1)$ and $(0, 1)$. Next let us denote by Q_n $(Q_n^-$, resp$)$ $n = 1, 2, \ldots$, the segment joining the points $\left(\frac{1}{n}, 1\right)$ and $\left(\frac{1}{n+1}, 1\right)\left(\left(\frac{1}{n}, -1\right) \text{and} \left(\frac{1}{n+1}, -1\right), \text{respectively}\right)$. Finally put $p = (0, 0)$, $q = (1, 0)$. Let us consider the set

$$M = \bigcup_{n=0}^{\infty} P_n \cap \bigcup_{n=1}^{\infty} Q_{2n} \cap \bigcup_{n=1}^{\infty} Q_{2n-1}^- .$$

If p' and q' are two distinct points of the plane we can choose a set $M(p', q')$ geometrically similar to M and such that p' corresponds to p and q' to q.

Now we are prepared to construct the spaces M_k. Put $M_1 = M$. It is easy to see that the order of disconnectedness of the topology τ/M is 1. Indeed, M is not locally connected at the points of the segment P_0. On the other hand the set P_0 is open with respect to the topology $(\tau/M)^*$. Having constructed, the set M_k $(k \geqq 1)$ let us denote by M_{k+1} the union of M_k and all sets $M(p', q')$ such that the segment $p'q'$ joining p' and q' si contained in the first axis and the set $M_k \cap p'q'$ contains only the points p' and q'. Using the fact that P_0 is $(\tau/M)^*$-open, we can prove by induction that the order of disconnectedness of the topology τ/M_k is k.

1.16. Example. There exists a subset M of the Euclidean plane such that the order of disconnectedness of the topology τ/M is ω_0.

Construction. Let $\{U_k\}$ be a disjoint locally finite sequence of non-void open sets in the Euclidean plane. According to the example 1.15 we can choose subsets $M_k \subset U_k$, $k = 1, 2, \ldots$, such that the order of disconnectedness of the topology τ/M_k is k. Put $M = \bigcup_{k=1}^{\infty} M_k$. Evidently the order of disconnectedness of the topology τ/M is ω_0.

Note. Using transfinite planes instead of the Euclidean plane, topologies of an arbitrary order of disconnectedness can be constructed. I have not been able to construct metrizable topologies with order of disconnectedness greater than $\omega_0 + 1$.

2. THE TOPOLOGY $c(\tau)$

In the whole section we assume that τ is a topology for a set P. The family of all connected subspaces of (P, τ) will be denoted by $\mathsf{C}(\tau)$. The symbol $\mathsf{F}\ \mathsf{C}(\tau)$ will be used to denote the family of all closed subspaces of (P, τ) belonging to $\mathsf{C}(\tau)$. Finally, the family of all locally connected sets from $\mathsf{C}(\tau)$ and $\mathsf{F}\ \mathsf{C}(\tau)$ will be denoted by $\mathsf{L}\ \mathsf{C}(\tau)$ and $\mathsf{LF}\ \mathsf{C}(\tau)$, respectively.

For convenience we shall introduce the concept of a feebly additive family.

2.1. Definition. A family $\mathfrak{M}$ of sets will be called feebly additive if

$$M_1 \in \mathfrak{M}\,, \quad M_2 \in \mathfrak{M}\,, \quad M_1 \cap M_2 \neq \emptyset \Rightarrow M_1 \cup M_2 \in \mathfrak{M}\,.$$

Let us recall that if N is a set and if $\mathfrak{M}$ is a family of sets, then the star of N in $\mathfrak{M}$, in symbols $S(N, \mathfrak{M})$, is the union of all $M \in \mathfrak{M}$ meeting N. The star of a point x is defined as the star of the one-point set (x). If N is a set and if $\mathfrak{M}$ is a family of sets, then the family of all $M \in \mathfrak{M}$, $M \subset N$ will be denoted by $\mathfrak{M}/N$. If is easy to see

2.1.1. If $x \in K \subset L$, then $S(x, \mathfrak{M}/K) \subset S(x, \mathfrak{M}/L)$.

2.1.2. If $x \in K \cap L$ then

$$S(x, \mathfrak{M}/K \cap L) \subset S(x, \mathfrak{M}/K) \cap S(x, \mathfrak{M}/L)\,.$$

