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THE TOPOLOGICAL STRUCTURE OF THE SET OF STABLE SOLUTIONS OF A DIFFERENTIAL SYSTEM

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It is shown that the set of points in which stable, equi-stable and uniformly stable solutions originate, may be characterised as a G_δ-set.

This paper is devoted to the study of the structure of the set of stable, equi-stable and uniformly stable solutions of the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(t, x) \tag{1}$$

$\left(\text{in vector notation, } x = [x_1, x_2, \ldots, x_n],\ \|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}\right)$, such that the components $X_i(t, x)$ are defined in the half-space $t \geqq 0$ and satisfy some existence condition (*e. g.*, the Carathéodory conditions). The structure of this set will be studied by determining the type of the intersection of a hyperplane $t = \text{const} \geqq 0$ with the solutions of (1). We will confine ourselves to the hyperplane $t = 0$; the results obtained for this case will also hold for any other case $t = \text{const} \geqq 0$.

The necessity of the condition, *i. e.* the statement "If M is the set of all points in which the stable, equi-stable or uniformly stable solutions of (1) originate then M is of type G_δ", will be proved for systems of curves more general than that of solutions of (1).

Denote by S any system of curves which satisfies the following two conditions:

1. To any point x_0 and real t_0 there exists (at least one) vector function $x(t) \in S$, defined and continuous on some interval (t_1, t_2) with $t_1 < t_0 < t_2$, and such that it passes through x_0 at time t_0, i. e. $x(t_0) = x_0$.

2. Let $x_1(t)$ and $x_2(t)$ be curves of S, defined on intervals (t_1, t_2) and (t_3, t_4) respectively. If there exists a real τ such that $x_1(\tau) = x_2(\tau)$ and $t_3 < \tau < t_2$, then the curve $z(t)$ defined by

$$\begin{aligned} z(t) &= x_1(t) \quad \text{for} \quad t_1 < t \leqq \tau\,, \\ z(t) &= x_2(t) \quad \text{for} \quad \tau \leqq t < t_4 \end{aligned}$$

also belongs to S.

It is easily shown that for each curve of S there exists at least one maximal domain, which is necessarily an open interval (a, b), where a may be $-\infty$ and b may be $+\infty$. This interval will be termed the interval of definition. If the interval of definition of a curve $x(t)$ of S is (a, b) with $b < +\infty$, then there cannot exist a finite limit $\lim_{t \to b-} x(t)$ (otherwise the curve $x(t)$ could be prolonged through b). In other words,

$$\sigma(x) = \lim_{t \to b-} \sup_{t < \alpha < \beta < b} \|x(\alpha) - x(\beta)\| > 0 .$$

We will now define the basic notions:

Definition 1. Let the interval of definition of a curve $\tilde{x}(t) \in S$ be (T_1, T_2) with $T_1 < 0 < T_2$, possibly $T_1 = -\infty$ or $T_2 = +\infty$. Then $\tilde{x}(t)$ is *stable* if to any $\varepsilon > 0$ there is a $\delta > 0$ such that if

$$\|\tilde{x}(0) - x\| < \delta$$

then every curve $x(t) \in S$ which has $x(0) = x$ is then defined on $\langle 0, T_2)$ and satisfies

$$\|\tilde{x}(t) - x(t)\| < \varepsilon \quad \text{for} \quad 0 \leqq t < T_2 .$$

Definition 2. Let $\tilde{x}(t)$ be a curve of S with an interval of definition (T_1, T_2), $T_1 < 0 < T_2$. The curve $\tilde{x}(t)$ is *equi-stable* if to any $\varepsilon > 0$, T_0 $(0 \leqq T_0 < T_2)$ there exists a $\delta(\varepsilon, T_0) > 0$ such that if

$$\|\tilde{x}(t_0) - x\| < \delta(\varepsilon, T_0)$$

for some $t_0 \in \langle 0, T_0)$, then every curve $x(t) \in S$ which has $x(t_0) = x$ is then defined on $\langle t_0, T_2)$ and satisfies

$$\|\tilde{x}(t) - x(t)\| < \varepsilon \quad \text{for} \quad t_0 \leqq t < T_2 .$$

Definition 3. Let the interval of definition of a curve $\tilde{x}(t) \in S$ be (T_1, T_2), $T_1 < 0 < T_2$. The curve $\tilde{x}(t)$ is *uniformly stable* if to any $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$\|\tilde{x}(t_0) - x\| < \delta$$

for some $t_0 \in \langle 0, T_2)$, then every curve $x(t) \in S$ which has $x(t_0) = x$ is then defined on $\langle t_0, T_2)$ and satisfies

$$\|\tilde{x}(t) - x(t)\| < \varepsilon \quad \text{for} \quad t_0 \leqq t < T_2 .$$

If we put $T_2 = +\infty$ in these definitions, we obtain the customary definitions of stability, equi-stability and uniform stability, *i. e.* definitions in which it is pre-assumed that the curves are defined for all $t \geqq 0$. Since the solutions of (1) may be defined on a finite interval only, it seems useful to generalise the customary definitions to curves with bounded domains in the manner of our definitions 1, 2 and 3. The case of stable solutions defined for all $t \geqq 0$ will be considered separately.

Equi-stability is usually formulated in other terms: a curve $x(t) \in S$ is equi-stable if to any real $\varepsilon > 0$, $t_0 \geqq 0$ there exists a $\delta(\varepsilon, t_0) > 0$ such that if $\|\tilde{x}(t_0) - x(t_0)\| < \delta(\varepsilon, t_0)$, $x(t) \in S$, then $\|\tilde{x}(t) - x(t)\| < \varepsilon$ for $t \geqq t_0$.

However, if the curves are solutions of a differential system satisfying the Carathéodory conditions

1. For fixed x, $X_i(t, x)$ are measurable in t.
2. For fixed t, $X_i(t, x)$ are continuous in x.
3. To every $\alpha > 0$ there exists a Lebesgue integrable $m(t)$ such that $|X_i(t, x)| \leqq m(t)$ for $|t| \leqq \alpha$, $\|x\| \leqq \alpha$; then the two definitions of equi-continuity are equivalent.

We will also need

Definition 4. First we define $T^{(\alpha)}$ for $0 \leqq \alpha \leqq 1$ and real T. If $T = \infty$, let $T^{(\alpha)} = \alpha/(1 - \alpha)$ for $0 \leqq \alpha < 1$, and $T^{(1)} = \infty$. If $T < \infty$, then let $T^{(\alpha)} = \alpha T$. Next we define sets $M_a^{(\alpha)}$ for $0 \leqq \alpha \leqq 1$ and $a \geqq 0$ or $a = \infty$. A point $\tilde{x}$ belongs to $M_a^{(\alpha)}$ if there exists a curve $\tilde{x}(t)$ in S which satisfies the following conditions:

1. The interval of definition of $\tilde{x}(t)$ is (T_1, T_2), where $T_1 < 0 < T_2$, $T_2 \geqq a$, and the curve $\tilde{x}(t)$ passes through $\tilde{x}$ at time $t = 0$, *i. e.* $\tilde{x}(0) = \tilde{x}$.
2. If $T_2^{(\alpha)} > 0$, let t_0 be any point from $\langle 0, T_2^{(\alpha)})$; if $T_2^{(\alpha)} = 0$, put $t_0 = 0$ also. To every $\varepsilon_n = 2^{-n}$ there exists a $\delta_\alpha^{(n)} > 0$ such that if $\|\tilde{x}(t_0) - x\| < \delta_\alpha^{(n)}$ then any curve $x(t) \in S$ with $x(t_0) = x$ is defined on $\langle t_0, T_2)$ and satisfies $\|\tilde{x}(t) - x(t)\| < \varepsilon_n$ for $t_0 \leqq t < T_2$.

Note. All curves of S which originate in the sets $M_a^{(\alpha)}$ satisfy a unicity condition for increasing t. In greater detail, if $x(t) \in S$ and $y(t) \in S$, if $x(0) = y(0) \in M_a^{(\alpha)}$, and if the intervals of definition of $x(t)$ and $y(t)$ are (T_x^1, T_x^2) and (T_y^1, T_y^2) respectively, then $T_x^2 = T_y^2$ and $x(t) = y(t)$ for $t \in \langle 0, T_x^2)$.

The sets $M_a^{(\alpha)}$ have the following meaning:

The set $M_0^{(0)}$ consists of all the points of the hyperplane $t = 0$ in which there originate *stable* curves in the sense of *Definition 1.*

The set $M_\infty^{(0)}$ consists of all the points of the hyperplane $t = 0$ in which there originate *stable* curves defined *for all* $t \geqq 0$.

The set $M_0^{(1)}$ consists of all the points of the hyperplane $t = 0$ in which there originate *uniformly* stable curves in the sense of *Definition 3.*

The set $M_\infty^{(1)}$ consists of all the points of the hyperplane $t = 0$ in which there originate *uniformly stable* curves, defined *for all* $t \geqq 0$.

The set

$$\prod_{n=1}^{\infty} M_0^{(\alpha_n)} \text{ for } 0 < \alpha_n < 1 . \quad \lim_{n\to\infty} \alpha_n = 1 \text{ monotonously},$$

consists of all the points of the hyperplane $t = 0$ in which there originate *equi-stable* curves in the sense of *Definition 2.*

The set

$$\prod_{n=1}^{\infty} M_\infty^{(\alpha_n)} \text{ for } 0 < \alpha_n < 1 , \quad \lim_{n\to\infty} \alpha_n = 1 \text{ monotonously},$$

consists of all the points of the hyperplane $t = 0$ in which there originate *equi-stable* curves, defined *for all* $t \geqq 0$.

Theorem 1. *The sets* $M_a^{(\alpha)}$ *with* $0 \leqq \alpha \leqq 1$ *and* $a \geqq 0$ *or* $a = \infty$ *are of type* G_δ.

Note. By a well-known property of G_δ-sets it immediately follows that the sets $\prod_{n=1}^{\infty} M_a^{(\alpha_n)}$ are of type G_δ.[1])

Proof. The δ-neighbourhood of $\tilde{x}$, *i. e.* the set of points x with $\|\tilde{x} - x\| < \delta$, will be denoted by $U(\tilde{x}, \delta)$. To any point $\tilde{x} \in M_a^{(\alpha)}$ there exists a $\delta_{1,\alpha}^{(n)}(\tilde{x}) > 0$ with the property described in Definition 4, condition 2 (it was there denoted by $\delta_\alpha^{(n)}$), and such that

$$(0,1) \quad \delta_{1,\alpha}^{(n)}(\tilde{x}) < \sigma(\tilde{x}) = \lim_{t \to T_2 -} \sup_{t < \mu < \nu < \tilde{T}_2} \|\tilde{x}(\mu) - \tilde{x}(\nu)\| , \quad \tilde{x}(0) = \tilde{x} , \quad \tilde{x}(t) \in S$$

if $\sigma(\tilde{x}) > 0$. Choose $\delta_{2,\alpha}^{(n)}(\tilde{x}) > 0$ such that if

$$\|\tilde{x}(t_0) - x(t_0)\| < \delta_{2,\alpha}^{(n)}(\tilde{x})$$

for arbitrary $t_0 \in \langle 0, T_2^{(\alpha)})$ when $\tilde{T}_2^{(\alpha)} > 0$ and $t_0 = 0$ when $\tilde{T}_2^{(\alpha)} = 0$, then

$$\|\tilde{x}(t) - x(t)\| < \tfrac{1}{3}\delta_{1,\alpha}^{(n)}(\tilde{x}) \quad \text{for} \quad t_0 \leqq t < \tilde{T}_2 .$$

Theorem 1 then immediately follows from the identity

$$M_a^{(\alpha)} = \prod_{n=1}^{\infty} \sum_{x \in M_a^{(\alpha)}} U(x, \delta_{2,\alpha}^{(n)}(x))$$

which we proceed to prove. The inclusion

$$M_a^{(\alpha)} \subset \prod_{n=1}^{\infty} \sum_{x \in M_a^{(\alpha)}} U(x, \delta_{2,\alpha}^{(n)}(x))$$

is obvious. Let x^* be a point of

$$(0,2) \qquad \prod_{n=1}^{\infty} \sum_{x \in M_a^{(\alpha)}} U(x, \delta_{2,\alpha}^{(n)}(x)) ;$$

we shall prove that then x^* belongs to $M_a^{(\alpha)}$. By the existence condition the definition of S, there exists a curve $x^*(t)$ with $x^*(0) = x^*$ and whose interval of definition is (T_1^*, T_2^*) with $T_1^* < 0 < T_2^*$, where T_1^* may be $-\infty$ and T_2^* may be $+\infty$. We must prove that

$$(0,3) \qquad T_2^* \geqq a .$$

The intervals of definition (T_1, T_2) of curves $x(t) \in S$ with $x(0) \in M_a^{(\alpha)}$ satisfy

$$(0,4) \qquad T_2 \geqq a .$$

Since x^* is in the set (0,2), there exists a point $\tilde{x} \in M_a^{(\alpha)}$ such that $x^* \in U(\tilde{x}, \delta_{2,\alpha}^{(n)}(\tilde{x}))$ (arbitrary n). Now, the inequality (0,3) will immediately follow from (0,4) if we prove the following statement:

[1]) J. Kisiński has remarked that for ordinary stability (*i. e.* for the sets $M_a^{(o)}$), our Theorem 1 follows from the theorem on the structure of zero sets of continuous functions. The uniformly stable and equi-stable cases cannot be reduced to this theorem.

Let $\tilde{x}$ be a point of $M_a^{(\alpha)}$, and x^* a point of $U(\tilde{x}, \delta_{2,\alpha}^{(n)}(\tilde{x}))$; let $x^*(t)$, $\tilde{x}(t)$ be curves of S with $x^*(0) = x^*$, $\tilde{x}(0) = \tilde{x}$, and with intervals of definition (T_1^*, T_2^*) and $(\tilde{T}_1, \tilde{T}_2)$ respectively; then $T_2^* = \tilde{T}_2$.

Proof. Assume $T_2^* > \tilde{T}_2$ (necessarily then $\tilde{T}_2 < \infty$). As $\tilde{x}(t)$ cannot be prolonged through $\tilde{T}_2$, we must have

$$\sigma(\tilde{x}) = \lim_{t \to \tilde{T}_2 -} \sup_{t < \mu < \nu < \tilde{T}_2} \|x(\mu) - x(\nu)\| > 0 .$$

Since $x^* \in U(x, \delta_{2,\alpha}^{(n)}(\tilde{x}))$, we have $\|x^* - \tilde{x}\| < \delta_{2,\alpha}^{(n)}(\tilde{x})$; using the construction of $\delta_{2,\alpha}^{(n)}$ and (0,1),

$$\|x^*(t) - \tilde{x}(t)\| < \tfrac{1}{3}\delta_{1,\alpha}^{(n)}(\tilde{x}) < \tfrac{1}{3}\sigma(\tilde{x}) \quad \text{for} \quad 0 \leqq t < \tilde{T}_2 .$$

Obviously for any μ, ν with $0 < \mu < \nu < T_2$

$$\text{(0,5)} \quad \|x^*(\mu) - x^*(\nu)\| > \|\tilde{x}(\mu) - \tilde{x}(\nu)\| - \|x^*(\mu) - \tilde{x}(\mu)\| - \|x^*(\nu) - \tilde{x}(\nu)\| > \\ > \|\tilde{x}(\mu) - \tilde{x}(\nu)\| - \tfrac{2}{3}\sigma(\tilde{x}) .$$

To any $\gamma, 0 < \gamma < \tilde{T}_2$, there obviously exist $\mu, \nu, \gamma < \mu < \nu < \tilde{T}_2$ such that $|\sigma(\tilde{x}) - \|\tilde{x}(\mu) - \tilde{x}(\nu)\|| < \frac{1}{4}\sigma(\tilde{x})$. Hence and from (0,5) we conclude

$$\|x^*(\mu) - x^*(\nu)\| > (\tfrac{3}{4} - \tfrac{2}{3})\,\sigma(\tilde{x}) = \tfrac{1}{12}\sigma(\tilde{x}) .$$

Thus the limit $\lim_{t \to \tilde{T}_2 -} x^*(t)$ does not exist, and $x^*(t)$ cannot be defined for $t \geqq T_2$. This proves $T_2^* \leqq \tilde{T}_2$. The second inequality $T_2^* \geqq \tilde{T}_2$ follows from condition 2 in Definition 4 (the curves $x(t)$ mus be defined for those $t \geqq 0$ for which $\tilde{x}(t)$ is), and we conclude $T_2^* = \tilde{T}_2$.

Now we pass to the proof that x^* satisfies the second condition in the definition of $M_a^{(\alpha)}$. Since x^* is in the set (0,2), we have for every n that

$$x^* \in \sum_{x \in M_a^{(\alpha)}} U(x, \delta_{2,\alpha}^{(n+1)}(x)) ,$$

so that $x^* \in U(\tilde{x}_n, \delta_{2,\alpha}^{(n+1)}(\tilde{x}_n))$ for some $\tilde{x}_n \in M_a^{(\alpha)}$. Take

$$\delta_{1,\alpha}^{(n)}(x^*) = \tfrac{1}{2}\delta_{1,\alpha}^{(n+1)}(\tilde{x}_n) .$$

Let x be any point satisfying

$$\text{(0,6)} \qquad \|x^*(t_0) - x\| < \delta_{1,\alpha}^{(n)}(x^*) = \tfrac{1}{2}\delta_{1,\alpha}^{(n+1)}(\tilde{x}_n)$$

for some t_0, where t_0 is arbitrarily chosen in $\langle 0, T_2^{*(\alpha)})$ if $T_2^{*(\alpha)} > 0$ and $t_0 = 0$ if $T_2^{*(\alpha)} = 0$. From (0,6) and $\|x^*(t_0) - \tilde{x}_n(t_0)\| < \frac{1}{3}\delta_{1,\alpha}^{(n+1)}(\tilde{x}_n)$ there follows

$$\|x - \tilde{x}_n(t_0)\| < \delta_{1,\alpha}^{(n+1)}(\tilde{x}_n) .$$

If we had $0 \leqq t_0 < \tilde{T}_{2,n}^{(\alpha)}$, then by Definition 4 we would also have

$$\|x(t) - \tilde{x}_n(t)\| < \varepsilon_{n+1} \quad \text{for} \quad t_0 \leqq t < \tilde{T}_{2,n}$$

with $x(t) \in S$, $x(t_0) = x$, and $(\tilde{T}_{1,n}, \tilde{T}_{2,n})$ the interval of definition of $\tilde{x}_n(t)$. But we have already proved that $T_2^* = \tilde{T}_{2,n}$, so that (see $T^{(\alpha)}$ in Definition 4) necessarily $T_2^{*(\alpha)} = \tilde{T}_{2,n}^{(\alpha)}$. Thus we have proved the following statement:

If x is any point with

$$\|x^*(t_0) - x\| < \delta_{1,\alpha}^{(n)}(x^*) = \tfrac{1}{2}\delta_{1,\alpha}^{(n+1)}(\tilde{x}_n)$$

($0 \leqq t_0 < T_2^{*(\alpha)}$ *if* $T_2^{*(\alpha)} > 0$ *and* $t_0 = 0$ *if* $T_2^{*(\alpha)} = 0$, *then*

$$\|x(t) - \tilde{x}_n(t)\| < \varepsilon_{n+1} \quad \textit{for} \quad t_0 \leqq t < \tilde{T}_{2,n} = T_2^* .$$

From $\|x^* - \tilde{x}\| < \delta_{2,\alpha}^{(n+1)}(\tilde{x}_n)$ there follows $\|x^*(t_0) - \tilde{x}(t_0)\| < \frac{1}{3}\delta_{1,\alpha}^{(n+1)}(\tilde{x}_n)$ and thence

$$\|x^*(t) - \tilde{x}_n(t)\| < \varepsilon_{n+1} \quad \text{for} \quad t_0 \leqq t < \tilde{T}_{2,n} = T_2^* .$$

Using this result, we may reformulate the preceding statement in the following manner:

If x is any point with

$$\|x^*(t_0) - x\| < \delta_{1,\alpha}^{(n)}(x^*) ,$$

then

$$\|x^*(t) - x(t)\| < 2\varepsilon_{n+1} = \varepsilon_n \quad \textit{for} \quad t_0 \leqq t < T_2^* .$$

Obviously this statement implies that x^* satisfies the second condition in the definition of sets $M_a^{(\alpha)}$, *i. e.* that $x^* \in M_a^{(\alpha)}$.

