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REMARKS CONCERNING THE INVARIANCE OF BAIRE SPACES UNDER MAPPINGS

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Let f be a mapping of a space P onto a space Q. Under what conditions on f may we assert that if P is a Baire space then Q is a Baire space. Analogously, under what conditions, if Q is a Baire space then P is a Baire space.

A space is said to be a Baire space if every non-void open subset of P is of the second category. The term Baire space was introduced in [1], Chapter 9. In F. HAUSDORFF [3], Baire spaces are called O_{II}-spaces. For the basic properties of Baire spaces see [1], Chapter 9, and [2].

Let f be a mapping of a space P onto a space Q. Under what conditions on f may we assert that if P is a Baire space then Q is a Baire space? It may be noticed that the image of a Baire space under a continuous mapping may fail to be a Baire space. Moreover, the image of a complete normed linear space under a linear continuous mapping f may fail to be a Baire space. This is the case when f is not open. The image of a Baire space under an open mapping (f is open if images of open sets are open) may fail to be a Baire space. For example, denoting by S the Euclidean plane, put

$$P_1 = \{(x, y);\ (x, y) \in S,\ y \neq 0\}\ , \quad P_2 = \{(x, 0);\ (x, 0) \in S,\ x \text{ rational}\}\ .$$

Consider $P = P_1 \cap P_2$ as a subspace of S. On the other hand, let us define a topology for the set P such that P_1 and P_2 are open and P_1 and P_2 are subspaces of S (in the relative topology). Denote this space by Q. It is easy to see that P is a Baire space, Q is not a Baire space and the identity mapping from P onto Q is open (since the inverse mapping is continuous).

On the other hand, the image of a Baire space under a continuous and open mapping is a Baire space. We shall prove a generalization of this theorem. We shall introduce concepts of almost continuous and feebly open mappings and we shall prove that the image of a Baire space under an almost continuous and feebly open mapping is a Baire space.

The following unsolved problem is more interesting: Let f be a continuous and open mapping of a space P onto a Baire space Q and let us suppose that the "point-inverses" of f are Baire spaces (we call a point-inverse of f every set of the form $f^{-1}[y]$, $y \in Q$). May we assert that P is a Baire space? In particular, may we assert that the topological product of two Baire spaces is a Baire space? In [2], theorem 1.7, the following result is proved:

(*) *Let P and Q be a Baires spaces. Suppose that P contains a dense countable set N such that every point of N is of a countable character. Then $P \times Q$ is a Baire space.*

This theorem is a generalization of the following older result:

(**) *If P and Q are separable metrizable spaces, then $P \times Q$ is a Baire space provided that P and Q are Baire spaces.*

In the present note we shall give a generalization of this result, assuming more generally that f is a continuous open mapping of a separable metrizable space R onto a Baire space S and the point-inverses of f are Baire spaces. Of course, (**) is an immediate consequence of this theorem. If is sufficient to put $P \times Q = R$, $Q = S$, f the projection of $P \times Q$ onto Q.

IMAGES OF BAIRE SPACES

Definition 1. Let f be a mapping of a space P onto a space Q. f will be called almost continuous if for every open subset V of Q

$$\overline{\operatorname{int} f^{-1}[V]} \supset f^{-1}[V] .$$

f will be called feebly continuous if

$$M \subset Q , \quad \operatorname{int} M \neq \emptyset \Rightarrow \operatorname{int} f^{-1}[M] \neq \emptyset .$$

f will be called feebly open if

$$N \subset P , \quad \operatorname{int} N \neq \emptyset \Rightarrow \operatorname{int} f[N] \neq \emptyset .$$

It is easy to prove.

Proposition 1. A mapping f of a space P onto a space Q is feebly open if and only if

$$N \text{ is dense in } Q \Rightarrow f^{-1}[N] \text{ is dense in } P .$$

The mapping f is feebly continuous if and only if

$$M \text{ is dense in } P \Rightarrow f[M] \text{ is dense in } Q .$$

Evidently, an almost continuous mapping is feebly continuous. In general the converse is not true. Indeed, it may be noticed that if f is a mapping of a space P onto a space Q and if R is an open subset of P with $f[R] = Q$, then f is feebly continuous provided that the restriction of f to R is a feebly continuous mapping. However, every one-to-one feebly continuous mapping is almost continuous. For clearness we shall prove

Proposition 2. A mapping f of a space P onto a space Q is almost continuous if and only if for every open subset U of P the restriction of f to U is a feebly open mapping.

Proof. Suppose that for every open subset U of P the restriction of f to U is feebly continuous. Let V be a non-void open subset of Q and put $H = f^{-1}[V]$, $U = \operatorname{int} H$. We have to prove $\bar{U} \supset H$. Suppose not and consider the restriction g of f to $W = P - \bar{U}$. Then g being feebly continuous, the interior U' of $g^{-1}[V \cap g[U]]$ (with respect to W) is non-void. Since W is open, U' is open in P and hence $U' \subset U$. This contradiction establishes the almost continuity of f. The converse implication is obvious.