2.1.3. If the family $\mathfrak{M}$ is feebly additive and both x and y belong to K, then either

$$S(x, \mathfrak{M}/K) = S(y, \mathfrak{M}/K)$$

or

$$S(x, \mathfrak{M}/K) \cap S(y, \mathfrak{M}/K) = \emptyset\,.$$

A family $\mathfrak{M}$ of subsets of a space (P, τ) contains arbitrarily small sets if for every x in P and every neighborhood U of x there exists a M in $\mathfrak{M}$ with $x \in M \subset U$. From the assertions 2.1.1, 2.1.2 and 2.1.3 we obtain at once the following

2.2. Theorem. *Let $\mathfrak{M}$ be a feebly additive family of subsets of a space (P, τ) and let $\mathfrak{M}$ contain arbitrarily small sets. The family*

2.2.1. $$\{S(x, \mathfrak{M}/U)\,;\quad x \in U \in \mathfrak{O}(\tau)\}$$

is an open sub-base for some topology for the set P which will be denoted by $\tau(\mathfrak{M})$. The topology $\tau(\mathfrak{M})$ is finer than τ.

2.3. Theorem. *Let $\mathfrak{M}$ be a feebly additive family of subsets of a space P and let $\mathfrak{M}$ contains arbitrarily small sets. If τ is metrizable, then $\tau(\mathfrak{M})$ is metrizable.*

Proof. Let φ be a metric for the space (P, τ). Without loss of generality we may assume $\varphi(x, y) \leqq 1$ for every x and y in P. Let us define a metric ϱ for the set P as follows:

2.3.1. Let $x, y \in P$. If there exists no $M \in \mathfrak{M}$ containing both x and y, then $\varrho(x, y) = 1$. In the opposite case let $\varrho(x, y)$ be the greatest lower bound of the set of diameters (with respect to φ) of all $M \in \mathfrak{M}$ containing both x and y.

It is easy to see that ϱ is a metric for the space $(P, \tau(\mathfrak{M}))$. The fact that ϱ is a metric is obvious. To prove that the topology $\tau(\mathfrak{M})$ is generated by ϱ, it is sufficient to notice that

$$S(x, \mathfrak{M}/U(x, \varphi, \varepsilon)) \subset U(x, \varrho, 2\varepsilon) \subset S(x, \mathfrak{M}/U(x, \varphi, 2\varepsilon))$$

where

$$U(x, \psi, \varepsilon) = \{y;\, y \in P,\, \psi(x, y) < \varepsilon\}\,,$$

ψ is a metric for P and ε is a positive real number. The proof of 2.3 is complete.

Examples. a) The family $\mathsf{C}(\tau)$ is feebly additive and contains arbitrarily small sets (one-point sets). Thus we have defined the topology $\tau(\mathsf{C}(\tau))$. Evidently, $\tau(\mathsf{C}(\tau))$ is the topology τ^* defined in the first section.

b) By a well-known theorem the family $\mathsf{LF}\,\mathsf{C}(\tau)$ is feebly additive and contains arbitrarily small sets. Thus we have defined the topology $\tau(\mathsf{LF}\,\mathsf{C}(\tau))$.

c) The family $\mathsf{A}(\tau)$ of all metrizable compact connected and locally connected subspaces of (P, τ) is feebly additive.

d) The family of all compact connected and locally connected subspaces of (P, τ) is feebly additive.

It may be noticed that the family $\mathsf{L}\,\mathsf{C}(\tau)$ is not feebly additive (in general). However, there exists a smallest feebly additive family containing $\mathsf{L}\,\mathsf{C}(\tau)$. Using this family, by 2.2 we obtain a topology which will be denoted by $c(\tau)$. The purpose of this section is to study this topology. We shall need a few propositions about the smallest feebly additive family containing a given family.