Thus we have proved that the sets in which stable, equi-stable and uniformly stable solutions originate, are of type G_δ.

The converse problem is that of constructing a differential system to a given G_δ-set in such a manner that stable solutions originate in the given set and unstable ones in its complement. Before passing to this problem, we shall examine the topological properties of such sets in the autonomous case.

AUTONOMOUS CASE

Let us consider the problem just mentioned for the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x) \tag{0,6}$$

in vector notation. Theorem 1 remains true, but, as will be seen, it is too weak. Much more can be said of the sets than their Baire class. First let us take the one-dimensional case; the situation is different for equations with or without unicity of solutions. Assume, then, that all solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \tag{0,7}$$

satisfy unicity conditions; here $f(x)$ is a continuous function. We will make use of the following two assertions:

I. Let x_1, x_2 be neighbouring zero points of $f(x)$; then all solutions with initial points in the open interval (x_1, x_2) are uniformly stable.

II. Let $\bar{\bar{x}}$ be the greatest, and $\bar{x}$ the smallest, zero point of $f(x)$. Then the solutions with initial points in the open interval $(\bar{\bar{x}}, +\infty)$ are either all stable or all unstable; a similar statement holds for $(-\infty, \bar{x})$.

Using I and II, the following theorem may be easily proved:

Theorem 2. *The set of points in which unstable solutions of* (0,7) *originate is composed of three subsets,*

1. *A closed interval* $\langle x^{**}, +\infty)$, *where* x^{**} *may be* $-\infty$ (*i. e. all solutions are unstable*) *or it may also be* $+\infty$ (*i. e. this subset is void*).

2. *A closed interval* $(-\infty, x^{*}\rangle$, *with* x^{*} *possibly* ∞ *or* $-\infty$.

3. *A countable subset* Q *of* (x^{*}, x^{**}) *such that, if* $x \in Q$, *then* x *is not simultaneously a limit point of both* $(-\infty, x)\, Q$ *and* $(x, \infty)\, Q$.

Note. From 3 it follows that Q is non-dense.

The properties 1 and 2 follow from assertion II directly. From assertion I it follows that the points x_0 not in the intervals $(-\infty, \bar{x}\rangle$ and $\langle\bar{\bar{x}}, \infty)$, and in which unstable solutions originate, can only be points such that $f(x_0) = 0$ and that $f(x)$ is non-zero in some one of $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$. Now we shall proceed to prove that the conditions of Theorem 2 are also sufficient.

Let there be given a set M composed of three subsets according to conditions 1,2 and 3. Define a function $f(x)$ thus:

$f(x) = x - x^{**}$ for $x \geqq x^{**}$ if the first subset is non-empty (*i. e.* for $x^{**} < \infty$),
$f(x) = x$ for all x if $x^{**} = -\infty$,
$f(x) = x - x^{*}$ for $x \leqq x^{*}$ if the second subset is non-empty.

The construction of $f(x)$ in (x^{*}, x^{**}) will be more involved. We assume that Q is non-empty — otherwise it would suffice to take $f(x) = 0$ for $x^{*} < x < x^{**}$. Now take any closed interval $J = \langle\xi, \eta\rangle$, $\xi \leqq \eta$ (possibly $\xi = -\infty$ or $\eta = +\infty$) with no limit points of Q in its interior and such that $Q(\xi, \eta) \neq \emptyset$. Define λ_1 and λ_2 in the following manner:

a) if ξ (or η) is a limit point of $Q(\xi, \eta)$ then let $\lambda_1 = \xi$ (or $\lambda_2 = \eta$);

b) if ξ (or η) is not a limit point of $Q(\xi, \eta)$ and $\xi \in Q$ (or $\eta \overline{\in} Q$) then let $\lambda_1 = \xi$ (or $\lambda_2 = \eta$);

c) if ξ (or η) is not a limit point of $Q(\xi, \eta)$ and $\xi \overline{\in} Q$ (or $\eta \in Q$) then let λ_1 be the least (or λ_2 the greatest) point of the set $Q(\xi, \eta)$.

Now define $f(x)$ in J thus:

$$f(\xi) = f(\eta) = f\left(\frac{\lambda_2 + \eta}{2}\right) = 0\,, \quad f(x) = 0 \quad \text{for} \quad x \in JQ\,,$$

$$f(x) < 0 \quad \text{for} \quad x \in (\xi, \lambda_1)\,, \quad f(x) > 0 \quad \text{for} \quad x \in \left(\lambda_1, \frac{\lambda_2 + \eta}{2}\right) - Q\,,$$

$$f(x) < 0 \quad \text{for} \quad x \in \left(\frac{\lambda_2 + \eta}{2}, \eta\right) \quad \text{(this last only if } \eta < +\infty\text{)}\,.$$

However, if J is a closed interval $\langle \xi, \eta \rangle$, $\xi \leqq \eta$, such that $Q(\xi, \eta) = \emptyset$, then define $f(x)$ thus: $f(\xi) = f(\eta) = 0$;

d) if $\xi \in Q$, $\eta \in Q$ then let $f\left(\dfrac{\xi + \eta}{2}\right) = 0$,

$$f(x) > 0 \text{ for } x \in \left(\xi, \frac{\xi + \eta}{2}\right), \quad f(x) < 0 \text{ for } x \in \left(\frac{\xi + \eta}{2}, \eta\right);$$

e) if $\xi \in Q$, $\eta \bar{\in} Q$ then let $f(x) > 0$ for $x \in (\xi, \eta)$;
f) if $\xi \bar{\in} Q$, $\eta \in Q$ then let $f(x) < 0$ for $x \in (\xi, \eta)$;
g) if $\xi \bar{\in} Q$, $\eta \bar{\in} Q$ then let $f(x) = 0$ for $x \in (\xi, \eta)$.

In all the cases a) to g) we further require that $f(x)$ satisfy a Lipschitz condition (with unity constant) and be bounded in absolute value by one.

It now remains to decompose E_1 into a system of closed intervals (possibly degenerate) in each of which $f(x)$ will be defined in the manner just described. The points of the set Q, countable by condition 3, may be arranged into a sequence $x_1, x_2, \ldots, x_n, \ldots$ Taking x_1, there exists a maximal closed interval J_1 such that $x_1 \in J_1$ and that its interior contains no limit point of the sets Q, $(-\infty, x^*\rangle$, $\langle x^{**}, x)$. Next take the first x_k not in J_1, and repeat the construction. Continuing in this way we obtain a sequence of intervals J_n. Every point of Q is in some J_n, and from condition 3 of Theorem 2 it follows that none of these intervals is degenerate. Finally, for any point x_0 not in this system of intervals nor in $(-\infty, x^*\rangle$ or $\langle x^{**}, +\infty)$ there is a maximal closed interval J such that the intersection of its interior with the sets Q, $(-\infty, x^*\rangle$ and $\langle x^{**}, +\infty)$ is empty (this interval is possibly degenerate). On these intervals J_n, J we define $f(x)$ in the manner described above; if J is degenerate, we put $f(x) = 0$ on J.

The function $f(x)$ thus defined on E_1 is single-valued (the intervals J_n, J may have common end-points, but $f(x)$ is zero there) and satisfies a Lipschitz condition.

We will now prove that the corresponding equation (0,7) has the desired properties. Obviously, solutions with initial points in $(-\infty, x^*\rangle$ or $\langle x^{**}, +\infty)$ are unstable. Next we must show that the solutions originating in points $x_n \in Q$ are also unstable.

1. If x_n is an interior points of some J_{k_n}, then by construction of $f(x)$ in J_n (see a), b), c)) the solution through x_n is unstable.

2. If x_n is the right (or left) end-point of J_{k_n}, then there are two alternatives,

α) x_n is not a limit point of QJ_{k_n}; then according to b) (or c)) the solution through x_n is unstable;

β) x_n is a limit point of QJ_{k_n}. Then according to condition 3 of Theorem 2, there is a $\delta > 0$ such that $(x_n - \delta, x_n)\, Q = \emptyset$ (or $(x_n, x_n + \delta)\, Q = \emptyset$). The interval $(x_n - \delta, x_n)$ (or $(x_n, x_n + \delta)$) is thus a subset of some J_k or J. By c), d), f) (or by b), d), e)) the solution through x_n is unstable.

It remains to show that all solutions with initial points x_0, $x_0 \bar{\in} Q$, $x_0 \bar{\in} (-\infty, x^*\rangle$, $x_0 \bar{\in} \langle x^{**}, +\infty)$, are uniformly stable.

If $f(x_0) \neq 0$, then uniform stability follows from assertion I or II and from boundedness of $f(x)$. However, we cannot exclude the possibility that $f(x_0) = 0$, $x_0 \notin Q$. By a) to g), from $f(x) < 0$ for $x < x_0$ it would follow that $x_0 \in Q$, contradicting our assumption; thus $f(x) \geqq 0$ for $x \leqq x_0$; similarly $f(x) \leqq 0$ for $x \geqq x_0$. Thence we conclude *uniform stability* of the solution through x_0.

Thus, in the *one-dimensional* autonomous case, we have succeeded in characterising the set of points in which unstable solutions originate. The following example will show that in the *two-dimensional* autonomous case topological conditions are not sufficient, and that the conditions which may serve to characterise the structure of sets of initial points of stable solutions, become very complicated. In this example use will be made of Theorem 3 to be proved later.

Example. Consider the set L of all points of E_1 with irrational coordinates. The set L is of type G_δ, so that, according to Theorem 3, there is a differential equation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = X(t, x_1)$$

such that $X(t, x_1) = 0$ for $t \leqq 0$; that $X(t, x_1)$ has continuous partial derivatives of all orders; that solutions of this equation are defined for all t; that solutions with initial points in L are *uniformly stable*; and that solutions originating in $E_1 - L$ are *unstable*. Let M be the set of all points on solutions which originate in L. Also consider a plane E_2 with coordinate axes x_1, x_2. The set M of the plane with axes x_1, t may be mapped into E_2 by the relation $x_2 = t$.

Next construct a system of differential equations

$$\text{(0,9)} \qquad \frac{\mathrm{d}x_1}{\mathrm{d}t} = X_1(x_1, x_2)\,, \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = X_2(x_1, x_2)\,,$$

such that X_1, X_2 are continuous, that solutions originating in M are equi-stable, and solutions originating in $E_2 - M$ are unstable. Since the solutions of the original equation have a finite first derivative, it suffices to put $X_1(x_1, x_2) = X(x_1, x_2)$ and $X_2(x_1, x_2) = 1$.

Now map the plane E_2 into a plane E_2^+ with axes z_1, z_2 thus:

$$z_1 = x_1 + f(x_2)\,, \quad z_2 = x_2\,.$$

The set M will be mapped onto a set which we shall denote by M_f.

1. Assume that the function f has a *continuous bounded first derivative*. Then the solutions of the system (0,9) are mapped onto curves in E_2^+ which are solutions of the system

$$\text{(0,10)} \qquad \frac{\mathrm{d}z_1}{\mathrm{d}t} = X_1(z_1 - f(z_2), z_2) + f'(z_2)\, X_2(z_1 - f(z_2), z_2) = Z_1(z_1, z_2)\,,$$

$$\frac{\mathrm{d}z_2}{\mathrm{d}t} = X_2(z_1 - f(z_2), z_2) = Z_2(z_1, z_2)\,.$$

Since $f(x_2)$ has a *bounded derivative*, the *equi-stable* solutions will be mapped into *equistable* solutions, and *unstable* into *unstable* ones.

2. Assume that f is continuous and does not have a derivative at any point. Let us attempt to construct functions Z_1, Z_2 which are continuous, the solutions of (0,10) originating in M_f are equi-stable, and the solutions originating in $E_2^+ - M_f$ are unstable. The set M_f will then be composed of uncountably many continuous curves $z_1(t)$, $z_2(t)$, which will not possess derivatives at any point, and which will not intersect each other. Assume that such functions Z_1, Z_2 exist. The set M_f is invariant with respect to the system (0,10) (take any point $[z_1, z_2] \in M_f$; the solution with this initial point is stable, so that, by definition of equi-stability in the autonomous case, all $[z_1(\Theta), z_2(\Theta)]$ with $\Theta \geqq 0$ are initial points of equi-stable solutions). Thus the set M_f consists of solutions of (0,10). But M_f consists of a system of non-intersecting curves without derivatives at any point. Thus the solutions of (0,10) which originate in M_f must correspond to single points, *i. e.* the functions Z_1, Z_2 must both be zero at points of M_f. Now, $L$is dense in E_1, so that M is dense in E_2 and thus M_f is dense in E_2^+. Thus we conclude that the functions Z_1, Z_2 are identically zero in E_2^+. On the other hand, the solutions with initial points in $E_2^+ - M_f$ are not unstable. From this contradiction it follows that, unless the function $f(x)$ is sufficiently smooth, systems (0,10) with the desired properties cannot be constructed.

We have reached the following situation. The set M_f consists of uncountably many continuous curves. *If these curves are sufficiently smooth, then we are able to construct a system* (0,10); *if these curves are not sufficiently smooth, then we are not able to construct a system* (0,10). It becomes clear that the conditions which characterise sets in which the stable solutions originate, become rather complicated in the autonomous case, and that in any case they are not of topological character.

Now take the non-autonomous case. In Theorem 1 we have proved that the set in which stable, equistable and uniformly stable solutions originate is a G_δ-set. In order to be able to state that this characterises the former sets, *i. e.* that these sets exhaust the class of G_δ-sets, we must, to any G_δ-set M, construct a differential system (1) such that the stable (or equi-stable, or uniformly stable) solutions originate in M, and that the unstable (non-equi-stable, non uniformly stable, respectively) solutions originate in $E - M$. We may also require that the notions of stability (equi-stability, uniform stability, respectively) correspond to solutions defined for all $t \geqq 0$ (*i. e.* that $T_2 = \infty$ in Definitions 1, 2, 3), and only resort to Definitions 1, 2, 3 if we do not succeed to construct such a system. In fact the differential system to be constructed will have its solutions defined for all $t \geqq 0$. The three different cases for the three types of stability will be lumped into one by constructing, for a given G_δ-set M, a differential system such that uniformly stable solutions originate in M and unstable solutions originate in the complement of M. The proof of Theorem 3 becomes extremely complicated in the three-dimensional case, and therefore will be performed for two dimensions only; the fundamental idea of the proof applies to the poly-dimensional case also. The system to be constructed will have one component identically zero; thus Theorem 3 is also easily formulated in the one-dimensional case, which, therefore, does not differ fundamentally from that in many dimensions.

Theorem 3. *Let $M \subset E_2$ be a G_δ-set. Then there exists a differential equation* (1) (*a vector equation with two components*) *such that every solution with initial point $x \in M$ is uniformly stable, and every solution with initial point $x \bar{\in} M$ is unstable. Furthermore, the function $X(t, x)$ has continuous partial derivatives of all orders.*

First, an outline of the proof. Let M be the given G_δ-set in the plane E_2 with axes x, y. We may assume it is the intersection of a decreasing sequence of open sets G_n^* in E_2. First, in E_3 (*i. e.* in three-dimensional space with coordinates t, x, y) we construct a system of two differential equations all of whose solutions are unstable; this step is prepared in Lemma 2. Next, in $\tilde{G}_1^*$ – this is the set of points on solutions with initial points in G_1^* – we leave a countable set of these solutions (this will serve to prove unstability of certain solutions), cancel the rest, and construct a new differential system in E_3. This new system will have the following properties:

1. In the complement of $\tilde{G}_1^*$ it coincides with the original system.
2. All the previously selected solutions remain solutions of the new system.
3. All the solutions of the new system are unstable again, but do not diverge very much; more precisely, there is a previously given $\alpha > 0$ such that, to any solution $x^*(t)$ with $x^*(0) \in G_1^*$ there is a $\delta > 0$ with $\sup_{t \geqq 0} \|x^*(t) - x(t)\| < \alpha$ whenever $\|x^*(0) - x(0)\| < \delta$. This is the object of lemma 11. The construction is repeated for the sets $G_2^*, \ldots, G_n^*, \ldots$ From $\lim_{n \to \infty} \alpha_n = 0$ it will follow that solutions originating in $M = \prod_{n=1}^{\infty} G_n^*$ are uniformly stable. As for the remaining solutions $x(t)$, their unstability follow directly from the construction if $x(0)$ is an interior point of some $G_n^* - G_{n+1}^*$. If $x \in G_n^* - G_{n+1}^*$ but is not an interior point, then to prove unstability, use must be made of the curves which had been left unchanged at each step.

It is rather difficult to construct a differential system with the desired properties to any given region $\tilde{G}_n^*$, since its frontier may be very complicated. To this end, in Lemma 8 a system of auxiliary surfaces is introduced, which decomposes $\tilde{G}_n^*$ into countably many tubes with simple boundaries and in which the subsequent construction becomes simpler. Another system of auxiliary surfaces forces the solutions not to diverge too far apart (Lemma 9). Finally, using Lemma 10, unstable solutions are constructed in these tubes.