Theorem 1. *Let us suppose that f is an almost continuous and feebly open maping of a space P onto a space Q. If P is a Baire space (of the second category in itselif) then Q is a Baire space (of the second category in itself, respectively).*

Proof. First suppose that P is a Baire space. To prove that Q is a Baire space it is sufficient to show that if $\{U_n\}$ is a sequence of open dense subsets of Q, then the set $U = \bigcap_{n=1}^{\infty} U_n$ is dense in Q. Put $V_n = \operatorname{int} f^{-1}[U_n]$ and $V = \bigcap_{n=1}^{\infty} V_n$. Since f is feebly open, the sets $f^{-1}[U_n]$ are dense in P. By almost continuity of f the sets V_n are dense in $f^{-1}[U_n]$, and consequently they are dense in P. As P is a Baire space, the set V is dense in P. Again by continuity of f, the set $f[V]$ is dense in Q. Since $f[V] \subset U$, U is dense in Q. Thus the proof of the assertion concerning Baire spaces is complete. The second assertion is an immediate consequence of the first.

Corrolary. Let f be a one-to-one feebly open and feebly continuous mapping of a space P onto a space Q. Then P is a Baire space if and only if Q is a Baire space. P is of the second category if and only if Q is of the second category.

Proof. Since f is one-to-one and feebly continuous, f is almost continuous. f being feebly open and one-to-one, the mapping f^{-1} is feebly continuous, and consequently, almost continuous.

INVERSE IMAGES OF BAIRE SPACES

We shall prove the following

Theorem 2. *Let us suppose that f is an open and continuous mapping of a metrizable separable space P onto a space Q. If Q is a Baire space and if the point-inverses f (that is, the sets of the form $f^{-1}[y]$, $y \in Q$) are Baire spaces, then P is a Baire space.*

The theorem 2 is an immediate consequence of the following

Theorem 2′. *Let f be an open and continuous mapping of a metrizable separable space P onto a space Q. If Q is of the second category (in itself) and if the point-*

inverses of f are of the second category (in themselves), then P is of the second category (in itself).

First we shall prove the following

Lemma. *Let f be an open and continuous mapping of a metrizable separable space P onto a space Q. Let F be a nowhere-dense closed subset of P. Denote by $M(F)$ the set of all $y \in Q$ for which the interior of $f^{-1}[y] \cap F$ is non-void. Then the set $M(F)$ is of the first category in Q.*

Proof of the lemma. Let $\{U_n\} = \{U_n;\ n = 1, 2, \dots\}$ be an open base for P. For every n, $n = 1, 2, \dots$ put

$$(*) \qquad M_n = \{y;\ y \in Q,\ \Phi \neq f^{-1}[y] \cap U_n \subset F\}$$

$\{U_n\}$ being an open base, we have at once

$$M(F) = \bigcup_{n=1}^{\infty} M_n .$$

Therefore it is sufficient to prove that the M_n are nowhere-dense in Q. We shall show that the M_n are closed and their interiors empty. Evidently

$$Q - M_n = \{y;\ y \in Q,\ f^{-1}[y] \cap U_n \cap (P - F) \neq \emptyset\} = \\ = f[U_n \cap (P - F)] .$$

The mapping f being open, the set $Q - M_n$ is open, and consequently, M_n is closed.

Now suppose $V = \operatorname{int} M_n \neq \emptyset$. According to the definition (*) of M_n

$$\Phi \neq f^{-1}[V] \cap U_n \subset F .$$

By continuity of f the set $f^{-1}[V]$ is open, and hence F is not nowhere-dense. This contradiction establishes the lemma.

Proof of the theorem 2′. Let us suppose that Q is of the second category (in itself) and the point-inverses of f are of the second category (in themselves). Finally, suppose that P is of the first category in itself. There exists a sequence $\{F_n\}$ of closed nowhere-dense subsets of P such that

$$P = \bigcup\{F_n;\ n = 1, 2, \dots\} .$$

According to the lemma the set $M = \bigcup_{n=1}^{\infty} M(F_n)$ is of the first category in Q. Thus we may choose a point y_0 in $Q - M$. Put $K = f^{-1}[y_0]$. The sequence $\{F_n \cap K\}$ of closed subsets of K covers K, and hence there exists an n such that the interior U (with respect to K) of $F_n \cap K$ is non-void. But this is impossible since M is the set of all $y \in Q$ for which there exists an n, $n = 1, 2, \dots$, such that the interior of $F_n \cap f^{-1}[y]$ (with respect to $f^{-1}[y]$) is non-void. The proof is complete.

References

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Резюме

ЗАМЕТКИ ОБ ОТОБРАЖЕНИЯХ, СОХРАНЯЮЩИХ ПРОСТРАНСТВА БЭРА

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Пространством Бэра называется топологическое пространство, в котором всякое непустое открытое множество второй категории. Рассматриваются следующие вопросы (1) и (2):

(1) Пусть f – отображение пространства Бэра P на пространство Q. При каких топологических условиях, касающихся отображения f, также Q есть пространство Бэра? Оказывается, что достаточно предполагать:

(a) Если V – открытое подмножество пространства Q, то

$$\overline{\operatorname{int} f^{-1}[V]} \supset f^{-1}[V] .$$

(b) Если U – непустое открытое подмножество пространства P, то int int$f[u] \neq 0$.

В частности достаточно предполагать, что f – непрерывное и открытое отображение.

(2) Пусть f – непрерывное и октрытое отображение пространства P на пространство Бэра Q и пусть полные прообразы точек являются пространствами Бэра. Вопрос, является ли P пространством Бэра, автору кажется интересным и сложным. Доказывается, что ответ положителен, если пространство P метризуемо и сепарабельно. Это – обобщение класической теоремы о том, что топологическое произведение двух метризуемых сепарабельных пространств Бэра является пространством Бэра.