2.4. The smallest feebly additive family. Let $\mathfrak{M}$ be a family of sets and let M be the union of the family $\mathfrak{M}$. There exists a smallest feebly additive family $\mathfrak{M}_\infty$ containing $\mathfrak{M}$. Indeed, the family of all subsets of M is feebly additive and the intersection of an arbitrary set of feebly additive families is feebly additive. Moreover, we have obtained that $\mathfrak{M}_\infty$ is a subfamily of the family of all subsets of M. Putting $\mathfrak{M}_0 = \mathfrak{M}$, let us define by induction the sequence $\{\mathfrak{M}_n\}$ of families of sets such that $\mathfrak{M}_{n+1}$ is the family of all sets of the form $K \cup L$ where both K and L belong to $\mathfrak{M}_n$ and $K \cap L \neq \emptyset$. It is easy to prove

2.4.1.
$$\mathfrak{M}_\infty = \bigcup_{n=1}^{\infty} \mathfrak{M}_n .$$

Indeed, by induction we obtain at once that for every n the family $\mathfrak{M}_n$ is contained in $\mathfrak{M}_\infty$. Thus we have proved the inclusion $\subset$. On the other hand the family $\bigcup_{n=1}^{\infty} \mathfrak{M}_n$ is feebly additive.

We shall need a description of stars $S(x, \mathfrak{M}_\infty)$ in terms of chains of sets from $\mathfrak{M}$. First let us recall that a chain (of sets) is a finite sequence $\alpha = \{M_0, \ldots, M_n\}$ of sets with $M_{i-1} \cap M_i \neq \emptyset$ $(i = 1, \ldots, n)$. A chain $\alpha = \{M_0, \ldots, M_n\}$ joins the points x and y in a family $\mathfrak{M}$ if $M_i \in \mathfrak{M}$ $(i = 0, 1, \ldots, n)$ and $x \in M_0$, $y \in M_n$.

Let us define iterated stars by induction as follows:

2.4.2.
$$S_0(x, \mathfrak{M}) = S(x, \mathfrak{M}) ; \quad S_{n+1}(x, \mathfrak{M}) = S(S_n(x, \mathfrak{M}), \mathfrak{M}) .$$

By induction it is easy to obtain that

2.4.3.
$$S_n(x, \mathfrak{M}) = S(x, \mathfrak{M}_n) .$$

and that

2.4.4. $S_n(x, M)$ is the set of all points y for which there exists a chain $\{M_0, \ldots, M_n\}$ joining x and y in $\mathfrak{M}$.

It follows

2.4.5.
$$S(x, \mathfrak{M}_\infty) = \bigcup_{n=0}^{\infty} S(x, \mathfrak{M}_n) = \bigcup_{n=0}^{\infty} S_n(x, \mathfrak{M}) .$$

2.4.6. $S(x, \mathfrak{M}_\infty)$ is the set of all points y for which there exists a finite chain joining x and y in $\mathfrak{M}$. Next, it is easy to see that if $\mathfrak{M}$ is a family of sets and K is a set, then

2.4.7.
$$\mathfrak{M}_n/K = (\mathfrak{M}/K)_n , \quad \mathfrak{M}_\infty/K = (\mathfrak{M}/K)_\infty .$$

Finally, if $\mathfrak{M}$ is a family of subsets of a space containing arbitrarily small sets, then $\mathfrak{M}_\infty$ also contains arbitrarily small sets.

2.5. The topology $\tau(\mathfrak{M}_\infty)$. Let $\mathfrak{M}$ be a family of subsets of a space (P, τ) containing arbitrarily small sets. By 2.4 and 2.2.2 we have defined the topology $\tau(\mathfrak{M}_\infty)$ which (by 2.3) is metrizable provided that τ is metrizable. Since the family $\mathfrak{M}_\infty$ is uniquely determined by $\mathfrak{M}$, the topology $\tau(\mathfrak{M}_\infty)$ will be denoted merely by $\tau(\mathfrak{M})$.

2.5.1. Let us suppose that if $M \in \mathfrak{M}$, U is an open subspace of $(M, \tau/M)$ and $x \in U \cap M$, then there exists a (τ/M) – neighborhood H of x belonging to $\mathfrak{M}$.

Then for every M in $\mathfrak{M}$,
$$\tau/M = \tau(\mathfrak{M})/M .$$

Indeed, if U is open in M and $x \in U$, then we may choose an open set V in (P, τ) with $V \cap M = U$. Evidently $S(x, V/\mathfrak{M}_\infty) \cap M \subset H$.