As will be noticed, the system of differential equations

$$\frac{dx}{dt} = X(t, x, y), \quad \frac{dy}{dt} = Y(t, x, y)$$

to be constructed for Theorem 3 will have $Y(t, x, y) \equiv 0$, *i. e.* the solutions move only in the direction of the x-axis. Obviously the proof of Theorem 3 is composed of a series of lemmas.

Before formulating Lemmas 1 and 2, let us consider a special case, in which the set M consists of a single point — *e. g.* the origin. After a suitable transformation, the solution passing through the origin is $x \equiv 0$. This solution is to be stable, so that $\|x(t)\| < \varepsilon$ for $t \geqq 0$ if $\|x(0)\| < \delta$; however, all solutions with $x(0) \neq 0$ must be unstable. Thus we must be able to construct a differential system (1), whose solutions with initial points in $\frac{1}{2}\delta < \|x\| < \delta$ are unstable and bounded: $\|x(t)\| < \varepsilon$ for $t \geqq 0$. First we will take $n = 1$ and, using Lemmas 1 and 2, construct a differential equation all of whose solutions are bounded and unstable. The following definition will be useful.

Definition 5. Let $d > 0$; let $\alpha(t)$, $\beta(t)$ be functions defined on some interval $\langle t_0, \infty)$ (different for different pairs of functions) and with continuous derivatives of all orders. We will say that $\alpha(t)$, $\beta(t)$ form a regular couple with respect to d if, for some $\eta > 0$, $\alpha(t) + \eta < \beta(t)$ for $t \in \langle t_0, \infty)$ and if there is a monotone sequence $t_n \to \infty$ such that $\alpha(t_n) + d + \eta < \beta(t_n)$.

Unless explicitly remarked, all functions will be assumed to possess derivatives of all orders.

Lemma 1. *Assume that $\xi^{00}(t)$, $\xi^{01}(t)$ are defined for $t \geqq 0$ and form a regular couple with respect to d. Then there exists a system of functions $\xi^{i,k}(t)$, $i = 0, 1, 2, \ldots$, $0 \leqq k \leqq 2^i$ such that:*

$$\xi^{i,k}(t),\ 0 \leqq k \leqq 2^i \text{ are defined for } t \geqq i\,;$$

$$\text{for } t \geqq i,\ \xi^{i,2k}(t) \text{ coincides with } \xi^{i-1,k}(t)\,;$$

$$\xi^{i,k}(t),\ \xi^{i,k+1}(t) \text{ form a regular couple with respect to } d\,;$$

to every curve $x = \xi^{i,2k+1}(t)$ there corresponds a point with coordinates $x = a^{i,2k+1}$, $t = 0$ which will be termed the fundamental point of the curve $x = \xi^{i,2k+1}(t)$; (to clarify the situation we describe the relation between these points and curves; in the subsequent construction, the functions $\xi^{i,2k+1}(t)$ will be defined for all $t \geqq 0$, and the points $x = a^{i,2k+1}$, $t = 0$ will be the initial points of these extended curves $x = \xi^{i,2k+1}(t)$);

the $a^{i,2k+1}$ with $i \geqq 1$, $k \geqq 0$ are all the dyadic rationals in the open interval $(\xi^0(0), \xi^{01}(0))$;

set $a^{i,2k} = a^{i-1,k}$; then to every curve $x = \xi^{i,k}(t)$ there corresponds a point $x = a^{i,k}$, $t = 0$, and to the curves $\xi^{i,2k}(t)$, $\xi^{i-1,k}(t)$ identical for $t \geqq i$, there corresponds the same point; the last property may then be formulated thus: $a^{i,k} < a^{i,k+1}$, i. e. the points $a^{i,k}$ are ordered similarly to the curves $\xi^{i,k}(t)$.

The proof of this lemma is quite obvious. For instance, the function $\xi^{11}(t)$ is constructed thus: let

$$\xi^{00}(t_n) + d + b < \xi^{01}(t_n)$$

for $0 \leqq t_0 < t_1 < \ldots < t_n < \ldots$, $\lim_{n\to\infty} t_n = \infty$. We choose points $\xi^{11}(t_n)$: for even n let $\xi^{11}(t_n)$ satisfy

$$\xi^{00}(t_n) + b' < \xi^{11}(t_n), \quad \xi^{11}(t_n) + d + b' < \xi^{01}(t_n),$$

where b' is sufficiently small positive; for n odd let $\xi^{11}(t_n)$ satisfy

$$\xi^{00}(t_n) + d + b' < \xi^{11}(t_n), \quad \xi^{11}(t_n) + b' < \xi^{01}(t_n).$$

For other values of $t \geqq 1$ choose $\xi^{11}(t)$ in such a manner that it has continuous derivatives of all orders and that

$$\xi^{00}(t) + b' < \xi^{11}(t), \quad \xi^{11}(t) + b' < \xi^{01}(t).$$

Since $\xi^{10}(t) = \xi^{00}(t)$ and $\xi^{12}(t) = \xi^{01}(t)$, the functions $\xi^{10}(t)$, $\xi^{11}(t)$ and the functions $\xi^{11}(t)$, $\xi^{12}(t)$ form regular couples with respect to d. The functions $\xi^{21}(t)$, $\xi^{23}(t)$ are constructed in the strips between $\xi^{10}(t)$, $\xi^{11}(t)$ and $\xi^{11}(t)$, $\xi^{12}(t)$ respectively, in the same manner. The set of points $x = a^{i,2k+1}$, $t = 0$ with the desired properties is then also constructed easily.

The point of the foregoing construction is the following: if a function $X^*(t, \xi)$ can be found, with continuous partials, defined in the region $t \geqq 0$, $\xi^{00}(t) \leqq \xi \leqq \xi^{01}(t)$ and such that $\xi^{i,k}(t)$ are solutions of the differential equation

$$\frac{d\xi}{dt} = X^*(t, \xi), \tag{1,1}$$

and if on extending the solutions $\xi^{i,k}(t)$ over the interval $0 \leqq t \leqq i$ we have $\xi^{i,2k+1}(0) = a^{i,2k+1}$, then any two solutions $\xi_1(t)$, $\xi_2(t)$ with $\xi^{00}(0) \leqq \xi_1(0) < \xi_2(0) \leqq \xi^{01}(0)$ will form a regular couple with respect to d. Thus all solutions will be unstable. Such a function $X^*(t, \xi)$ will be constructed in the following lemma, for the case of two equations.

Lemma 2. *There exists a function $X^{(0)}(t, x, y)$ defined in the half-space $t \geqq 0$, with continuous partials of all orders and such that all solutions of the system*

$$\frac{dx}{dt} = X^{(0)}(t, x, y), \quad \frac{dy}{dt} = 0 \tag{2,1}$$

are bounded and unstable.

Proof. First construct the function $X^*(t, x)$ considered above. In Lemma 1 put $\xi^{00}(t) = 0$ and $\xi^{01}(t) = 1$ for $t \geqq 0$. This is a couple regular with respect to $d = \frac{1}{2}$. According to Lemma 1, there exist a system of functions $\xi^{i,k}(t)$ of the properties described there. In the regions $t \geqq 0$, $x \leqq 0$ and $t \geqq 0$, $x \geqq 1$ and $t \leqq 0$ put $X^*(t, x) = 0$.

The construction of $X^*(t, x)$ in the half-strip $0 \leqq x \leqq 1$, $t \geqq 0$ will proceed by induction – the n-th step will consist in the construction in the rectangle $0 \leqq x \leqq 1$,

$n \leqq t \leqq n + 1$. We will require that the curves $\xi^{i,k}(t)$ be solutions of the differential equation

$$\frac{dx}{dt} = X^*(t, x) \tag{2,2}$$

and that the initial point of solution $\xi^{i,k}(t)$ (prolongable over all $t \geqq 0$) be the point $x = a^{i,k}$.

First step: before constructing $X^*(t, x)$ in the first rectangle, we must define the function $\xi^{11}(t)$ (heterofore defined for $t \geqq 1$) on the interval $\langle 0, 1 \rangle$. As already stated, we require that:

1. $\xi^{11}(0) = a^{11}$; 2. $\xi^{11}(t)$ possess derivatives of all orders; 3. $\xi^{00}(t) < \xi^{11}(t) < \xi^{01}(t)$; 4. $\dfrac{d^k \xi^{11}(0)}{dt^k} = 0$ for all $k \geqq 1$.

On the curve $\xi^{11}(t)$, the function $X^*(t, x)$ is defined in such a manner that this curve is a solution of (2,2). The function $X^*(t, x)$ may then be extended over $0 \leqq t \leqq 1$, $0 \leqq x \leqq 1$ with continuous partials of all orders, using general theorems (see [1]).

Induction. Assume that $X^*(t, x)$ is already defined in the first n rectangles, *i. e.* on the region $0 \leqq t \leqq n$, $0 \leqq x \leqq 1$ (with continuous partials of all orders), and on the curves $x = \xi^{i,k}(t)$, $i \leqq n$. Further assume that the functions $\xi^{i,2k+1}(t)$ with $i \leqq n$ are defined for all $t \geqq 0$, have derivatives of all orders, satisfy $\xi^{i,2k+1}(0) = a^{i,2k+1}$, and $\xi^{i,2k+1}(t)$ is a solution of (2,2) for $0 \leqq t \leqq n$.

First we must extend the domain of definition of $\xi^{n+1,2k+1}(t)$ to all $t \geqq 0$. In the interval $\langle 0, n \rangle$ identify $\xi^{n+1,2k+1}(t)$ with that solution of (2,2) which has initial point $x = a^{n+1,2k+1}$, $t = 0$. In the interval $\langle n, n + 1 \rangle$ we extend the function $\xi^{n+1,2k+1}(t)$ in such a manner that it has derivatives of all orders and satisfies

$$\xi^{n,k}(t) < \xi^{n+1,2k+1}(t) < \xi^{n,k+1}(t) .$$

Such an extension is possible (see [1]). On the curves $\xi^{n+1,2k+1}(t)$ define $X^*(t, x)$ so that $\xi^{n+1,2k+1}(t)$ are solutions of (2,2). Finally $X^*(t, x)$ is extended to the region $n \leqq t \leqq n + 1$, $0 \leqq x \leqq 1$ in such manner that it possesses partials of all orders.

We have thus defined a function $X^*(t, x)$ with partials of all orders, and with zero values in the regions $t \geqq 0$, $x \leqq 0$ and $t \geqq 0$, $x \geqq 1$ and $t \leqq 0$. In the note following Lemma 1 it was remarked that all solutions $\xi(t)$ with $0 \leqq \xi(0) \leqq 1$ are unstable. Finally, let $[x]$ denote the integral part of a real number x. In the half-space $t \geqq 0$ define a function $X^{(0)}(t, x, y)$ by

$$X^{(0)}(t, x, y) = X^*(t, x - [x]) .$$

This proves Lemma 2.

The system (2,1) has the following property:

Definition 6. Let $X(t, x, y)$ be defined in the half-space $t \geqq 0$. We will say that the system of differential equations

$$\text{(2,2)} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = X(t, x, y), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

has *property* (A) in an open set $G \subset E_2$, if for any point $[x_0, y_0]$ and any open set G_1 with $[x_0, y_0] \in G_1 \subset G$ there holds

$$\inf \varrho([t, x(t), y_0], [\tau, x, y]) > 0$$

where $x = x(t)$, $y = y_0$, $x(0) = x_0$ is a solution of (2,2), $[\tau, x, y]$ is any point on $\widetilde{h(G_1)}$[2]) and ϱ is the ordinary Euclidean distance function.

Note. The system (2,1) has property (A) in E_2. The function $X^{(0)}(t, x, y)$ is independent of y, so that the course of integral curves does not depend on the y-coordinate of the initial point. For any pair of different integral curves $x(t)$, $x^*(t)$ with equal y-coordinates of initial points, there cannot be

$$\liminf_{t \to \infty} |x(t) - x^*(t)| = 0$$

since they form a regular couple with respect to d.

Before passing to the formulation and proof of Lemma 8, some auxiliary lemmas are necessary. The following one is concerned with decomposition of an open set into a system of rectangles with a special property.

Lemma 3. *Any open set G in E_2 can be decomposed into a countable system of rectangles in such a manner that their interiors are disjoint, the limit points of the system are precisely the boundary points of G, and the x-coordinates of their vertices are dyadic rationals.*

Construction. For positive integral n let G_n consists of those points of G whose distance from the boundary of G is greater than $1/2^n$. For integers n, m let $K^1_{n,m}$ be the rectangle with vertices $[n, m]$, $[n, m + 1]$, $[n + 1, m]$, $[n + 1, m + 1]$. Let K^1 be the system of those $K^1_{n,m}$ whose closure is in G_1. More generally, let $K^l_{n,m}$ be the rectangle with vertices

$$\left[\frac{n}{2^{l-1}}, \frac{m}{2^{l-1}}\right], \quad \left[\frac{n}{2^{l-1}}, \frac{m+1}{2^{l-1}}\right], \quad \left[\frac{n+1}{2^{l-1}}, \frac{m}{2^{l-1}}\right], \quad \left[\frac{n+1}{2^{l-1}}, \frac{m+1}{2^{l-1}}\right],$$

and let K^l consist of those rectangles $K^l_{n,m}$ whose closures are in G_l, and which are not subsets of any rectangles from $K^1, \ldots, K^{l-1}$. Finally, let the system K be the union of the systems $K^1, K^2, \ldots, K^n, \ldots$

The following notation will be useful.

Definition 7. $O(x_1, x_2, y_1, y_2)$ $(x_1 \leqq x_2, y_1 < y_2)$ denotes the rectangle with vertices $[x_1, y_1]$, $[x_1, y_2]$, $[x_2, y_1]$, $[x_2, y_2]$. The segment joining points $[x_1, y_1]$, $[x_1, y_2]$ (points $[x_2, y_1]$, $[x_2, y_2]$) will be termed the front (end, respectively) x-edge

[2]) $h(G)$ is the boundary of G. The set $\widetilde{h(G)}$ consists of points on solutions of (2,2) with initial points on $h(G)$.

of the rectangle O. Similarly, the segment joining points $[x_1, y_1]$, $[x_2, y_1]$ (points $[x_1, y_2]$, $[x_2, y_2]$) will be termed the front (end) y-edge of O.

Note to Lemma 3. In each rectangle (square) of the system K, consider the points which lie on one third and two thirds of each y-edge. Denote the set of these points by Z. Let us state explicitly some important properties of this set.

1. No point of Z lies on an x-edge of any rectangle of the system K.
2. Every point on the boundary of G is the limit point of pairs of points from Z with equal y-coordinates and such that their distance converges to zero.
3. No point of G is a limit point of Z.

Not to have to repeat this list, we formulate the following definition.

Definition 8. A set Z will be said to have *property* (B) *in G with respect* to the system K, if K is a system of rectangles which satisfies, in G, the conditions of Lemma 3, if the conditions 2. and 3. just listed are satisfied, and if the x-coordinates of points of Z are not dyadic rationals (thence follows condition 1).

For the purposes of lemmas to follow, we will use the following notation. Let there be given two functions $x_1(t)$, $x_2(t)$ defined for all $t \geqq 0$ and such that $x_1(0) = x_2(0)$, and also a pair of real numbers $y_1 < y_2$. The *surface* $x(t, y)$, as a function of t, y, defined in the set $t \geqq 0$ $y_1 \leqq y \leqq y_2$ by $x(t, y) = x_1(t)$ will be *denoted by* $P(x_1(t), y_1, y_2)$. The *surface* $x(t, y)$, defined in the set $t \geqq 0$, $y_1 \leqq y \leqq y_2$ by

$$x(t, y) = x_1(t) + \frac{x_2(t) - x_1(t)}{y_2 - y_1}(y - y_1)$$

will be called *of type R* and *denoted by* $R(x_1(t), x_2(t), y_1, y_2)$. The curves $[t, x_1(t), y_1]$, $[t, x_2(t), y_2]$ for $t \geqq 0$, will be termed boundary curves (front, end). Note that $x(t, y)$ has continuous partials of all orders. Let $R(x_1(t), x_2(t), y_1, y_2)$ be arbitrary. Consider the following two surfaces. Set $x_m(t) = \min(x_1(t), x_2(t))$, $x_M(t) = \max(x_1(t), x_2(t))$, and denote the surfaces $P(x_m(t), y_1, y_2)$, $P(x_M(t), y_1, y_2)$ by $P^1(R)$, $P^2(R)$ respectively. If we move in the direction of the x-axis, then the surface R lies between the surfaces $P^1(R)$, $P^2(R)$ and these latter surfaces act as a type of buffer. Note however, that $P^1(R)$, $P^2(R)$ do not necessarily possess partial derivatives.

If S, N are arbitrary sets in E_3, then the distance between S and N is defined by

$$\varrho(S, N) = \inf \varrho([t_1, x_1, y_1], [t_2, x_2, y_2])$$

with $[t_1, x_1, y_1]$ an arbitrary point of S, and $[t_2, x_2, y_2]$ an arbitrary point of N; ϱ is the usual Euclidean distance-function.

Finally, the *segment* $V(x_0, y_1, y_2)$ is the set of points $[0, x_0, y]$ with $y_1 \leqq y \leqq y_2$; the points $[0, x_0, y_1]$ and $[0, x_0, y_2]$ are the end points of the segment $V(x_0, y_1, y_2)$. The fundamental segment of a surface $P(x_1(t), y_1, y_2)$ or $R(x_1(t), x_2(t), y_1, y_2)$ is the segment $V(x_1(0), y_1, y_2)$ (recall $x_1(0) = x_2(0)$).

We may now formulate the following lemma.

Lemma 4. *Let $P(x_1(t), y_1, y_2)$ be a surface and N a set such that $\varrho(P(x_1(t), y_1, y_2), N) > 0$. Then there exists a $\Delta\alpha > 0$ such that for the surface $P(x_1(t), y_1 - \Delta\alpha, y_2 + \Delta\alpha)$ there again holds $\varrho(P(x_1(t), y_1 - \Delta\alpha, y_2 + \Delta\alpha), N) > 0$.*

The proof is obvious.

Now we shall proceed to the construction of the auxiliary surfaces. For the sake of clarity the construction is divided into several lemmas.