2.5.2. *If* $\mathsf{C}(\tau(\mathfrak{M})) \subset \mathfrak{M}$, *then* $\tau(\mathfrak{M})$ *is a locally connected topology.*

Proof. It is sufficient to prove that the sets of the form $S(x, \mathfrak{M}_\infty/U)$ with open U are $\tau(\mathfrak{M})$-connected. By 2.4.5 and 2.4.6 we have

$$S(x, \mathfrak{M}_\infty/U) = \bigcup_{n=0}^{\infty} S_n(x, \mathfrak{M}/U)\,.$$

The set $S_0 = S_0(x, \mathfrak{M}/U)$ is $\tau(\mathfrak{M})$-connected as a union of $\tau(\mathfrak{M})$-connected sets containing a common point x. By induction, S_{n+1} is $\tau(\mathfrak{M})$-connected as a union of the $\tau(\mathfrak{M})$-connected set S_n and $\tau(\mathfrak{M})$-connected sets meeting S_n. Since $S_n \subset S_{n+1}$, $n = 0, 1, 2, \ldots$, the set $\bigcup_{n=1}^{\infty} S_n$ is $\tau(\mathfrak{M})$-connected. The proof is complete.

2.6. The topology $c(\tau)$. The topology $\tau((\mathsf{L}\,\mathsf{C}(\tau))_\infty) = \tau(\mathsf{L}\,\mathsf{C}(\tau))$ will be denoted merely by $c(\tau)$. According to the preceding results we have at once $\tau \geqq c(\tau)$, $c(\tau)/K = \tau/K$ for every $K \in \mathsf{L}\,\mathsf{C}(\tau)$ (by 2.5.1), $c(\tau)$ is locally connected (by 2.5.2). If τ is metrizable, then $c(\tau)$ is metrizable (by 2.3). According to the definition of $s(\tau)$ we have $c(\tau) \leqq s(\tau)$.

2.6.1. *The topology* $c(\tau)$ *is connected if and only if every two points may be joined by a finite chain in* $\mathsf{L}\,\mathsf{C}(\tau)$.

Proof. As already known, every two points may be joined in $\mathsf{L}\,\mathsf{C}(\tau)$ if and only if

$$S(x) = S(x, (\mathsf{L}\,\mathsf{C}(\tau))_\infty) = P$$

for every x in P. If $S(x) = P$ for some x in P, then P is $c(\tau)$-connected since the sets of the form $S(x)$ are $c(\tau)$-connected (see 2.5.1 and the proof of 2.5.2). Conversely, suppose $S(x) \neq P$ for some x in P. By definition the sets of the form $S(y)$ are $c(\tau)$-open and by 2.1.3 either $S(y_1) = S(y_2)$ or $S(y_1) \cap S(y_2) = \emptyset$. Thus $S(x)$ is $c(\tau)$-open; since

$$S(x) = P - \bigcup\{S(y);\, y \in P - S(x)\}\,,$$

$S(x)$ is $c(\tau)$-closed. If follows that the space $(P, c(\tau))$ is not connected. The proof is complete.

2.6.2. **An unsolved problem.** I do not know whether the equality $s(\tau) = c(\tau)$ holds.

3. THE FINEST TOPOLOGY AGREEING WITH A GIVEN TOPOLOGY ON LOCALLY CONNECTED SUBSPACES

In this section we assume that τ is a topology for a set P. We shall use the notation from section 2. As we know from sections 1 and 2, the topologies τ, $s(\tau)$ and $c(\tau)$ agree on sets belonging to $\mathsf{L}\,\mathsf{C}(\tau)$. In this section we shall investigate the finest topology agreeing with τ on sets from a given subfamily of $\mathsf{L}\,\mathsf{C}(\tau)$. First we shall prove the following

3.1. Theorem. *Let $\mathfrak{N}$ be a subfamily of* $\mathsf{L}\,\mathsf{C}(\tau)$. *Let us denote by T the family of all topologies τ' for the set P agreeing with τ on all sets belonging to $\mathfrak{N}$, i. e., for which*

3.1.1. $$K \in \mathfrak{N} \Rightarrow \tau/K = \tau'/K .$$

Denote inf T *by τ_0. Then $\tau_0 \in T$ and*

3.1.2. $$\tau_0 = \sup \{\tau_K; K \in \mathfrak{N}\} .$$

The topology τ_0 is locally connected and a set $U \subset P$ is τ_0-open if and only if the set $U \cap K$ is (τ/K)-open for every K in $\mathfrak{N}$. Analogously, a set $F \subset P$ is τ_0-closed if and only if the set $F \cap K$ is (τ/K)-closed for every K in $\mathfrak{N}$. Finally, $x \in \tau_0[M]$ if and only if there exists a K in $\mathfrak{N}$ such that $x \in (\tau/K)[K \cap M]$.

Proof. First let us consider the topology $\tau' = \sup \{\tau_K; K \in \mathfrak{N}\}$. If $K \in \mathfrak{N}$, then $\tau_K \leqq \tau_0$ since is the largest topology agreeing with τ on the set K. If follows that $\tau' \leqq \tau_0$. On the other hand $\tau_0/K \geqq \tau/K$ for every K in $\mathfrak{N}$, and consequently, $\tau' \geqq \tau_0$. Thus the equality 3.1.2 is proved. Simultaneously we have $\tau_0 \in T$. The topologies τ/K, $K \in \mathfrak{N}$, being locally connected, the topologies τ_K are locally connected. Combining 1.1 and 3.1.2 we obtain at once that τ_0 is locally connected. The description of τ_0-open and τ_0-closed sets follows at once from 3.1.2. Indeed, it is easy to show that the family $\mathfrak{D}$ of all sets $U \subset P$ for which $U \cap K$ is (τ/K)-open for every K in $\mathfrak{N}$ is the family of all τ'-open sets for some topology τ' for P. By 3.1.2 we obtain the equality $\tau' = \tau_0$. Besides, the definition of τ_0 by 3.1.2 is a well-known and standard method of definition of topologies.

3.2. *Let $\mathfrak{N}$ be a subfamily of* $\mathsf{L}\,\mathsf{C}(\tau)$ *and let τ_0 be the topology from* 3.1. *The space (P, τ_0) is connected if and only if every two points of P may be joined by a finite chain in $\mathfrak{N}$.*

Proof. Choose a point x in P and consider the set

$$S(x) = S(x, \mathfrak{N}_\infty) .$$

By 2.4.5 the set $S(x)$ is the set of all points $y \tau P$ for which there exists a finite chain in $\mathfrak{N}$ joing x and y. By 3.1 the set $S(x)$ is τ_0-open. Since

$$S(x) = \bigcup\{S(y); y \in (P - S(x))\}$$

the set $S(x)$ is closed. By induction we may conclude from 2.4.5 that $S(x)$ is τ_0-connected. Thus, if $S(x) = P$, then P is τ_0-connected, and conversely, if $S(x) \neq P$, then τ_0 is not connected. The proof is complete.

It is easy to see:

3.3. If U is an open subset of (P, τ_0), then the relativization of τ_0 to U is the largest topology agreeing with τ on sets from $\mathfrak{N} \cap U$. From 3.2 we have at once that U is $\tau(\mathfrak{N})$-connected if and only if U is τ_0-connected.

3.4. The topology $m(\tau)$. Putting $\mathfrak{N} = \mathsf{L}\,\mathsf{C}(\tau)$ in 3.1 we obtain a topology which will be denoted by $m(\tau)$. From 3.1 we can again conclude that τ is locally connected provided that for every $x \in \tau[M]$, $M \subset P$, there exists a K in $\mathsf{L}\,\mathsf{C}(\tau)$ with $x \in (\tau/K)$ . . $[K \cap M]$. Combining 3.2 and 2.6.1 we obtain:

3.4.1. The topology $m(\tau)$ is connected if and only if the topology $c(\tau)$ is connected. That is, $m(\tau)$ is connected if and only if every two points of P may be joined by a finite chain in $\mathsf{L}\,\mathsf{C}(\tau)$. More generally, (by 3.3).