Lemma 5. Assumptions: *Let there be given a system of differential equations*

$$(5,1) \qquad \frac{dx}{dt} = X(t, x, y), \quad \frac{dy}{dt} = 0$$

an open set $G \subset E_2$ and a finite number of segments

$$V_1 = V(x_1, y_1, y_2), \ \ldots, \ V_n = V(x_n, y_1, y_2),$$
$$x^* \leqq x_1 < \ldots < x_n \leqq x^{**}, \quad V_1 \subset G, \ldots, V_n \subset G.$$

Let there be given surfaces

$$P_1^i = P(x_{1i}(t), y_1, y_2), \ \ldots, \ P_n^i = P(x_{ni}(t), y_1, y_2),$$
$$x_{ki}(0) = x_k, \quad x_{k1}(t) \leqq x_{k2}(t), \quad i = 1, 2$$

($x_{ki}(t)$ need not possess derivatives) with the following properties:

$$(5,2) \qquad \varrho(P_k^i, \widetilde{h(G)}) > 0, \quad k = 1, \ldots, n, \ i = 1, 2, \text{[3]}$$

$$(5,3) \qquad \varrho(P_k^i, P_l^j) > 0, \quad k \neq l, \quad 1 \leqq k \leqq n, \quad 1 \leqq l \leqq n, \quad i, j = 1, 2.$$

Let there be given a segment $V_{n+1} = V(\sigma, y_1, y_2)$, $x^* \leqq \sigma \leqq x^{**}$, $V_{n+1} \subset G$ different from all the preceding segments, and such that for the surface $P(\sigma(t), y_1, y_2)$ there holds

$$(5,4) \qquad \varrho(P(\sigma(t), y_1, y_2), \widetilde{h(G)}) > 0$$

where $x = \sigma(t)$, $y = \frac{1}{2}(y_1 + y_2)$ is a solution of (5,1) with initial point $x = \sigma$, $y = \frac{1}{2}(y_1 + y_2)$.

Conclusion. *There exists a surface $P(x(t), y_1, y_2)$ originating in the segment V_{n+1} such that $x(t)$ has derivatives of all orders and*

$$(5,5) \qquad \varrho(P(x(t), y_1, y_2), \widetilde{h(G)}) > 0,$$

$$(5,6) \qquad \varrho(P(x(t), y_1, y_2), P_k^i) > 0$$

for $k = 1, \ldots, n$, $i = 1, 2$.

Proof. If the surface $P(\sigma(t), y_1, y_2)$ satisfies the inequality (5,6), then it suffices

[3]) $h(G)$ is the boundary of G, $\widetilde{h(G)}$ is the set of points of solutions of (2,2) with initial points in $h(G)$.

to put $x(t) = \sigma(t)$. However, $P(\sigma(t), y_1, y_2)$ may have zero distance from some of the surfaces P_k^i. In that case, put

$$\gamma \leqq \min \varrho(P_k^i, P_l^j)\,, \quad k \neq l\,, \quad 1 \leqq k \leqq n\,, \quad 1 \leqq l \leqq n\,, \quad i, j = 1, 2\,,$$
$$\gamma \leqq \min \varrho(P_k^i, \widetilde{h(G)})\,, \quad k = 1, \ldots, n\,, \quad i = 1, 2\,,$$
$$\gamma \leqq \varrho(V_k, V_{n+1})\,, \quad k = 1, \ldots, n\,.$$

Using (5,2), (5,3) and the fact that V_{n+1} does not coincide with any of the V_k ($k = 1, \ldots, n$), we may choose $\gamma > 0$. Now take the least T such that, for some k, i,

$$\varrho\left(\left[T, \sigma(T), \frac{y_1 + y_2}{2}\right], \left[T, x_{k,i}(T), \frac{y_1 + y_2}{2}\right]\right) = \frac{\gamma}{2}\,, \tag{5,7}$$

where the point $[T, x_{k,i}(T), \frac{1}{2}(y_1 + y_2)]$ belongs to the surface $P_k^i = P(x_{k,i}(t), y_1, y_2)$. Using the properties of γ, there is at most one such pair of indices i, k.

Assume $i = 2$ (the case $i = 1$ has a similar proof). In [1] there is perfomed the construction of a function $x^*(t)$ with continuous derivatives of all orders and such that for $t > 0$

$$x_{k,2}(t) + \tfrac{1}{3}\gamma < x^*(t) < x_{k,2}(t) + \tfrac{2}{3}\gamma$$

($x_{k,2}(t)$ need not be differentiable). In the same paper it is shown that there exists a function $x(t)$ and a $\delta > 0$ such that $x(t)$ has continuous derivatives of all orders and

$$x(t) = \sigma(t) \text{ for } 0 \leqq t \leqq T, \quad x(t) = x^*(t) \text{ for } t \geqq T + \delta\,,$$
$$x_{k,2}(t) + \tfrac{1}{4}\gamma < x(t) < x_{k,2}(t) + \tfrac{3}{4}\gamma \text{ for } T < t < T + \delta\,.$$

We will now demonstrate (5,6) for the surface $P(x(t), y_1, y_2)$. First take $l = k$, $i = 2$; thus we shall prove

$$\varrho(P_k^2, P(x(t), y_1, y_2)) > 0\,,$$

where k is that index for which (5,7) holds. According to (5,7),

$$x(t) = \sigma(t) \geqq x_{k,2}(t) + \tfrac{1}{2}\gamma \quad \text{for} \quad 0 \leqq t \leqq T$$

and

$$x(t) \geqq x_{k,2}(t) + \tfrac{1}{4}\gamma \quad \text{for} \quad t \geqq T\,.$$

This implies the inequality to be proved. Since

$$x_{k,1}(t) \leqq x_{k,2}(t) < x_{k,2}(t) + \tfrac{1}{4}\gamma < x(t)\,,$$

the inequality (5,6) also holds for $l = k$, $i = 1$.

For the remaining indices $l \neq k$, $i = 1, 2$, the inequality (5,6) is proved thus: by (5,7), in $0 \leqq t \leqq T$ there holds

$$|x_{k,2}(t) - x(t)| \geqq \tfrac{1}{2}\gamma\,, \quad l \neq k\,, \quad 1 \leqq l \leqq n\,, \quad i = 1, 2,$$

and for $t \geqq T$ there holds $|x_{k,2}(t) - x(t)| < \frac{3}{4}\gamma$. From this and $\varrho(P_k^i, P_l^2) > \gamma$, $l \neq k$ there easily follows (5,6).

Now let us prove (5,5). In the set $0 \leqq t \leqq T$, $y_1 \leqq y \leqq y_2$, the surface $P(x(t), y_1, y_2)$ coincides with $P(\sigma(t), y_1, y_2)$. Using (5,4), the part of $P(x(t), y_1, y_2)$ which lies above $0 \leqq t \leqq T$, $y_1 \leqq y \leqq y_2$ has a positive distance from $\widetilde{h(G)}$. In the set $t \geqq T$ $y_1 \leqq y \leqq y_2$ there holds $x_{k,2}(t) + \frac{1}{4}\gamma < x(t) < x_{k,2}(t) + \frac{3}{4}\gamma$, so that as

$$\varrho(P_k^2, \widetilde{h(G)}) \geqq \gamma ,$$

this part of the surface also has a positive distance from $\widetilde{h(G)}$. Thus both parts of the surface have a positive distance from $\widetilde{h(G)}$, *i. e.*

$$\varrho(P(x(t), y_1, y_2), \widetilde{h(G)}) > 0 .$$

This proves Lemma 5.

In the following Lemma 6 we will consider a more complicated case.

Lemma 6. Assumptions: *Let there be given a differential system* (5,1), *an open set* $G \subset E_2$, *a rectangle* $O(x^*, x^{**}, y_0, y_1)$, *and three groups of segments*:

1. *segments* $V_l = V(x_l, y_0, y_1)$, $l = 1, \ldots, n$, $x^* \leqq x_l \leqq x^{**}$, $y_0 < y_1$, $V_l \subset G$ (*disjoint and contained in the rectangle* $O(x^*, x^{**}, y_0, y_1)$),
2. *segments* $V_q = V(x_q, y_{-q}, y_0)$, $q = n + 1, \ldots, m$, $x^* \leqq x_q \leqq x^{**}$, $y_{-q} \leqq y_0$, $V_q \subset G$ (*disjoint*),
3. *segments* $V_s = V(x_s, y_1, y_s)$, $s = m + 1, \ldots, r$, $x^* \leqq x_s \leqq x^{**}$, $y_1 \leqq y_s$, $V_s \subset G$ (*disjoint*).

In these segments there originate surfaces $R_1, \ldots, R_r$, *which satisfy*

$$\varrho(P^i(R_k), \widetilde{h(G)}) > 0 \quad \textit{for} \quad k = 1, \ldots, r,\ i = 1, 2 . \tag{6,1}$$

The next assumption will be denoted by (6,2):

6,2) *If the segments* V_l, V_q $(l = 1, \ldots, n,\ q = n + 1, \ldots, m)$ *or the segments* V_l, V_s $(l = 1, \ldots, n,\ s = m + 1, \ldots, r)$ *have a common end point, then the surfaces* R_l, R_q *or* R_l, R_s *respectively, have a common boundary curve.*

If the segments V_k, V_h $(k = 1, \ldots, r,\ h = 1, \ldots, r)$ *are disjoint, then*

$$\varrho(P^i(R_k), P^j(R_h)) > 0 , \quad i, j = 1, 2 . \tag{6,3}$$

Let there be given a segment $V_{r+1} = V(\sigma, y_0, y_1)$, $V_{r+1} \subset G$, *different from all the segments* V_l $(l = 1, \ldots, n)$ (*it may have a common end point with the segments* V_q, $q = n + 1, \ldots, m$, V_s, $s = m + 1, \ldots, r$) *such that*

$$\varrho(P(\sigma(t), y_0, y_1), \widetilde{h(G)}) > 0 \tag{6,4}$$

where $x = \sigma(t)$, $y = \frac{1}{2}(y_0 + y_1)$ *is a solution of* (5,1) *with initial point* $x = \sigma$, $y = \frac{1}{2}(y_0 + y_1)$.

Conclusion. *The segment* V_{r+1} *can be decomposed into five segments*

$$V_{r+1}^{(e)}(\sigma, \eta_e, \eta_{e+1}) , \quad y_0 = \eta_1 \leqq \eta_2 \leqq \ldots \leqq \eta_{\iota} = y_1 ,$$

in which there originate surfaces $R^{(e)}_{r+1}$ *with*

(6,5) $$\varrho(P^i(R^{(e)}_{r+1}), \widetilde{h(G)}) > 0\,, \quad e = 1, \dots, 5\,, \ i = 1, 2\,.$$

(6,6) *If the segment* V_{r+1} *has a common front point (or rear point) with some segment* V_k, $k = 1, \dots, r$, *then the surface* $R^{(1)}_{r+1}$ *(or* $R^{(5)}_{n+1}$, *respectively) has a common boundary curve with the surface* R_k *originating in the segment* V_k.

If the segment V_{r+1} *is disjoint with* V_k, $k = 1, \dots, r$, *then*

(6,7) $$\varrho(P^i(R^{(e)}_{r+1}), P^j(R_k)) > 0 \quad for \quad k = 1, \dots, r,\ e = 1, \dots, 5,\ i, j = 1, 2.$$

(6,8) *The surfaces* $R^{(e)}_{r+1}$, $R^{(e+1)}_{r+1}$ *have common boundary curves.*

Let us make some remarks before passing to the proof of Lemma 6. The relation of the rectangle $O(x^*, x^{**}, y_0, y_1)$ to the segments V_k is the following: the segments of the first group lie in the rectangle, the segments of the second and third groups lie outside this rectangle and intersect it in a single point only. The significance of this rectangle, unimportant for the proof of Lemma 6, will appear in Lemma 8. In order to prevent possible misunderstanding, we emphasise that the inclusion $O(x^*, x^{**}, y_0, y_1) \subset G$ need not hold, and that the points x_k $(k = 1, \dots, r)$ need not be ordered by magnitude. We do not even exclude the case that to some x_l $(1 \leqq l \leqq n)$ there is a q or s $(n + 1 \leqq q \leqq m,\ m + 1 \leqq s \leqq r)$ such that $x_l = x_q$ or $x_l = x_s$.

Proof of Lemma 6. Using Lemma 5, we construct a surface $P(x(t), y_0, y_1)$ which has

(6,9) $$\varrho(P(x(t), y_0, y_1), \widetilde{h(G)}) > 0\,,$$

(6,10) $$\varrho(P(x(t), y_0, y_1), P^i(R_l)) > 0\,, \quad l = 1, \dots, n\,.$$

This surface $P(x(t), y_0, y_1)$ may, however, intersect the surfaces R_q or R_s $(q = n + 1, \dots, m,\ s = m + 1, \dots, r)$ in the sense that their boundary curves intersect. In such a case we must change the boundaries of the surface $P(x(t), y_0, y_1)$.

Assume, then, that for some q, $n + 1 \leqq q \leqq m$,

$$\varrho(P(x(t), y_0, y_1), P^i(R_q)) = 0$$

(the case of an s, $m + 1 \leqq s \leqq r$, is similar).

According to Lemma 4, to every surface

$$P^i(R_q) = P(\xi^i_q(t), y_{-q}, y_0)\,, \quad q = n + 1, \dots, m,\ i = 1, 2$$

there exists an $\alpha^i_q > 0$ such that

$$\varrho(P(\xi^i_q(t), y_{-q} - \alpha^i_q, y_0 + \alpha^i_q), \widetilde{h(G)}) > 0\,, \quad q = n + 1, \dots, m,\ i = 1, 2$$

($\xi^1_q(t)$ is the minimum, and $\xi^2_q(t)$ the maximum, of the functions that describe the boundary curves of the surface R_q; see Definition 4).

Take $\alpha > 0$ with $\alpha < \alpha^i_q$, $\alpha < \frac{1}{2}(y_1 - y_0)$. Define surfaces $P^i_{\alpha,k}$, $k = 1, \dots, m$, thus: If the segment $V(x_l, y_0, y_0 + \alpha)$, $1 \leqq l \leqq n$ has no common end point with any of the segments V_q, $q = n + 1, \ldots, m$ then let $P^i_{\alpha,l}$ be that part of $P^i(R_l)$ which originates in the segments $V(x_l, y_0, y_0 + \alpha)$. If however $V(x_l, y_0, y_0 + \alpha)$, $1 \leqq l \leqq n$, has a common end point with some V_q, $n + 1 \leqq q \leqq m$, then let $P^i_{\alpha,l} \equiv P(\eta^i_l(t), y_0, y_0 + \alpha)$, where

$$\eta^1_l(t) = \min\left(\xi^1_l(t), \xi^1_q(t)\right), \quad \eta^2_l(t) = \max\left(\xi^2_l(t), \xi^2_q(t)\right).$$

For $q = n + 1, \ldots, m$, if V_q has no common end point with any of the segments V_l, $1 \leqq l \leqq n$, then set

$$P^i_{\alpha,q} = P(\xi^i_q(t), y_0, y_0 + \alpha).$$

According to (6,1),

$$\varrho(P^i_{\alpha,k}, \widetilde{h(G)}) > 0, \quad 1 \leqq k \leqq m, \ i = 1, 2, \tag{6,11}$$

and according to (6,3),

$$\varrho(P^i_{\alpha,k}, P^j_{\alpha,h}) > 0 \quad \text{for} \quad k \neq h, \ 1 \leqq k \leqq m, \ 1 \leqq h \leqq m, \ i, j = 1, 2. \tag{6,12}$$

There are now two cases.

a) The segment $V_{r+1} = V(\sigma, y_0, y_1)$ does not intersect any V_q, $q = n + 1, \ldots, m$. Then, using Lemma 5, there exists a surface $P(\hat{x}(t), y_0, y_0 + \alpha)$ with

$$\varrho(P(\hat{x}(t), y_0, y_0 + \alpha), \widetilde{h(G)}) > 0, \tag{6,13}$$

$$\varrho(P(\hat{x}(t), y_0, y_0 + \alpha), P^i_{\alpha,k}) > 0, \quad 1 \leqq k \leqq m, \ i = 1, 2. \tag{6,14}$$

The segments V_n of Lemma 5 are those of the segments $V(x_l, y_0, y_0 + \alpha)$, $l = 1, \ldots, n$ and $V(x_q, y_0, y_0 + \alpha)$, $q = n + 1, \ldots, m$ that are not in the first group. According to (6,11), (6,12) and (6,4) respectively, the inequalities (5,2), (5,3) and (5,4) subsist.

b) The segment V_{r+1} intersects some V_q, $n + 1 \leqq q \leqq m$, in which there originates the surface R_q. If the boundary curves of R_q are $\lambda^1_q(t)$, $\lambda^2_q(t)$, we may write

$$R_q \equiv R(\lambda^1_q(t), \lambda^2_q(t), y_{-q}, y_0).$$

Set $\hat{x}(t) = \lambda^2_q(t)$. Then the surface $P(\hat{x}(t), y_0, y_0 + \alpha)$ will satisfy (6,13) and (6,14). Indeed, if there were

$$\varrho(P(\lambda^2_q(t), y_0, y_0 + \alpha), \widetilde{h(G)}) = 0,$$

then we would also have

$$\varrho(P(\xi^i_q(t), y_0, y_0 + \alpha), \widetilde{h(G)}) = \varrho(P^i_{\alpha,q}, \widetilde{h(G)}) = 0$$

for some i, where

$$\xi^1_q(t) = \min\left(\lambda^1_q(t), \lambda^2_q(t)\right), \quad \xi^2_q(t) = \max\left(\lambda^1_q(t), \lambda^2_q(t)\right);$$

but, by construction of α, this is impossible. Similarly for (6,14): if there were

$$\varrho(P(\lambda^2_q(t), y_0, y_0 + \alpha), P^i_{\alpha,k}) = 0$$

for some k, $1 \leqq k \leqq m$, $k \neq q$, and $i = 1, 2$, then we would also have

$$\varrho(P^j_{\alpha,q}, P^i_{\alpha,k}) = 0$$

for some j, in contradiction with (6,12).

Now set $\eta_1 = y_0$, $\eta_2 = y_0 + \frac{1}{3}\alpha$, $\eta_3 = y_0 + \frac{2}{3}\alpha$, $\eta_4 = y_1$, a construct the surfaces

$$R^{(1)}_{r+1} = P(\hat{x}(t), \eta_1, \eta_2)\,, \quad R^{(2)}_{r+1} = R(\hat{x}(t), x(t), \eta_2, \eta_3)\,, \quad R^{(3)}_{r+1} = P(x(t), \eta_3, \eta_4)\,.$$

By construction of $R^{(e)}_{r+1}$ (see b)), (6,6) is evidently satisfied, and the surfaces $R^{(e)}_{r+1}$, $R^{(e+1)}_{r+1}$ have a common boundary curve. Using (6,9) and (6,13), we see that (6,5) is satisfied. If

$$\varrho(P(x(t), y_0, y_1), P^i(R_s)) > 0 \quad \text{for} \quad s = m + 1, \ldots, r,\ i = 1, 2\,,$$

then (6,7) for $e = 3$ is a consequence of (6,10). The inequality (6,7) for $e = 2$ is a consequence of (6,10) and the fact that the domains of definition of the surfaces $P^i(R^{(2)}_{r+1})$ is $t \geqq 0$, $\eta_2 \leqq y \leqq \eta_3$ and those of the surfaces $P^i(R_q)$ or $P^i(R_s)$ are $t \geqq 0$, $y_{-q} \leqq \leqq y \leqq y_0 < \eta_2$ and $t \geqq 0$, $\eta_3 < y_1 \leqq y \leqq y_s$ respectively. The inequality (6,7) for $e = 1$ is a consequence of (6,14). If there were $\varrho(P(x(t), y_0, y_1), P^i(R_s)) = 0$ for some s, $s = m + 1, \ldots, r$, $i = 1, 2$, then it would be necessary to change the rear boundary of $P(x(t), y_0, y_1)$ in a similar manner.

In the following lemma we shall weaken the assumption (6,4).

Lemma 7. *Make all the assumptions of Lemma 6 except that (6,4) is replaced by*

$$V(\sigma, y_0, y_1) \subset G\,. \tag{7,1}$$

Assume that the system (5,1) has property (A) in the open set G. Then the conclusions of Lemma 6 hold, with the following changes: The segment $V(\sigma, y_0, y_1)$ can be decomposed into a finite system of subsegments $V^{(1)}_{r+1}, \ldots, V^{(P)}_{r+1}$, in which there originate surfaces $R^{(1)}_{r+1}, \ldots, R^{(P)}_{r+1}$ with the properties described in Lemma 6. (There their number was reduced to five.)

Proof. Using property (A), the segment $V(\sigma, y_0, y_1)$ may be decomposed into a finite set of segments

$$V(\sigma, \eta_1, \eta_2), \ldots, V(\sigma, \eta_\alpha, \eta_{\alpha+1})\,, \quad y_0 = \eta_1 < \eta_2 < \ldots < \eta_\alpha < \eta_{\alpha+1} = y_1$$

in such a manner that the surfaces $P(\sigma_k(t), \eta_k, \eta_{k+1})$ (where $x = \sigma_k(t)$, $y = \frac{1}{2}(\eta_k + \eta_{k+1})$ is a solution of (5,1) with initial point $x = \sigma$, $y = \frac{1}{2}(\eta_k + \eta_{k+1})$) satisfy

$$\varrho(P(\sigma_k(t), \eta_k, \eta_{k+1}), \widetilde{h(G)}) > 0\,. \tag{7,2}$$

Consider, in turn, the rectangles $O(x^*, x^{**}, \eta_1, \eta_2)$, $O(x^*, x^{**}, \eta_2, \eta_3), \ldots$, $O(x^*, x^{**}, \eta_\alpha, \eta_{\alpha+1})$. By (7,2), in each of them (6,4) holds, and the construction of Lemma 6 may be performed.

Before formulating Lemma 8, another definition will be necessary.

Definition 9. A surface (a function $x(t, y)$ in variables t, y) defined in the half-strip $t \geqq 0$, $y_1 \leqq y \leqq y_2$, will be termed of type R^+ if there exist a real $t_0 \geqq 0$ and a function $\alpha(t) > 0$ with $\lim_{t \to \infty} \alpha(t) = 0$, with derivatives of all orders, and such that for $t \geqq t_0$ there holds $\alpha(t) < \frac{1}{2}(y_2 - y_1)$ and

$$x(t, y) = \lambda(t, y) \quad \text{for} \quad y_1 + \alpha(t) \leqq y \leqq y_2 - \alpha(t)\,,$$
$$\min(x_1(t), \lambda(t, y)) \leqq x(t, y) \leqq \max(x_1(t), \lambda(t, y)) \quad \text{for} \quad y_1 \leqq y \leqq y_1 + \alpha(t)\,,$$
$$\min(x_2(t), \lambda(t, y)) \leqq x(t, y) \leqq \max(x_2(t), \lambda(t, y)) \quad \text{for} \quad y_2 - \alpha(t) \leqq y \leqq y_2\,,$$

where

$$\lambda(t, y) = x_1(t) + \frac{x_2(t) - x_1(t)}{y_2 - y_1}(y - y_1), \quad x_1(t) = x(t, y_1), \quad x_2(t) = x(t, y_2),$$

with $x(t, y)$ possessing partials of all orders in $t \geqq 0$, $y_1 \leqq y \leqq y_2$, and such that $\dfrac{\partial^k x(t, y)}{\partial y^k} = 0$ for $k \geqq 1$, $y = y_i$, $i = 1, 2$.

If several such surfaces are considered simultaneously, we will denote them by $R_1^+, \ldots, R_n^+, \ldots$

Note to Definition 9. We will perform the construction of a surface of type R^+, given a fundamental segment $V(x_0, y_1, y_2)$, the boundary curves $x_1(t)$, $x_2(t)$ (with $x_1(0) = x_2(0) = x_0$, defined for all $t \geqq 0$), the function $\alpha(t)$ and the $t_0 \geqq 0$.

Set

$$x(t, y) = x_1(t) + \frac{x_2(t) - x_1(t)}{y_2 - y_1}\Bigg[C_1(y_1, y_1 + \alpha(t)) \int_{y_1}^{y}\int_{y_1}^{\xi} f(\eta, y_1, y_1 + \alpha(t))\, d\eta\, d\xi +$$
$$C_2(y_1, y_1 + \alpha(t)) \int_{y_1}^{y}\int_{y_1}^{\xi} \eta f(\eta, y_1, y_1 + \alpha(t))\, d\eta\, d\xi\Bigg]$$

for $y_1 \leqq y \leqq \frac{1}{2}(y_1 + y_2)$, $t \geqq t_0$, and $x(t, y) = x_1(t) + x_2(t) - x(t, y_1 + y_2 - y)$ for $\frac{1}{2}(y_1 + y_2) < y \leqq y_2$, $t \geqq t_0$, where the function f is defined as follows (denoting the second and third variables by u, v). It suffices to define $f(\eta, u, v)$ for $u < v$, by

$$f(\eta, u, v) = e^{[1/(u-\eta)]+[1/(\eta-v)]} \quad \text{for} \quad u < \eta < v,$$
$$f(\eta, u, v) = 0 \qquad \text{for} \quad \eta \leqq u \quad \text{or} \quad \eta \geqq v.$$

It is easily shown that

$$x\left(t, \frac{1}{2}(y_1 + y_2)\right) = \frac{x_1(t) + x_2(t)}{2} \quad \text{and} \quad \frac{\partial x(t, y)}{\partial y} = \frac{x_2(t) - x_1(t)}{y_2 - y_1}$$

for $y_1 + \alpha(t) \leqq y \leqq y_2 - \alpha(t)$. The $C_1(u, v)$ and $C_2(u, v)$ are determined by the system of equations

$$C_1(u, v)\int_u^v f(\eta, u, v)\, d\eta + C_2(u, v)\int_u^v \eta f(\eta, u, v)\, d\eta = 1,$$

$$C_1(u, v)\int_u^v\int_u^{\xi} f(\eta, u, v)\, d\eta\, d\xi + C_2(u, v)\int_u^v\int_u^{\xi} \eta f(\eta, u, v)\, d\eta\, d\xi = v - u.$$

Since for $u < v$ the determinant of the system,

$$\begin{vmatrix} \displaystyle\int_u^v f(\eta, u, v)\, d\eta, & \displaystyle\int_u^v \eta f(\eta, u, v)\, d\eta \\ \displaystyle\int_u^v\int_u^{\xi} f(\eta, u, v)\, d\eta\, d\xi, & \displaystyle\int_u^v\int_u^{\xi} \eta f(\eta, u, v)\, d\eta\, d\xi \end{vmatrix} =$$
$$= -\int_u^v\int_u^{\xi} f(\xi, u, v) f(\eta, u, v)(\xi - \eta)^2\, d\eta\, d\xi$$

is non-zero, the coefficients $C_1(u, v)$, $C_2(u, v)$ are determined uniquely and have continuous partials of all orders for $u < v$. Since $x_1(t)$, $x_2(t)$, $\alpha(t)$ also have continuous derivatives of all orders, it follows that the function $x(t, y)$ has continuous partials of all orders. From the above relations it easily follows that

$$x(t, y) = x_1(t) + \frac{x_2(t) - x_1(t)}{y_2 - y_1}(y - y_1)$$

for $t \geqq t_0$, $y_1 + \alpha(t) \leqq y \leqq y_2 - \alpha(t)$. Since the function

$$\omega(t, y) = C_1(u, v)\int_u^y\int_u^\xi f(\eta, u, v)\,\mathrm{d}\eta\,\mathrm{d}\xi + C_2(u, v)\int_u^y\int_u^\xi \eta f(\eta, u, v)\,\mathrm{d}\eta\,\mathrm{d}\xi$$

has at most one zero point of the second partial derivative $\dfrac{\partial^2\omega(t, y)}{\partial y^2}$ in the open interval (u, v), we have

$$0 \leqq \omega(t, y) \leqq y - u\,;$$

thence we conclude the inequalities of Definition 9. In addition, from the expression for $x(t, y)$ it easily follows that $\dfrac{\partial^k x(t, y)}{\partial y^k} = 0$ for $k \geqq 1$, $y = y_i$, $i = 1, 2$, $t \geqq 0$.

Lemma 8. Assumptions: *There is given a system of differential equations*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(t, x, y)\,, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 0\,, \tag{8,1}$$

and an open set $G_1 \subset E_2$. The system (8,1) has property (A) in G_1. There are given a system of rectangles K_1 and a set Z_1 such that Z_1 has property (B) in G_1 with respect to K_1. The solutions of (8,1) with initial points on the x-edges of rectangles of K_1 generate surfaces of type R^+ (single surfaces of this system will be denoted by R_n^+).

Conclusion. *To every open set $G_2 \subset G_1$ there can be constructed a system of rectangles K_2 and a set Z_2 such that Z_2 has property (B) in G_2 with respect to K_2, and that there exist surfaces L_n originating in the x-edges of rectangles of K_2 with the following properties:*

If the x-edge of a rectangle of K_2 is not a subset of an x-edge of some rectangle of K_1, then the surface L_n originating in this edge is of type R (see Definition 8, p. 277).

If it is such a subset, then the surface L_n is part of one of the surfaces of type R^+ mentioned above.

(8,2) *If V_m, V_n (fundamental segments of the surfaces L_m, L_n) have a common end point, then the surfaces L_m, L_n have in common the boundary curve with this point as initial.*

If V_m, V_n are disjoint, then

$$\varrho(L_m, L_n) > 0\,. \tag{8,3}$$

Finally, there hold

$$(8,4) \qquad \varrho(L_n, \widetilde{h(G_2)}) > 0\,,$$

$$(8,5) \qquad \varrho(L_n, \hat{Z}) > 0$$

where $Z = Z_1 + Z_2$ *and* $\hat{Z}$ *is the system of solutions of* (8,1) *with initial points in* Z.

Proof. First note that the surfaces R_n^+ (generated by integral curves of the system (8,1) with initial points on x-edges of rectangles of K_1) have the properties (8,2), (8,3), (8,5) with Z replaced by Z_1; this is a consequence of property (A) and of the continuous dependence of solutions on the initial point.

Denote by $K^{(1)}$ a system of rectangles which is finer than the system K_1 and simultaneously finer than the system obtained by decomposing G_2 according to Lemma 3 (the rectangles of $K^{(1)}$ are the intersections of rectangles from K_1 with the rectangles obtained by application of Lemma 3).

For the system $K^{(1)}$, construct the set Z_2 according to the note to Lemma 3. Now number by positive integers the front x-edges of those rectangles from $K^{(1)}$, in which the surfaces R^+ do not originate (*i. e.* the edges which are not subsets of x-edges of rectangles from K_1).

Construction of the first auxiliary surface. Let the first front x-edge $V(x_1, y_1^{(1)}, y_2^{(1)})$ belong to the rectangle $O_n(\Theta_1^{(n)}, \Theta_2^{(n)}, \eta_1^{(n)}, \eta_2^{(n)}) \in K_1$. Use Lemma 7, with

$$(G_2 - Z) \,.\, \underset{[x,y]}{\exists}\, (\Theta_1^{(n)} < x < \Theta_2^{(n)})$$

instead of G, and $O(\Theta_1^{(n)}, \Theta_2^{(n)}, y_1^{(1)}, y_2^{(1)})$ instead of the rectangle $O(x^*, x^{**}, y_0, y_1)$.

If $y_1^{(1)} = \eta_1^{(n)}$ (or $y_2^{(1)} = \eta_2^{(n)}$), then for the surfaces originating in segments V_q, $q = n + 1, \ldots, m$ (or V_s, $s = m + 1, \ldots, r$) in Lemma 7, take the curves

$$x = \xi_q(t)\,, \quad y = \eta_1^{(n)} \quad (\text{or } y = \eta_2^{(n)})$$

which are the boundary curves of surfaces R_n^+, and whose fundamental points are on an y-edge of the rectangle

$$O(\Theta_1^{(n)}, \Theta_2^{(n)}, y_1^{(1)}, y_2^{(1)})\,.$$

According to Lemma 7, the segment $V(x_1, y_1^{(1)}, y_2^{(1)})$ decomposes into a finite set of segments

$$V(x_1, \zeta_k, \zeta_{k+1})\,, \quad y_1^1 = \zeta_1 < \zeta_2 < \ldots < \zeta_{m_1} = y_2^{(1)}\,;$$

in the individual segments $V(x_1, \zeta_k, \zeta_{k+1})$ there originate surfaces $R_1^{(k)}$ with

$$\varrho(P^i(R_1^{(k)}), \widetilde{h(G)}) > 0\,, \quad k = 1, \ldots, m_1\,, \; i = 1, 2\,.$$

Either the curve

$$x = \xi_k(t)\,, \quad y = y_1^{(n)} \quad (\text{or } y = y_2^{(n)})$$

is a boundary curve of the surface $R_1^{(1)}$ (or $R_1^{(m_1)}$), or

$$\varrho(P^i(R_1^{(k)}), P(\xi_k(t), y_i^{(1)}, y_i^{(1)})) > 0$$

where $P(\xi_k(t), y_i^{(1)}, y_i^{(1)})$ is really the curve $x = \xi_k(t)$, $y = y_i^{(1)}$. By definition of G, of surfaces of type R^+, and from the last two inequalities, we easily conclude that

$$R_1^{(k)}, R_1^{(k+1)} \text{ have a common boundary curve },$$

$$\varrho(P^i(R_1^{(k)}), \widetilde{h(G_2)}) > 0 , \quad \varrho(P^i(R_1^{(k)}), \hat{Z}) > 0 , \quad k = 1, \ldots, m_1, \ i = 1, 2 ,$$

either $R_1^{(k)}$, R_h^+ have a common boundary curve or

$$\varrho(P^i(R_1^{(k)}), R_h^+) > 0 , \quad k = 1, \ldots, m_1, \ i = 1, 2 .$$

From the system $K^{(1)}$ we form a new system $K^{(2)}$ by subdividing all rectangles of $K^{(1)}$ by straight lines $y = \zeta_k$, $k = 1, \ldots, m_1$.

Induction. Now assume: The set G_2 is decomposed by a system of rectangles $K^{(n)}$, where $K^{(n)}$ is finer than all the systems $K^{(n-1)}, \ldots, K^{(1)}$ and every rectangle of $K^{(1)}$ has been subdivided into a finite number of rectangles. The system $K^{(n)}$ decomposes each of the first n x-edges V_k, $k = 1, \ldots, n$, into subsegments $V_k^{(s,n)}$, $s = 1, \ldots, p_k^{(n)}$, in which there originate surfaces $R_k^{(s)}$, $k = 1, \ldots, n$, $s = 1, \ldots, p_k^{(n)}$ with the following properties ($V_k^{(s,n)}$ is the fundamental segment of the surface $R_k^{(s)}$; according to Definition 4 of surfaces R and to the construction presently being performed, an index n at $R_k^{(s)}$ would be superfluous):

(8,6) If $V_k^{(s,n)}$, $V_l^{(d,n)}$ have a common point, then the surfaces $R_k^{(s)}$, $R_l^{(d)}$ have in common the boundary curve with this point as initial;

(8,7) if $V_k^{(s,n)}$, $V_l^{(d,n)}$ are disjoint, then

$$\varrho(P^i(R_k^{(s)}), P^j(R_l^{(d)})) > 0 ;$$

for every surface $R_k^{(s)}$, $k = 1, \ldots, n$, $s = 1, \ldots, p_k^{(n)}$ there hold

$$\varrho(P^i(R_k^{(s)}), \widetilde{h(G_2)}) > 0 , \tag{8,8}$$

$$\varrho(P^i(R_k^{(s)}), \hat{Z}) > 0 ; \tag{8,9}$$

(8,10) if $V_k^{(s,n)}$, V_h^+ have a common point, then the surfaces $R_k^{(s)}$, R_h^+ have in common the boundary curve with this point as initial;

(8,11) if $V_k^{(s,n)}$, V_h^+ are disjoint, then

$$\varrho(P^i(R_k^{(s)}), R_h^+) > 0 .$$

For the inductive step we further assume:

(8,12) The system $K^{(n)}$ decomposes the $(n + 1)$-st front x-edge of a rectangle from $K^{(1)}$ into σ subsegments such that:

If we denote

$$V_{n+1}^{(l)} \equiv V(\gamma_{n+1}, \mu_{n+1}^{(l)}, \mu_{n+1}^{(l+1)}) , \quad l = 1, \ldots, \sigma + 1 ,$$

then for every segment

$$V_k^{(s,n)} = V(\gamma_k, \mu_k^{(s,n)}, \mu_k^{(s+1,n)}) , \quad (k = 1, \ldots, n, \ s = 1, \ldots, p_k^{(n)}$$

and strip

$$H = \underset{[x,y]}{\exists} \left(\mu_{n+1}^{(l)} \leqq y \leqq \mu_{n+1}^{(l+1)}\right)$$

there holds the following relation:

α) either all the points $[\gamma_k, y]$, $\mu_{n+1}^{(l)} \leqq y \leqq \mu_{n+1}^{(l+1)}$, lie on the segment $V_k^{(s,n)}$,

β) or the segment $V_k^{(s,n)}$ has precisely one point in common with H,

γ) or finally $V_k^{(s,n)}$ and H are disjoint.

The auxiliary surface originating in the segments $V_{n+1}^{(l)}$ will be constructed in turn. First, using Lemma 7, construct the surfaces originating in the segment $V_{n+1}^{(1)}$.