3.4.2. An open subset U of $(P, m(\tau))$ is connected if and only if U is $c(\tau)$-connected.

3.5. Example. There exists a topology τ such that $c(\tau) \neq m(\tau)$.

Construction. Let P' be the set of all countable ordinals. Let us denote by J the interval $\{x; 0 \leqq x < 1\}$ of real numbers. The usual ordering of the sets P' and J is denoted by $>$. We define an ordering $\succ$ for the set $P = P' \times J$ such that $(\gamma, x) \succ (\delta, y)$ if and only if either $\gamma > \delta$ or $\gamma = \delta$ and $x > y$. The points $(\gamma, 0)$ will be denoted merely by γ. Denote by τ_2 the order topology for the ordered set P. Next, denote by τ_1 the usual topology (i. e. the order topology) for the set Z of all irrational numbers of the unit interval $\langle 0, 1\rangle$ of real numbers. Finally, let us choose an element Ω such that

$$\Omega \text{ non} \in Z \cup P \cup Z \times P .$$

We define a topology τ for the set

$$R = (Z \times P) \cup (\Omega)$$

as follows:

a) The set $Z \times P$ is τ-open and the relativization of τ to $Z \times P$ is the product topology $\tau_1 \times \tau_2$.

b) The family of all sets of the form

$$U(\gamma) = (\Omega) \cup \{(z, x); z \in Z, x \in P, x \geqq \gamma\} ,$$

where γ $(= (\gamma, 0)) \in P$, is a local base at the point Ω.

Evidently, τ is a Hausdorff topology for the set R. Moreover it is easily proved that the topology τ is completely regular. Indeed, R is a subspace of the one-point compactification of the locally compact completely regular space $\langle 0, 1\rangle \times P$. We shall prove that $m(\tau) \neq c(\tau)$, more precise, that $m(\tau)$ and $c(\tau)$ are not homeomorphic at the point Ω.

3.5.1. If $Q \subset Z \times P$ and if the relative topology τ/Q is connected, then there exists a z_0 in Z such that $Q \subset (z_0) \times P$, and moreover, Q is an interval in $(z_0) \times P$ (of course, we define $(z_0, x) > (z_0, y)$ if and only if $x > y$).

The proof of this proposition follows at once from the facts that the topology τ_1 is totally disconnected and that a subset M of P is connected if and only if M is an interval. The sets of the form $z \times P$, $z \in Z$, are locally connected and connected. Thus

3.5.2. $$S(\Omega, \mathsf{L}\,\mathsf{C}(\tau)/U(\gamma)) = U(\gamma) \quad (\gamma = (0, \gamma) \in P) .$$

As an immediate corollary of 3.5.2 we have

3.5.3. The topologies τ and $c(\tau)$ are homeomorphic at the point Ω.

Now to prove $m(\tau) \neq c(\tau)$ it is sufficient to show that

3.5.4. The topologies τ and $m(\tau)$ are not homeomorphic at the point Ω.

Thus it remains to prove 3.5.4. Let us choose a subset Z' of Z of potency $\aleph_1$. There exists a one-to-one mapping f from P' (the set of all countable ordinals) onto Z'. Let us consider the set

$$T = (\Omega) \cup \bigcup\{(f(\delta), x); x \in P, x \succ (0, \delta)\} \cup \bigcup_{z \in Z - Z'} (z) \times P .$$

The set T is not a τ-neighborhood of the point Ω because it contains no set of the form $U(\gamma)$. However, the set T is $m(\tau)$-open. According to theorem 3.1 it is sufficient to show that for every $Q \in \mathsf{L}\,\mathsf{C}(\tau)$ the set $Q \cap T$ is open with respect to the relativization of τ to Q. If $Q \subset Z \times P$, then according to 3.5.1 there exists a z_0 in Z such that $Q \cap R$ is an interval in $z_0 \times P$. $T \cap (z_0 \times P)$ being an open interval, the set $T \cap Q$ is an open interval in Q, and consequently $T \cap Q$ is open in Q. There remains the case $\Omega \in Q$. The topology τ/Q being connected, there exists a subset Z'' of Z and a mapping x from Z'' in P such that

$$Q' - Q \subset \{(z, x(z)); z \in Z''\}$$

where

$$Q' = (\Omega) \cup \bigcup_{z \in Z''} \{(z, x); x \in P, x \succ x(z)\} .$$

We shall prove that the set Z'' contains no accumulation point of Z'', that is, Z'' is discrete with respect to the relative topology. Let us suppose that there exists a sequence $\{z_n\}$ in Z'' converging to a point $z_0 \in Z''$. Put