If $V_{n+1}^{(1)}$ is in the rectangle

$$O(\Theta_1^{(m)}, \Theta_2^{(m)}, \eta_1^{(m)}, \eta_2^{(m)})$$

then in Lemma 7 we take the set

$$(G_2 - Z) \,.\, \exists(\Theta_1^{(m)} < x < \Theta_2^{(m)})$$

for the set G, and the rectangle

$$O(\Theta_1^{(m)}, \Theta_2^{(m)}, \mu_{n+1}^{(1)}, \mu_{n+1}^{(2)})$$

for the rectangle $O(x^*, x^{**}, y_0, y_1)$ considered there. The surfaces $R_k^{(s)}$, $k = 1, \dots, n$, $s = 1, \dots, p_k^{(n)}$, and R_k^+ already constructed are grouped according to the property α), β) or γ) in (8,12) that they satisfy. Those $R_k^{(s)}$ that satisfy α) are placed in the first group, *i. e.* the group of surfaces whose fundamental segments are of the form $V(x_k, y_0, y_1)$, $k = 1, \dots, n$. The surfaces that satisfy β) are placed in the second and third groups; those that satisfy γ) are not considered at all. If

$$\mu_{n+1}^{(1)} = \eta_1^{(m)} \quad (\text{or } \mu_{n+1}^{(2)} = \eta_2^{(m)})$$

then the boundary curves of surfaces R_h^+,

$$x = \xi_k(t)\,, \quad y = \eta_1^{(m)} \quad (\text{or } y = \eta_2^{(m)})$$

whose fundamental points are on an y-edge of the rectangle $O(\Theta_1^{(m)}, \Theta_2^{(m)}, \mu_{n+1}^{(1)}, \mu_{n+1}^{(2)})$, are also placed in the second and third groups of Lemma 7 (as in the construction of the first surface). According to Lemma 7, the segment $V(\gamma_{n+1}, \mu_{n+1}^{(1)}, \mu_{n+1}^{(2)})$ decomposes into a finite number of subsegments

$$V(\gamma_{n+1}, \zeta_k, \zeta_{k+1})\,, \quad \mu_{n+1}^{(1)} = \zeta_1 < \zeta_2 < \dots < \zeta_{u+1} = \mu_{n+1}^{(2)}\,;$$

in the individual segments there originate surfaces $R_{n+1}^{(s)}$ such that:

$$\varrho(P^i(R_{n+1}^{(s)}), \widetilde{h(G)}) > 0\,, \quad s = 1, \dots, u,\ i = 1, 2\,;$$

(8,13) If $V_{n+1}^{(1)}$ (or $V_{n+1}^{(u)}$) and $V_k^{(l,n)}$ have a common point, then the surfaces $R_{n+1}^{(1)}$ (or $R_{n+1}^{(u)}$) and $R_k^{(l)}$ have a common boundary curve;

(8,14) If $V_{n+1}^{(s)}$, $V_k^{(l,n)}$ are disjoint, then

$$\varrho(P^i(R_{n+1}^{(s)}), P^j(R_k^{(l)})) > 0\,, \quad s = 1, \dots, u,\ k = 1, \dots, n,\ l = 1, \dots, p_k^{(n)},\ i, j = 1, 2\,;$$

(8,15) Each couple of surfaces $R_{n+1}^{(s)}$, $R_{n+1}^{(s+1)}$ have a common boundary curve.

As in the construction of the first auxiliary surface, we conclude that

(8,16) $$\varrho(P^i(R^{(s)}_{n+1}), \widetilde{h(G_2)}) > 0\,, \quad s = 1, \ldots, u,\ i = 1, 2\,;$$

(8,17) $$\varrho(P^i(R^{(s)}_{n+1}), \hat{Z}) > 0\,.$$

(8,18) If $V^{(s)}_{n+1}$, V^+_h have a common point, then the surfaces $R^{(s)}_{n+1}$, R^+_h have a common boundary curve;

(8,19) If $V^{(s)}_{n+1}$, V^+_h are disjoint, then

$$\varrho(P^i(R^{(s)}_{n+1}), R^+_h) > 0\,.$$

The segments $V^{(2)}_{n+1}, \ldots, V^{(\sigma)}_{n+1}$ are dealt with in precisely the same manner. Thus we obtain a decomposition of V_{n+1} into a finite number ($p^{(n+1)}_{n+1}$) of subsegments, in which there originate surfaces $R^{(s)}_{n+1}$, $s = 1, \ldots, p^{(n+1)}_{n+1}$ with properties (8,13) to (8,19)), possibly after renumbering). Taking into account (8,13) to (8,19), we see that (8,16) and (8,11) hold for $n + 1$ also.

It remains to define $K^{(n+1)}$ and to show that (8,12) holds for $n + 1$. The segment V_{n+1} has been subdivided into $p^{(n+1)}_{n+1}$ subsegments

$$V(\gamma_{n+1}, \mu^{(l)}_{n+1}, \mu^{(l+1)}_{n+1})\,, \quad l = 1, \ldots, p^{(n+1)}_{n+1}\,.$$

The straight lines

(8,20) $$y = \mu^{(l)}_{n+1}\,, \quad l = 1, \ldots, p^{(n+1)}_{n+1} + 1$$

decompose the rectangles of $K^{(n)}$. Those rectangles from $K^{(n)}$, in whose both x-edges there originate surfaces already constructed, are left unchanged; the remaining rectangles are subdivided by the straight lines mentioned; the resulting system is denoted by $K^{(n+1)}$. By construction, property (8,12) holds for $K^{(n+1)}$ also. Let us show that the sequence of systems $K^{(n)}$ subdivides every rectangle of $K^{(1)}$ into a finite number of parts, *i. e.* that to every rectangle of $K^{(1)}$ there is an index n_0 such that $K^{(n)}$ with $n > n_0$ do not subdivide this rectangle.

Let the integer associated with the front x-edge of this rectangle be k_1; the rear x-edge is decomposed into a finite number of front x-edges of rectangles of $K^{(1)}$; to these then corresponds a finite set of positive integers $k_2, \ldots, k_r$. According to (8,20) we may then put $n_0 = \max(k_1, \ldots, k_r)$. The sequence $K^{(n)}$ defines a limit system of rectangles, which we denote by K_2. By construction, in those x-edges of rectangles of this system that are not subsets of x-edges of rectangles of K_1 there originate surfaces $R^{(s)}_k$ satisfying (8,2) to (8,5).

If an x-edge of a rectangle of K_2 is a subset of an x-edge of some rectangle of K_1, then the solutions of (8,1) with initial points on this x-edge form, by assumption, a surface of type R^+; this is the surface which will be denoted by L_n. It remains to prove that these latter surfaces together with the surfaces $R^{(s)}_k$ already constructed, satisfy (8,2) to (8,5).

As has been mentioned at the beginning of this proof, and by (8,10), (8,11), evidently (8,2) and (8,3) hold. Since the fundamental segment V^+ of R^+ is in G_2 and by property (A), the relation (8,4) holds. It was mentioned that (8,5) holds if $\hat{Z}$ is replaced by $\hat{Z}_1$; all that there remains to prove is that

$$\varrho(R^+, \hat{Z}_2) > 0 .$$

Since the fundamental segment V^+ of the surface R^+ is a subset of G_2, no point of V^+ is a limit point of Z_2; using property (A), we obtain this last inequality.

Note also the following property: if the x-coordinates of vertices of rectangles from K_1 are dyadic rationals, then so are the x-coordinates of vertices of rectangles from K_2; this is so since, except for the application of Lemma 3, all the subdividing lines were parallel to the x-axis.

Note to Lemma 8. All the surfaces constructed in Lemma 8 were such that if they are described as functions $x(t, y)$ of variables t, y then they have continuous derivatives of all orders. However, if two of these surfaces have a common boundary curve, they need not have a common tangent plane along this curve. In the sequel it will be required that these surfaces consist of solutions of the system (8,1) with $X(t, x, y)$ having continuous derivatives of all orders. Thus it will be necessary to change these surfaces in the neighbourhood of common boundary curves so as to obtain surfaces sufficiently smooth. Such surfaces are of type R^+. Using the note to Definition 9 we may successively change our surfaces in such a manner that we obtain surfaces of type R^+, leave the boundary curves unchanged, and preserve properties (8,2) to (8,5).

In the next lemma, a system of surfaces will be constructed, such that the distance between them does not increase overmuch. The point is that if we also construct a differential system whose solutions do not intersect the surfaces of this system, then the distance between solutions will not increase overmuch also. However, a different notion of distance than the one used heretofore will be introduced.

Definition 10. Let $S_1(x = x_1(t, y))$, $S_2(x = x_2(t, y))$ be two surfaces defined over the set $t \geqq 0$, $y_1 \leqq y \leqq y_2$; then their outer distance on the half-line $t \geqq t_0$ is defined as

$$\varrho^{(2)}_{\langle t_0, +\infty)}(S_1, S_2) = \sup |x_1(t, y) - x_2(t, z)| \quad \text{for} \quad t \geqq t_0,\ y \in \langle y_1, y_2 \rangle\,,\ z \in \langle y_1, y_2 \rangle\,.$$

Lemma 9. Assumptions: *There is given a differential system* (8,1) *and an open set* G. *The solutions of* (8,1) *are bounded and the system has property* (A) *in* G. *There is given a set* Z *and a system of rectangles* K *such that*: Z *has property* (B) *in* G *with respect to* K, *the* x*-coordinates of vertices of rectangles from* K *are dyadic rationals, and in the* x*-edges of these rectangles there originate surfaces of type* R^+ *which consist of integral curves of the system* (8,1). *The fundamental segments* V *of these surfaces of type* R^+ *and the* x*-edges of rectangles from* K *have the following relation*:

Every x*-edge of any rectangle from* K *either consists of a finite number of fundamental segments of these surfaces, or it is a subset of some fundamental segment.*

These surfaces of type R^+ have the following properties:

(9,1) *If V_m, V_n (fundamental segments of surfaces R_m^+, R_n^+) have a common end-point, then R_m^+, R_n^+ have in common the boundary curve with this point as initial.*

If V_m, V_n are disjoint, then

$$\varrho(R_m^+, R_n^+) > 0\,, \tag{9,2}$$

$$\varrho(R_n^+, \widetilde{h(G)}) > 0\,, \tag{9,3}$$

$$\varrho(R_n^+, \hat{Z}) > 0\,. \tag{9,4}$$

Conclusion. *To every positive real η there exists a system of rectangles K', finer than K, and such that: every rectangle of K consists of a finite number of rectangles from K'; the x-coordinates of vertices of rectangles of K' are again dyadic rationals; the lengths of y-edges of rectangles of K' are less than η; in the x-edges of rectangles from K' there originate surfaces of type R^+ which, together with the given surfaces, again satisfy* (9,1) *to* (9,4); *for every rectangle, if R_1^+ (R_2^+) is the surface originating in the front (rear) x-edge, then*

$$\varrho^{(2)}_{\langle t_0,\infty)}(R_1^+, R_2^+) < \eta \tag{9,5}$$

(t_0 may vary for different rectangles).

Proof. Use positive integers to number the rectangles from K. In each in turn perform the following construction. Denote the n-th rectangle by $O(x_1, x_2, y_1, y_2)$. In its x-edges there originate a finite number of surfaces of type R^+; in its y-edges there originate either boundary curves of the remaining or newly constructed (in preceding steps) surfaces of type R^+, or a finite number of curves from $\hat{Z}$.[4] Denote these curves by $x = \xi_n(t)$, $y = y_i$, $i = 1, 2$. Then

$$\varrho(P(\xi_n(t), y_i, y_i), R_j^+) > 0\,, \quad i, j = 1, 2\,; \tag{9,6}$$

$$\varrho(P(\xi_n(t), y_i, y_i), P(\xi_m(t), y_i, y_i)) > 0\,, \quad n \neq m,\ i = 1, 2\,. \tag{9,7}$$

If both

$$\varrho^{(2)}_{\langle t_0,\infty)}(R_1^+, R_2^+) > \eta$$

and the y-edges of O are less than η, then O is left unchanged. Assume then that at least one of these assumptions is not true. Decompose $O(x_1, x_2, y_1, y_2)$ by straight lines parallel to the x-axis, and passing through the end points of the fundamental segments V_l constituting the x-edges of the rectangle O. In these smaller rectangles we perform, in turn, the following construction:

Let $O^{(1)}(x_1, x_2, y_1, y)$ be the first of these rectangles. Choose $t_0 \geqq 0$ so that for $t \geqq t_0$ the function $\alpha(t)$ is sufficiently small (see Definition 9 of surfaces originating in x-edges of $O^{(1)}$). For each curve $x = \xi_n(t)$, $y = y_i$, $i = 1, 2$ consider the segment

[4]) $\hat{Z}$ is the system of solutions of the system (8,1) with initial points in Z.

$V(\xi_n(0), y_i, \gamma)$. If there exists a curve $x = \xi_k(t)$, $y = y_j$, $j = -i + 3$ originating in the other end point of this segment (this is possible only if $\gamma = y_2$, *i. e.* if the rectangle O has not been decomposed at all, $O \equiv O^{(1)}$), then we construct a surface of type R^+ with boundary curves $x = \xi_n(t)$, $y = y_i$ and $x = \xi_k(t)$, $y = y_j$ in such a manner that it originates in the segment $V(\xi_n(0), y_1, y_2)$. On the other hand, if no curve $x = \xi_l(t)$, $y = y_j$ originates in the other end point of $V(\xi_n(0), y_1, \gamma)$, then we may construct such a curve that satisfies (9,6) and (9,7). The surfaces thus constructed obviously satisfy (9,1) to (9,3). Among the surfaces which originate in the segments $V(\xi_n(0), y_1, y_2)$ just constructed, there do not occur any of the curves $x = \xi_n(t)$, $y = y_i$ (*i. e.* boundary curves of the given or constructed surfaces R^+ or curves of $\hat{Z}$). Now, the distance between surfaces R_1^+ and R_2^+ does not tend to infinity; thus by further construction of surfaces of type R^+ (with fundamental segments $V(x, \eta_1, \eta_2)$ such that x is a dyadic rational) and by further subdivision of O by straight lines $y = \eta_i$, $y_1 = \eta_1 < \eta_2 < \ldots < \eta_l = y_2$ we can obtain that (9,5) holds for neighbouring surfaces with $\eta/(p + 1)$ instead of η (p is the number of curves of $\hat{Z}$ originating in the y-edges of O), and that (9,1) to (9,3) hold.

Now exclude those surfaces which have at least one boundary curve in the set $\hat{Z}$; then the outer distance of neighbouring surfaces will be less than η, and (9,4) will hold. In this construction each rectangle of K is decomposed into a finite number of new rectangles. The system consisting of these latter is denoted by K'. The x-coordinates of vertices of rectangles from K are again dyadic rationals, since this holds for those of K, and in the construction either the x-edges were prolonged or the new vertices were chosen with dyadic rational x-coordinates.

In the next step there will be constructed a system of surfaces to ensure instability of solutions. Let K by the system of rectangles constructed in Lemma 9. In the y-edges of every rectangle from K there originates a finite number of curves $x = \xi_n(t)$, $y = \text{const}$. These are either boundary curves of surfaces considered in Lemma 9 or curves from the set $\hat{Z}$. According to (9,2), (9,4), each pair of curves $x = \xi_n(t)$, $y = \text{const}$ form a regular couple of functions with respect to some $d > 0$. According to Lemma 1, between any two neighbouring curves $x = \xi_n(t)$, $y = \text{const}$ we may construct a system of curves $x = \xi^{i,k}(t)$, $y = \text{const}$ which form regular couples with respect to d and whose fundamental points have dyadic rational x-coordinate. Thus on surfaces $y = \text{const}$ originating in the y-edges of rectangles from K, we have defined systems of curves in accordance with Lemma 1.

As in the proof of Lemma 9, consider any rectangle $O(x_1, x_2, y_1, y_2)$. Decompose it by straight lines parallel to the x axis, passing through the end points of fundamental segments which constitute the x-edges of O. Thus we obtain rectangles $O^{(i)}(x_1, x_2, \zeta_i, \zeta_{i+1})$, $y_1 = \zeta_1$, $y_2 = \zeta_{r+1}$. As shown above, construct systems of curves $x = \xi^{i,k}(t)$, $y = \text{const}$, originating in the y-edges of the rectangles $O^{(i)}$. Above every such rectangle, connect the curves $x = \xi^{i,k}(t)$, $y = \zeta_s$ and $x = \xi^{j,l}(t)$, $y = \zeta_{s+1}$ by a curve of type R if the x-coordinates of their fundamental points are equal. Such surfaces are constructed starting at t such that both boundary curves are defined and that the functions $\alpha(t)$ corresponding to all near surfaces of type R^+ (see Definition 9) are small enough to ensure that the constructed surface does not intersect the surfaces constructed previously. The segment $V(\xi^{i,k}(0), y_1, y_2)$ will be termed the fundamental segment of this surface. Finally, this construction is performed in all rectangles of K.

Note that the surfaces just constructed — to be denoted by N — have the following property:

(10,1) If, in the direction of the positive x axis, the fundamental segment $V(R)$ of the surface R is *before* (*behind*) the fundamental segment $V(N)$ of some surface N, then the whole surface R is *before* (*behind*) the surface N.

Similarly: if $x = \xi_n(0)$, $y = y_i$ is the fundamental point of the curve $x = \xi_n(t)$, $y = y_i$ which is either a boundary curve of some R or a curve of $\hat{Z}$, and if the point $x = \xi_n(0)$, $y = y_i$ is *before* (*behind*) the fundamental segment $V(N)$, then the whole curve $x = \xi_n(t)$, $y = y_i$ is *before* (*behind*) the surface N. As in the note to Lemma 8, the surfaces of type R just constructed may be replaced by surfaces of type R^+. We formulate our conclusions in the following lemma.

Lemma 10. *Given a system of surfaces according to Lemma 9, there exists a system of surfaces N (these are of type R^+, but are defined only for $t \geqq t_0$, with t_0 possibly varying for different surfaces) which, together with the given surfaces, satisfy* (9,1) *to* (9,5), (10,1) *and such that the boundary curves of the surfaces N form a system of curves which satisfies the assumptions of Lemma 1.*

Proof. According to the foregoing construction, the system of surfaces N together with the system of surfaces R_n^+ satisfies (9,1) to (9,3), (9,5), (10,1). To prove (9,4), note that as a consequence of property (B), the x-coordinates of points of Z are not dyadic rational, while the fundamental segments $V(x, y_1, y_2)$ of surfaces N have x dyadic rational (*i. e.* the points of Z are not on the fundamental segments of surfaces N). From (10,1) then (9,4) follows easily.

The meaning of property (10,1) is that it allows us to extend the domains of definition of surfaces N, viz. $t \geqq t_n$, $y \in \langle y_1^{(n)}, y_2^{(n)} \rangle$, to half-strips $t \geqq 0$, $y \in \langle y_1^{(n)}, y_2^{(n)} \rangle$, while preserving properties (9,1) to (9,4). We will describe in brief such a construction for some given surface N_1. Consider the fundamental segment $V(N_1)$ of this surface. It lies in some rectangle O of the system K'. Denote by S_1, S_2 the surfaces which originate in the front and rear edges of O. The surface N_1 is to be prolonged so as to remain between S_1 and S_2. This will render properties (9,3) and (9,4), except for the curves of $\hat{Z}$[5]) with initial points on O. If the segment $V(N_1)$ has a common front (rear) end point with the fundamental segment of some surface R_n^+, then by construction, the boundary curve of this surface coincides with the curve $\xi^{i,k}(t)$ corresponding to this end point. Thus for the boundary curve of N_1 there must be chosen the boundary

[5]) The definition of $\hat{Z}$ is in Lemma 8.

curve of the surface R_n^+. We will then have (9,1) and (9,4). On the other hand, if the segment $V(N_1)$ does not have a common end point with the fundamental segment of any of the surfaces R_n^+, then the boundary curves of N (*i. e.* the curves $\xi^{i,k}(t)$) are to be prolonged over all $t \geqq 0$ in such a manner that they intersect no curve of $\tilde{Z}$ and no boundary curve of any surface R_n^+ with initial point in O. According to properties (9,1) to (9,4) of the system of surfaces R_n^+, such a prolongation is possible.

The next lemma has already been described in greater detail; it forms the inductive step in the proof of Theorem 3.

Lemma 11. Assumptions: *Given a system of two differential equations*

$$\frac{dx}{dt} = X_1(t, x, y), \quad \frac{dy}{dt} = 0 \tag{11,1}$$

and an open set G_1, $G_1 \subset E_2$. The solutions of (11,1) *are bounded and have property* (A) *in G_1. There is given a system of rectangles K_1 and a set Z_1 such that Z_1 has property* (B) *in G_1 with respect to K_1. The solutions of* (11,1) *with initial points in x-edges of rectangles of K_1 constitute surfaces of type R^+ (single surfaces will be denoted by R_n^+).*

Conclusion. *To every open set G_2, $G_2 \subset G_1$, and every $\eta > 0$ there may be constructed a function $X_2(t, x, y)$, a system of rectangles K_2 and a set Z_2 with the following properties*:

K_2 is finer than K_1 in G_2,

the y-edges of rectangles of K_2 have lengths less than η,

the set Z_2 has property (B) *in G_2 with respect to K_2,*

the function $X_2(t, x, y)$ is defined for $t \geqq 0$ and has continuous partials of all orders,

$$X_2(t, x, y) = X_1(t, x, y) \quad \text{in} \quad \widetilde{E_2 - G_2}\,, \tag{11,2}$$

$$X_2(t, x, y) = X_1(t, x, y) \quad \text{in} \quad \underset{[t,x,y]}{\exists} \left(0 \leqq t \leqq \left[\tfrac{1}{\eta}\right]\right)^{6}). \tag{11,3}$$

(11,4) *The surfaces of type R^+ which originate in the x-edges of rectangles from K_1 are again composed of solutions of the differential system*

$$\frac{dx}{dt} = X_2(t, x, y), \quad \frac{dy}{dt} = 0\,. \tag{11,5}$$

Let us denote by $\hat{Z}$ the set of solutions of (11,1) *with initial points in the set Z, $Z = Z_1 + Z_2$. The curves of $\hat{Z}$ are again solutions of* (11,5).

The solutions of (11,5) *with initial points on the x-edges of rectangles from K_2 constitute surfaces of type R^+ which satisfy* (9,1) *to* (9,5). *All the solutions of* (11,5) *with initial points in any given rectangle $O \in K_2$ are uniformly unstable, i. e. to*

[6] $[x]$ is the integral part of x.

every rectangle $O \in K_2$ there exists an $\alpha > 0$ such that for any pair of solutions $x = x_1(t)$, $y = y_1$ and $x = x_2(t)$, $y = y_1$ (with coinciding y-components) with initial points in O, there holds

$$\sup_{t \geq 0} |x_1(t) - x_2(t)| > \alpha .$$

The solutions of (11,5) *are bounded and have property* (A) *in* G_2.

Proof. First apply Lemma 8 to obtain a set Z_2 and a system of rectangles K', with surfaces of type R originating in the front x-edges of these rectangles and having the properties described in the lemma. Using the note to Lemma 8, these surfaces of type R are changed to surfaces of type R^+. Next, apply Lemma 9 to obtain a system of rectangles K_2 such that in the x-edges of these rectangles there originate surfaces of type R^+ satisfying (9,1) to (9,5). The surfaces thus constructed using Lemmas 8 and 9 will be denoted by PR_n^+ (these are the auxiliary surfaces). Finally apply Lemma 10 to obtain a system of surfaces N to be used in the proof of instability (surfaces of this system will be denoted by NR_n^+). To distinguish between the given surfaces of type R^+ and the remaining ones, denote by DR_n^+ the surfaces of the assumptions in this lemma. Also choose a sequence of integers t_n: $t_1 > [1/\eta]$, $t_{n+1} = t_n + 1$, which will be made to correspond to the rectangles from K_2.

We now proceed to define the function $X_2(t, x, y)$. In order to satisfy (11,2) to (11,4), (11,6), put

$$X_2(t, x, y) = X_1(t, x, y)$$

on the sets $\widetilde{E_2 - G_2}$, $\underset{[t,x,y]}{\exists} (0 \leq t \leq [1/\eta])$, $\tilde{Z}$ and on the surfaces DR_n^+.

To satisfy (11,7), on each surface PR_n^+ put

$$X_2(t, x, y) = \frac{\partial x(t, y)}{\partial t} \quad \text{for} \quad t \geq t_n + 1 \tag{11,8}$$

where $x(t, y)$ is the functional description of the surface PR_n^+; t_n is the integer which has been made to correspond to the rectangle from K_2, in whose front x-edge the given surface PR_n^+ originates. The identity (11,8) then means that, starting from some instant, the surfaces PR_n^+ will consist of solutions of (11,5).

Similarly define $X_2(t, x, y)$ on every surface NR_n^+ by the identity (11,8), but only for $t \geq t_m + \tau_n + 1$ (here t_m is the integer corresponding to the rectangle containing the fundamental segment of NR_n^+, and τ_n is an integer such that NR_n^+ is defined for $t \geq \tau_n$). Thus the function $X_2(t, x, y)$ has been defined on the sets

$$\widetilde{E_2 - G_2}\,, \quad \underset{[t,x,y]}{\exists} (0 \leq t \leq [1/\eta])\,, \quad \tilde{Z}\,, \quad DR_n^+$$

– these constitute a connected set – and also on the surfaces PR_n^+, NR_n^+. Using property (B), since the sequence t_n and τ_n diverge to infinity, the union of these sets is closed. It is required to extend the function $X_2(t, x, y)$ over the complete half-space $t \geq 0$ in such a manner that it have continuous partials of all orders, that (11,7) hold, and that the solutions with initial points on a fundamental segment of any NR_n^+

entirely belong to the surface NR_n^+. The construction of such a function $X_2(t, x, y)$ will be performed in several steps.

Obviously $X_2(t, x, y) = X_1(t, x, y)$ in the parallellepiped $0 \leqq t \leqq t_1$. We proceed to define $X_2(t, x, y)$ in $t_1 \leqq t \leqq t_1 + 1$. However, first we must extend the domain of definition of some of the surfaces NR_n^+ to the half-strip $t \geqq 0$, and change the surface PR_1^+ somewhat (PR_1^+ is the surface on which $X_2(t, x, y)$ has been defined for $t \geqq$ $\geqq t_1 + 1$). Next consider those surfaces NR_n^+ on which $X_2(t, x, y)$ is defined for $t \geqq t_1 + 1$. There is at most a finite number of such surfaces (possibly none); denote them by $NR_1^+, \ldots, NR_k^+$. The integral curves of the system (11,5) which have initial points on the fundamental segments $V(PR_1^+)$, $V(NR_1^+)$, ..., $V(NR_k^+)$, constitute surfaces defined for $0 \leqq t \leqq t_1$; denote them by PR_1', NR_1', ..., NR_k'. According to property (10,1), the surfaces PR_1' and PR_1^+, and also NR_i' and NR_i^+, may be connected in the parallellepiped $t_1 \leqq t \leqq t_1 + 1$ in such a manner that the resulting surfaces are of type R^+, in the parallellepiped $0 \leqq t \leqq t_1$ they coincide with the surfaces PR_1', NR_1', ..., NR_k' respectively, and in the half-space $t \geqq t_1 + 1$ they coincide with the surfaces PR_1^+, NR_1^+, ..., NR_k^+ respectively (considered as functions $x_i(t, y)$ they have continuous partials of all orders and satisfy (9,1) to (9,4) – *cf.* the note to Lemma 11 to follow). In order to satisfy (11,7), on the just constructed parts of surfaces put

$$X_2(t, x, y) = \frac{\partial x_i(t, y)}{\partial t}\,.$$

The function $X_2(t, x, y)$ is thus defined on a closed set, and can be extended to the complete parallellepiped $t_1 \leqq t \leqq t_1 + 1$ with continuous partials of all orders. The construction in all the other parallellepipeds $n \leqq t \leqq n + 1$ proceeds in an entirely similar manner. Thus we obtain a function $X_2(t, x, y)$ defined in the complete half-space $t \geqq 0$, with continuous partials of all orders, and satisfying (11,2) to (11,4), (11,6), (11,7). We will show that all solutions with initial points in any given $O_n \in K_2$ are uniformly unstable.

Let $x = x_1(t)$, $y = y_1$ and $x = x_2(t)$, $y = y_1$ be two solutions of the system (11,5) with initial points in $O_n = O(x_1^{(n)}, x_2^{(n)}, y_1^{(n)}, y_2^{(n)})$. Take dyadic rational ξ_1, ξ_2 with $x_1(0) < \xi_1 < \xi_2 < x_2(0)$. In the segments $V(\xi_1, y_1^{(n)}, y_2^{n)})$, $V(\xi_2, y_1^{(n)}, y_2^{(n)})$ there originate surfaces NR_1^+, NR_2^+ of type N, whose front (rear) boundary curves

$$\begin{gathered} x = \xi_1(t)\,, \quad y = y_1^{(n)}\,, \quad x = \xi_2(t)\,, \quad y = y_1^{(n)} \\ (x = \eta_1(t)\,, \quad y = y_2^{(n)}\,, \quad x = \eta_2(t)\,, \quad y = y_2^{(n)}) \end{gathered}$$

form a regular couple of functions with respect to d_1 (d_2), these latter numbers being independent of ξ_1, ξ_2 (see the last proposition in Lemma 10).

1. Assume that $y_1^{(n)} \neq y_1 \neq y_2^{(n)}$. To the surfaces NR_1^+, NR_2^+ there corresponds a τ such that $y_1^{(n)} + \alpha(t) < y_1 < y_2^{(n)} - \alpha(t)$ for $t \geqq \tau$ (see Definition 10). Since the curves

$$\begin{gathered} x = \xi_1(t)\,, \quad y = y_1^{(n)}\,, \quad x = \xi_2(t)\,, \quad y = y_1^{(n)} \\ (x = \eta_1(t)\,, \quad y = y_2^{(n)}\,, \quad x = \eta_2(t)\,, \quad y = y_2^{(n)}) \end{gathered}$$

form regular couples with respect to d_1 (d_2), there exist $\tau_1 \geqq \tau$, $\tau_2 \geqq \tau$ such that $\xi_2(\tau_1) > \xi_1(\tau_1) + d_1$, $\eta_2(\tau_2) > \eta_1(\tau_2) + d_2$. Since the solutions $x = x_1(t)$, $y = y_1$, $x = x_2(t)$, $y = y_1$ do not intersect NR_1^+, NR_2^+, it immediately follows that

$$x_2(t) \geqq \xi_2(t) + (\eta_2(t) - \xi_2(t)) \cdot \frac{y_1 - y_1^{(n)}}{y_2^{(n)} - y_1^{(n)}},$$

$$x_1(t) \leqq \xi_1(t) + (\eta_1(t) - \xi_1(t)) \cdot \frac{y_1 - y_1^{(n)}}{y_2^{(n)} - y_1^{(n)}},$$

and

$$x_2(t) - x_1(t) \geqq (\xi_2(t) - \xi_1(t)) \frac{y_2^{(n)} - y_1}{y_2^{(n)} - y_1^{(n)}} + (\eta_2(t) - \eta_1(t)) \cdot \frac{y_1 - y_1^{(n)}}{y_2^{(n)} - y_1^{(n)}}.$$

If

$$y_1^{(n)} < y_1 \leqq \frac{y_1^{(n)} + y_2^{(n)}}{2},$$

put $t = \tau_1$ to obtain $x_2(\tau_1) - x_1(\tau_1) \geqq \frac{1}{2}d_1$. If

$$\frac{y_1^{(n)} + y_2^{(n)}}{2} \leqq y_1 < y_2^{(n)}$$

put $t = \tau_2$ to obtain $x_2(\tau_2) - x_1(\tau_2) \geqq \frac{1}{2}d_2$.

2. Assume that $y_1 = y_1^{(n)}$ (or $y_1 = y_2^{(n)}$). Then, using regularity of the boundary curves with respect to d_1 (d_2) (see Lemma 10), we have

$$\sup_{t \geqq 0} |x_2(t) - x_1(t)| \geqq \min(d_1, d_2).$$

Thus in both cases,

$$\sup_{t \geqq 0} |x_2(t) - x_1(t)| > \tfrac{1}{2} \min(d_1, d_2).$$

The solutions of (11,5) are bounded. In the set $\widetilde{E_2 - G_2}$, the systems (11,5) and (11,1) coincide, and in $\tilde{G}_2$ there remain unchanged the surfaces DR_n^+ which consist of solutions of the original system (11,1).

It remains to prove that the system (11,5) has property (A) in G_2. Take any open $G \subset G_2$, and any point $[0, x_0, y_0] \in G$. Consider the square $K \equiv O(x_0 - \alpha, x_0 + \alpha, y_0 - \alpha, y_0 + \alpha)$ with center $[x_0, y_0]$ and such that $K \subset G$.

The integral curves of (11,5) with initial points on the front (rear) x-edges of K constitute surfaces; denote them by S_1 (S_2). Using Definition 2, and since the second component of the differential system is zero, it suffices to prove that

$$\varrho(P(x(t), y_0, y_0), S_i) > 0\,, \quad i = 1, 2$$

where $x = x(t)$, $y = y_0$ is the solution of (11,5) with initial point $x = x_0$, $y = y_0$, and where $P(x(t), y_0, y_0)$ denotes in fact the curve $x = x(t)$, $y = y_0$.

Let $i = 2$ (the proof for $i = 1$ is similar). Take dyadic rational ξ_1, ξ_2 with $x_0 < < \xi_1 < \xi_2 < x_0 + \alpha$. The system of rectangles K_2 decomposes the segments

$V(\xi_i, y_0 - \alpha, y_0 + \alpha)$, $i = 1, 2$, into a finite number of subsegments, in which there originate surfaces $NR_k^{+(i)}$, $i = 1, 2$. The surfaces $NR_k^{+(1)}$, $NR_l^{+(2)}$ have a positive distance (see Lemma 10). Set

$$\alpha = \min_{k,l} \varrho(NR_k^{+(1)}, NR_l^{+(2)}) .$$

Since solutions are uniquely determined by their initial points, the solution $x = x(t)$, $y = y_0$, *i. e.* the curve with initial point x_0, y_0, will lie entirely in front of the surfaces $NR_k^{+(1)}$; the surface S_2 will lie behind the surfaces $NR_l^{+(2)}$. It follows that

$$\varrho(P(x(t), y_0, y_0), S_2) \geqq \min_{k,l} \varrho(NR_k^{+(i)}, NR_l^{+(2)}) = \alpha > 0 .$$

This concludes the proof of Lemma 11. We must however still formulate the note concerning the process of connecting surfaces which was used in the proof of Lemma 11.

Note to Lemma 11. Let there be given a system of differential equations (11,1), and an open set $G \subset E_2$. Further, let there be given a system of rectangles K and a set Z such that Z has property (B) in G with respect to K; and also a system of surfaces of type R^+ with the following relation of their fundamental segments to the x-edges of rectangles of K: Either the fundamental segment is composed of a finite number of x-edges of rectangles from K, or the fundamental segment is part of some x-edge.

The system of surfaces of type R^+ is divided into two groups:

1. A system of distinguished surfaces, to be denoted by SR_n^+, which are composed of integral curves of the system (11,1).

2. The remaining surfaces, to be denoted by LR_n^+, and which need not be composed of integral curves of (11,1). The system of all these surfaces satisfies (9,1) to (9,4). $\hat{Z}$ is the system of solutions of (11,1) with initial points in Z.