$$x_0 = \sup \{x(z_n);\ n = 1, 2, \ldots\} .$$

Evidently, the relative topology τ/Q is not locally connected at the points of the set $\{(z_0, x); x \in P, x > x_0\}$. This contradiction shows that the set Z'' is discrete. Since the relative topology τ_1/Z'' is metrizable and separable, the set Z'' is countable. Thus the set $Z' \cap Z''$ is countable and we may choose a point δ in P' such that

$$z \in Z' \cap Z'' \Rightarrow f^{-1}[z] < (0, \delta) .$$

It follows that

$$U(\delta) \cap Q \subset T \cap Q ,$$

that is, $T \cap Q$ is a (τ/Q)-neighborhood of the point Ω. Evidently the set $Q \cap T - (\Omega)$ is (τ/Q)-open. It follows that the set $Q \cap T$ is (τ/Q)-open. The proof is complete.

I do not know whether in general $c(\tau) = m(\tau)$ for all metrizable topologies τ. However it is easy to prove the equality $c(\tau) = m(\tau)$ for complete metrizable topologies τ. First we shall prove the following

3.6. Theorem. *Let us suppose that the space* (P, τ) *is metrizable. Denote by* $\mathsf{A}(\tau)$ *the family of all compact* $A \in \mathsf{L}\,\mathsf{C}(\tau)$. *Let* τ_0 *denote the topology* $\sup\{\tau_A; A \in \mathsf{A}(\tau)\}$. *We have* $\tau(\mathsf{A}(\tau)) = \tau_0$.

Proof. Let us choose a metric φ for the space (P, τ) such that $\varphi(x, y) \leqq 1$ for every x and y in P. Let ϱ be the metric defined in 2.3.1 (of course, we put $\mathfrak{M} = \mathsf{A}(\tau)$).

According to the proof of 2.3, ϱ is a metric for the space $(P, \tau(\mathsf{A}(\tau)))$. We shall prove that ϱ is a metric for (P, τ_0). Since $\tau(\mathfrak{A}) \geqq \tau_0$, it is sufficient to prove that if $\lim_{n\to\infty} \varrho(x_n, x) = 0$, then x_n converges to x with respect to the topology τ_0. Let us suppose $\lim_{n\to\infty} \varrho(x_n, x) = 0$. Without loss of generality we may assume $\varrho(x_n, x) < 1$ for every $n = 1, 2, \ldots$ According to the definition of ϱ we may select $K_n \in \mathfrak{A}(\tau)$, $n = 1, 2, \ldots$, so that $x \in K_n$, $x_n \in K_n$ and $\varphi(K_n) \leqq 2\varrho(x_n, x)$. Let us consider the set

$$K = \bigcup\{K_n;\ n = 1, 2, \ldots\}\,.$$

It is easy to see that $K \in \mathsf{A}(\tau)$. The compactness of K is obvious. Indeed, if M is an infinite subset of K, then either $M \subset K_n$ for some n, or $x \in \overline{M}$, since $\lim_{n\to\infty} \varphi(K_n) = 0$. K is connected as a union of connected sets containing a common point. Finally, K is locally connected because if $y \in K - x$, then a neighborhood of y is contained in the union of a finite number of K_n and if $y = x$, then every neighborhood of y contains all K_n for sufficiently large n. From $K \in \mathsf{A}(\tau)$ it follows that

$$\tau(\mathsf{A}(\tau))/K = \tau/K = \tau_0/K\,.$$

In consequence the sequence $\{x_n\}$ converges to x with respect to the topology τ_0; this completes the proof.

3.7. Theorem. *Let us suppose that the space (P, τ) is complete metrizable. Then*

$$c(\tau) = \tau(\mathsf{A}(\tau)) = m(\tau) = \sup\,\{\tau_A;\ A \in \mathsf{A}(\tau)\}\,.$$

Proof. According to the preceding theorem it is sufficient to prove $c(\tau) = \tau(\mathsf{A}(\tau))$. For every M in $\mathsf{L}\,\mathsf{C}(\tau)$ let us denote by M^* the set of all $x \in P$ for which $M \cup (x)$ belongs to $\mathsf{L}\,\mathsf{C}(\tau)$. Evidently $M \subset M^* \subset \overline{M}$, M^* belongs to $\mathsf{L}\,\mathsf{C}(\tau)$ and M^* is a G_δ in P. Thus τ/M^* is a complete metrizable topology. Now if x and y are points of M^*, then according to the well-known theorem of Mazurkiewicz-Moore-Menger there exists an arc $A \subset M^*$ from x to y. Let $\{M_0, \ldots, M_n\}$ be a chain in $\mathsf{L}\,\mathsf{C}(\tau)$ joining the points $y_0 \in M_0$ and $y_{n+1} \in M_n$. Then the chain $\{M_0^*, \ldots, M_n^*\}$ also joins y_0 and y_{n+1} in $\mathsf{L}\,\mathsf{C}(\tau)$ and $\bigcup_{i=0}^{n} M_i^* \subset \tau[\bigcup_{i=0}^{n} M_i]$. Let us choose $y_i \in M_{i-1}^* \cap M_i^*$, $i = 1, \ldots, n$. Next let us choose arcs $A_i \subset M_i^*$, $i = 0, \ldots, n$ from y_i to y_{i+1}. Then $A = \bigcup_{i=0}^{n} A_i$ belongs to $A(\tau)$ and $A \subset \tau[\bigcup_{i=0}^{n} M_i]$. From the proof of 2.3 we can conclude at once that $c(\tau) = \tau(\mathfrak{A})$. The proof is complete.

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Резюме

ЛОКАЛЬНО СВЯЗНЫЕ ТОПОЛОГИИ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

В статье рассматриваются некоторые локально связные топологии соответствующие любой даной топологии.

Пусть τ и σ топологические структуры (топологии) в множестве P. Мы будем говорить, что δ тоньше σ, или σ грубее τ, и пишем $\tau \leqq \sigma$, или $\sigma \geqq \tau$, если всякое σ-открытое множество является одновременно τ-открытым. В дальнейшем τ всегда обозначает топологию в множестве P.

(1) Топология $\varrho(\tau)$: Пересечение любого семейства локально связных топологий является локально связной топологией. В частности, пересечение всех локально связных топологий $\sigma \leqq \tau$ является локально связной топологией. Обозначим её $s(\tau)$.

Топологию $s(\tau)$ возможно определить также следующим образом:

Если σ топология, то система всех связных компонент всех σ-открытых множеств является отркытым базисом для некоторой топологии, которую мы будем обозначать σ^*. Далее определяем $\tau^1 = \tau^*$,

$$\tau^\alpha = \inf\{(\tau^\beta)^* \; ; \quad \beta < \alpha\}$$

для всякого ординального числа α. Оказывается, что

$$s(\tau) = \inf\{\tau^\alpha\} .$$

(2) Топология $c(\tau)$: $\mathsf{LC}(\tau)$ обозначает систему всех связных и локально связных подмножеств пространства (P, τ). Если отркрытое множество U содержит точку x, пусть $S_0(x, U)$ – звезда точки x относительно систем всех $M \in \mathsf{LC}(\tau)$, $M \subset U$, $S_{n+1}(x, U)$ – звезда множества $S_n(x, U)$ относительно систем всех $M \in \mathsf{LC}(\tau)$, $M \subset U$. Положим

$$S_\infty(x, U) = \bigcup_{n=0}^{\infty} S_n(x, U) .$$

Существует топология $c(\tau)$ так, что для любой точки $x \in P$ система

$$\{S_\infty(x, U) \; ; \quad U \text{ открыто}, \quad x \in U\}$$

является полным базисом окрестностей в точке x для топологии $c(\tau)$. Оказывается, что топология $c(\tau)$ локально связна; следовательно, $c(\tau) \leqq s(\tau)$. Автору неизвестно, имеет ли место равенство $s(\tau) = c(\tau)$. Если τ метризуема, то также $c(\tau)$ метризуема.

(3) Топология $m(\tau)$: Если $M \subset P$, то τ_M обозначает самую тонкую топологию в P, индуцирующую на M ту же топологию как τ; иначе говоря, $N \subset P$ является τ_M-открытым, если и только множество $M \cap N$ открыто в подпространстве M пространства (P, τ). Пусть $m(\tau)$-пересечение всех τ_M, $M \in LC(\tau)$. Оказывается, что топология $m(\tau)$ локально связна и $m(\tau) \leqq c(\tau)$. Существует τ так, что $m(\tau) \neq c(\tau)$. Если (P, τ) является топологически полным метризуемым пространством, то $c(\tau) = m(\tau)$.