Assume that there is a real $t_0 > 0$ and a function $X_2(t, x, y)$ defined on $\widetilde{E_2 - G}$ and $\underset{[t,x,y]}{\exists}$ $(0 \leqq t \leqq t_0)$ such that

$$X_2(t, x, y) = X_1(t, x, y)$$

in $\widetilde{E_2 - G}$, on the surfaces SR_n^+ (if $X_2(t, x, y)$ is defined there), and on the curves of the system $\hat{Z}$. Take any one of the surfaces LR_n^+ denote it by LR_1^+, and its fundamental segment by VL_1. The integral curves of the system

$$\frac{dx}{dt} = X_2(t, x, y) , \quad \frac{dy}{dt} = 0$$

which have initial points on VL_1 are defined for $0 \leqq t \leqq t_0$, and constitute a surface which we will denote by U. Since the surfaces SR_n^+ are also composed of integral curves of this latter differential system (since $X_2(t, x, y) = X_1(t, x, y)$ on these surfaces), the system of surfaces SR_n^+ and U satisfy (9,1) to (9,4) on the interval $0 \leqq t \leqq$ $\leqq t_0$. The problem is to connect the surfaces U and LR_1^+, *i. e.* to construct a surface J

of type R^+ which coincides with LR_1^+ for $t \geqq t_0 + 1$, and coincides with U for $0 \leqq \leqq t \leqq t_0$, and retain properties (9,1) to (9,4). We will assume that the front point of the fundamental segment LV_1 is the rear point of the fundamental segment SV of some surface SR_n^+, and that the rear point of LV_1 lies on no fundamental segment of any SR_n^+. The other cases are quite similar. Using (9,1), the surfaces SR_n^+, LR_1^+ have a common boundary curve. Since U and SR_n^+ are composed of solutions of the new differential system, they also have a common boundary curve. Since (9,1) is to hold for the surface J also, the surfaces J and LR_1^+ must have a common boundary curve. Let $x_1(t, y)$, $x_2(t, y)$ be the functional descriptions of the surfaces U, LR_1^+ respectively. Thus we must have $x_1(t, y_1) = x_2(t, y_1)$, where y_1 is the y-coordinate of the front point of the segment LV_1. Consider the set of points $[t, x, y]$: $0 \leqq t \leqq t_0$, $y_1 \leqq y \leqq \leqq y_2$, $x_1(t, y) \leqq x \leqq x_2(t, y)$. This set is bounded, from the front, by the surface U, and from behind by the surface LR_1^+. Since both these surfaces satisfy (9,3) and (9,4), and since $x_1(0, y) = x_2(0, y) = x^*$ (they have a common fundamental segment), this set is disjoint with $\widetilde{h(G)}$ and $\tilde{Z}$. It can be shown that there is a $\delta > 0$ such that a δ-neighbourhood of this set is disjoint with $\widetilde{h(G)}$ and $\tilde{Z}$. There obviously exists a continuous function $x(t, y)$ such that

$$x(t, y) = x_2(t, y) \quad \text{for} \quad t \geqq t_0 + \delta\,, \quad y_1 \leqq y \leqq y_2\,,$$
$$x(t, y) = x_1(t, y) \quad \text{for} \quad 0 \leqq t \leqq t_0\,, \quad y_1 \leqq y \leqq y_2\,,$$
$$x(t, y_1) = x_1(t, y_1) = x_2(t, y_2)\,,$$

and that the points $[t, x(t, y), y]$ with $t_0 \leqq t \leqq t_0 + \delta$, $y_1 \leqq y \leqq y_2$ are in the δ-neighbourhood of the above set. The function $x(t, y)$ describes a surface J_1 with properties (9,1), (9,3), (9,4). However, (9,2) need not be satisfied, since this surface might intersect some surface SR^+. In any case, it intersects at most a finite number of surfaces SR^+, say $SR_1^+, \ldots, SR_p^+$ (since the limit points of surfaces SR^+ are in $\widetilde{h(G)}$). Consider first the surface SR_1^+.

Set

$$\alpha_1 = \min\,[\varrho(SR_1^+, \widetilde{h(G)}), \varrho(SR_1^+, \tilde{Z}), \varrho(SR_1^+, SR_k^+), \varrho(SR_1^+, LR_1^+)] > 0$$

for all indices k except such that the fundamental segments $V(SR_1^+)$ and $V(SR_k^+)$ intersect. Replace every point on the surface $x(t, y)$ whose distance from SR_1^+ is less than $\frac{1}{2}\alpha_1$, by a point with the same t, y-coordinates and whose x-coordinate is $\frac{1}{2}\alpha_1$ less or greater than that of the corresponding point on SR_1^+, and which lies on the same side of SR_1^+ as the original point. The surface J_2 thus constructed, together with the remaining surfaces except $SR_2^+, \ldots, SR_p^+$, satisfies (9,1) to (9,4). On performing a similar construction with all the surfaces $SR_2^+, \ldots, SR_p^+$ in turn, we finally obtain a surface which satisfies (9,1) to (9,4). However, the resulting surface need not have continuous partials; but it does have a positive distance in $t_0 \leqq t \leqq t_0 + 1$, $y_1 \leqq y \leqq y_2$ from all the surfaces SR_n^+ except those with which it has a common boundary curve. Using [1], we may approximate our surface by a surface having continuous partials of all orders (as a function of t, y). If the surface constructed above has a common boundary curve with some surface of type SR^+ then this curve, as a function of t, has continuous derivatives of all orders. The surface LR_1^+ is then included into the group of surface SR_n^+, and the construction is repeated for the remaining surfaces LR_k^+. When applying this note in the proof of Lemma 11, the surfaces $PR_1^+, NR_1^+, \ldots, NR_k^+$ for which the process of connection is to be performed, are put into the group LR^+, and DR_n^+ together with those of the surfaces PR_n^+ and NR_n^+ for which the process of connection has already been performed, are put into the group SR^+. In addition it is necessary to remark that according to the construction following Lemma 10, and using property (10,1), the domains of definition of the surfaces N may be extended in the manner necessary for the application of this note.

With this the proof of Lemma 11 is definitely completed, and the preparations of the proof of Theorem 3 are concluded.

As has already been mentioned, the set M is the intersection of a decreasing sequence of open sets G_n^*, $n = 1, 2, \ldots$ The proof is performed by induction. Obviously we may add the set $G_0^* = E_2$ to the sequence G_n^*. The O-th step in the induction process has been prepared in Lemma 2. We obtain a differential system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X^{(0)}(t, x, y)\,, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

with property (A) in E_2 (see note (2,1)); all of its solutions are bounded and uniformly unstable.

For the sake of clarity, let us also perform the first step, applying Lemma 11. Take the set G_1^*. For the set G_1 of Lemma 11 take E_2; for the system K_1 of Lemma 11 take the system of squares with integral coordinates of vertices. The set $Z^{(0)}$ to this system K_1 is constructed according to the note to Lemma 3. For the set G_2 of Lemma 11 take G_1^*, and choose $\eta_1 = \frac{1}{2}$. Using Lemma 11, construct a function $X^{(1)}(t, x, y)$, a system of rectangles $K^{(1)}$, and a set $Z^{(1)}$ which has property (B) in G_1^* with respect to K_1, and such that (11,2) to (11,4), (11,6), (11,7) are satisfied. The solutions of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X^{(1)}(t, x, y)\,, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

are bounded and have property (A) in G_1^*. For the next inductive step set $Z_1 = Z^{(0)} + Z^{(1)}$.

Induction. Assume that to the set G_n^* there have been constructed: a function $X^{(n)}(t, x, y)$, a system of rectangles $K^{(n)}$ and a set Z_n with property (B) in G_n^* with respect to $K^{(n)}$. The solutions of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X^{(n)}(t, x, y), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \tag{12,1}$$

are bounded and have property (A). The solutions of this system with initial points on the x-edges of rectangles from $K^{(n)}$ constitute surfaces of type R^+ and satisfy (11,2) to

(11,4), (11,6), (11,7). $\hat{Z}_n$ is the system of solutions of (12,1) with initial points in Z_n. Next, apply Lemma 11 again. For the set G_1 of the Lemma take G_n^*; for the system K_1 take $K^{(n)}$; for the set Z_1 take Z_n; for the set G_2 of the Lemma take G_{n+1}^*, and choose $\eta_n = 1/2^n$. On applying Lemma 11 we obtain a function $X^{(n+1)}(t, x, y)$, a system of rectangles $K^{(n+1)}$ and a set $Z^{(n+1)}$ ($Z_{n+1} = Z^{(n+1)} + Z_n$) which satisfy the assumptions of Lemma 11. Among these we emphasise the properties (11,2) to (11,4), (11,6), (11,7), and that the solutions of the differential system

$$\frac{dx}{dt} = X^{(n+1)}(t, x, y), \quad \frac{dy}{dt} = 0$$

are bounded and have property (A) in G_{n+1}^*. Since

$$X^{(n+1)}(t, x, y) = X^{(n)}(t, x, y) \quad \text{in} \quad \underset{[t,x,y]}{\exists} \left(0 \leqq t \leqq \left[\frac{1}{\eta_n}\right]\right)$$

and since $\lim_{n\to\infty} \eta_n = 0$, there exists a limit function $X(t, x, y)$ defined in the complete half-space and with continuous partials of all orders. Let us now show that the differential system

$$\frac{dx}{dt} = X(t, x, y), \quad \frac{dy}{dt} = 0 \tag{12,2}$$

satisfies the assumptions of Lemma 3.

First we prove that solutions with initial points in the set $M = \prod_{n=0}^{\infty} G_n^*$ are uniformly stable. Take any point $[x_0, y_0] \in M$ and any $\varepsilon > 0$. To this ε there is an index n_0 such that $2\eta_{n_0} < \varepsilon$. The set $G_{n_0}^*$ decomposes into a system of rectangles $K^{(n_0)}$. Assume that the selected point $[0, x_0, y_0]$ is in one rectangle

$$O_1 \equiv O(x_1^{(1)}, x_2^{(1)}, y_1^{(1)}, y_2^{(1)}) \in K^{(n_0)}$$

only (*i. e.* that it is an interior point; in the general case, it may belong to at most four rectangles — the proof is then analogous). Next take a sufficiently large index $n_1 > n_0$ such that for the rectangle

$$O_2 \equiv O(x_1^{(2)}, x_2^{(2)}, y_1^{(2)}, y_2^{(2)}) \in K^{(n_1)}, \quad [0, x_0, y_0] \in O_2$$

there holds $y_1^{(1)} < y_1^{(2)} < y_2^{(2)} < y_2^{(1)}$ (again we only consider the case that the point belongs to one rectangle only, the other cases being similar). Such an index n_1 always exists, since for $n > n_0$ the system $K^{(n)}$ is finer than the system $K^{(n_0)}$, and since the lengths of y-edges of rectangles from $K^{(n)}$ are less than $\eta_n = 1/2^n$ (see the assumptions in Lemma 11). The surfaces composed of solutions of the differential system

$$\frac{dx}{dt} = X^{(n_0)}(t, x, y), \quad \frac{dy}{dt} = 0$$

with initial points on the front and rear x-edges of the rectangle O_1 are of type R^+;

denote them by R_1^+, R_2^+ respectively. According to Lemma 11, they are also composed of solutions of every system (12,1) with $n \geqq n_0$ and thus of solutions of the system (12,2) also.

Denote by R_3^+, R_4^+ the surfaces composed of solutions of the differential system (12,1) with $n = n_1$ which have initial points on the x-edges of the rectangle O_2. Using property (11,7) (and its consequence (9,2)), we obtain

$$\varrho(R_1^+, R_3^+) > 0\,, \quad \varrho(R_2^+, R_4^+) > 0\,.$$

Set

$$\delta_1 = \min\left[\varrho(R_1^+, R_3^+), \varrho(R_2^+, R_4^+)\right].$$

To the rectangles O_1, O_2 there exist t_1, t_2 such that

$$\varrho^{(2)}_{<t_1,\infty)}(R_1^+, R_2^+) < \eta_{n_0}\,, \quad \varrho^{(2)}_{<t_2,\infty)}\,, (R_3^+, R_4^+) < \eta_{n_1}\,,$$

according to (9,5) (see property (11,7)).

Set $\tau = \max(t_1, t_2)$. Let $[T_1, \xi, \eta]$, $T_1 \geqq \tau$, be any point whose distance from the solution $x = x_0(t)$, $y = y_0$ $(x_0(0) = x_0)$ is less than δ_1, *i. e.*

$$[(x_0(T_1) - \xi)^2 + (\eta - y_0)^2]^{\frac{1}{2}} < \delta_1\,.$$

Since the solution $x = x_0(t)$, $y = y_0$ remains in $\tilde{O}_2$ (the surfaces R_3^+, R_4^+ are composed of solutions of (12,2)), the point $[T_1, \xi, \eta]$ is in the set $\tilde{O}_1$; therefore the solution $x = \xi(t)$, $y = \eta$ of (12,2) with this point as initial also remains in $\tilde{O}_1$. Since

$$\varrho^{(2)}_{<\tau,\infty)}(R_1^+, R_2^+) < \eta_{n_0}\,,$$

we have

$$[(x_0(t) - \xi(t))^2 + (\eta - y_0)^2]^{\frac{1}{2}} < \eta_{n_0} < \varepsilon \quad \text{for} \quad t \geqq T_1\,.$$

We have thus proved uniform stability for $t \geqq \tau$. Since solutions vary continuously with the initial conditions, we may choose $\delta > 0$ such that

$$[(x_0(t) - \xi(t))^2 + (\eta - y_0)^2]^{\frac{1}{2}} < \delta_1 \quad \text{for} \quad T_2 \leqq t \leqq \tau$$

whenever

$$[(x_0(T_2) - \xi(T_2))^2 + (\eta - y_0)^2]^{\frac{1}{2}} < \delta\,.$$

By choice of δ_1 we must have

$$[(x_0(t) - \xi(t))^2 + (\eta - y_0)^2]^{\frac{1}{2}} < \eta_{n_0} < \varepsilon \quad \text{for} \quad t \geqq \tau$$

also. This proves the uniform stability of the solution $x = x_0(t)$, $y = y_0$.

It remains to show that the solutions with initial points $[x_0, y_0] \,\bar{\in}\, M$ are unstable. There is an n with $[x_0, y_0] \in G_n^* - G_{n+1}^*$. There are then two cases.

1. The point $[x_0, y_0]$ is not a boundary point of G_{n+1}^*; thus it is an interior point of the set $G_n^* - G_{n+1}^*$. By construction of $X(t, x, y)$, we have $X(t, x, y) = X^{(n)}(t, x, y)$ in the set $\widetilde{E_2 - G_{n+1}^*}$ ($\widetilde{E_2 - G_{n+1}^*}$ is the set of solutions of (12,1) with initial points in $E_2 - G_{n+1}^*$); and the construction of $X^{(n)}(t, x, y)$ was such as to obtain

unstable solutions originating in G_n^* (see Lemma 11). Since $[t, x_1(t), y_1] \in \widetilde{G_n^* - G_{n+1}^*}$, $[t, x_0(t), y_0] \in \widetilde{G_n^* - G_{n+1}^*}$, the following constructions do not change this situation.

2. The point $[x_0, y_0]$ is a limit point of the set G_{n+1}^*; it then belongs to some rectangle $O \in K^{(n)}$. According to Lemma 11, the solutions of the system (12,1) are uniformly unstable in O. Further, the point $[x_0, y_0]$ is on the boundary of G_{n+1}^*, so that it is the limit point of pairs of points from the set $Z^{(n+1)}$. The curves of $\hat{Z}^{(n+1)}$ are solutions of (12,1), so that by Lemma 11 they are also solutions of the differential system (12,2). Now consider only those pairs of points from $Z^{(n+1)}$ which are in the rectangle O. The point $[x_0, y_0]$ is thus a limit point of a certain sequence of pairs of points from the set $Z^{(n+1)}$ with equal y-coordinates; the solutions of the system (12,2) with these points as initial have, using their uniform instability, an outer distance greater than a certain $\alpha > 0$ corresponding to the rectangle O. This implies instability of the solution $x = x_0(t)$, $y = y_0$.

References

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Резюме

ТОПОЛОГИЧЕСКАЯ СТРУКТУРА МНОЖЕСТВА УСТОЙЧИВЫХ РЕШЕНИЙ

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В этой статье исследована структура множества устойчивых, эквиустойчивых и равномерно устойчивых решений системы дифференциальных уравнений (1) (написаны при помощи векторов), которые выполняют какое-нибудь требование для существования решений. (Например, условие Каратеодори.) Это исследование можно провести с различных точек зрения. Здесь я буду выше упомянутые множества устойчивых, эквиустойчивых и равномерно устойчивых решений исследовать так, что определю структуру начальных точек этих решений. Начальные точки будут при этом находиться в многообразии $t = 0$. Наверно можно многообразие $t = 0$ заменить любым многообразием $t = \text{const}$.

Пусть $N_a^{(1)}$ – множество всех точек в многообразии $t = 0$, из которых выходят устойчивые решения данного дифференциального уравнения (1), $N_a^{(2)}$ – множество всех точек, в многообразии $t = 0$, из которых выходят эквиустойчивые решения данного дифференциального уравнения (1) и $N_a^{(3)}$ – множество всех точек,

в многообразии $t = 0$, из которых выходят равномерно устойчивые решения данного дифференциального уравнения (1); при этом если все эти решения подчинены требованию, чтобы были определены для всех $t \geqq 0$, то положим $a = \infty$ и если только требовать, чтобы были определены на некотором произвольном конечном промежутке $\langle 0, T)$ то положим $a = 0$. В статье доказывается эта теорема:

Теорема 1. *Все множества $N_a^{(i)}$, $i = 1, 2, 3$, $a = 0$ или $a = \infty$, являются множествами типа G_δ.*

Эта теорема доказывается для более общих систем кривых, чем решения дифференциальных уравнений если устойчивость, эквиустойчивость и равномерная устойчивость определена соответствующим образом.

Однако этим незавершена характеристика этих множеств. Для того, чтобы мы могли утверждать, что упомянутые множества характеризованы, необходимо построить к любому множеству N типа G_δ дифференциальное уравнение (1) так, чтобы из множества N выходили устойчивые или эквиустойчивые или равномерно устойчивые решения и при этом из дополнения этого множества N выходили решения, которые не являются устойчивыми или эквиустойчивыми или равномерно устойчивыми. В соответствии с известными соотношениями между устойчивостью, эквиустойчивостью и равномерной устойчивостью можно эти три случая свести к одному случаю.

Теорема 3. *Какое бы ни было множество N типа G_δ существует дифференциальное уравнение* (1), *компоненты правых частей $X_i(t, x)$ которого имеют частные производные всех порядков по переменным $t, x_1, \ldots, x_n$; такие, что*:

1. *Из множества N выходят равномерно устойчивые решения.*

2. *Из множества $E - N$ выходят неустойчивые решения.*

Эта теорема доказана в двумерном пространстве. Для случая n-мерного пространства суть доказательства остается правильной, но уже для $n = 3$ оно станет слишком громоздким. Система построена так, что правая часть одного дифференциального уравнения. тождественно равна нулю. Из этого вытекает, что теорему 3. можно легко сформулировать и для случая одного дифференциального уравнения.

Формулированная проблема частично разрешена также в автономном случае. Для случая одного дифференциального уравнения имеет место теорема:

Теорема 2. *Множество точек, из которых выходят неустойчивые решения уравнения $\mathrm{d}x/\mathrm{d}t = f(x)$ характеризовано тем, что оно состоит из трех частей*:

1. *Из полузамкнутого интервала $\langle x^{**}, \infty)$, при этом x^{**} может быть равно $x^{**} = -\infty$ (т. е. все решения неустойчивые) или может быть равно $+\infty$ (т. е. эта часть отсутствует).*

2. *Из полузамкнутого интервала* $(-\infty, x^*\rangle$, опять может быть $x^* = -\infty$ *или* $x^* = +\infty$.

3. *Из счетного множества Q точек, расположенных в промежутке* (x^*, x^{**}), *которое имеет это свойство*: *Если $x \in Q$, то точка x не может быть одновременно предельной точкой множеств* $(-\infty, x)\, Q$, $(x, \infty)\, Q$.

Приведенный пример показывает, что уже в двумерном автономном случае нельзя ограничиться условиями топологического характера, и характер условий, при помощи которых мы могли бы определить структуру этого множества, очень сложный.