Miloslav Jiřina

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ON REGULAR CONDITIONAL PROBABILITIES

MILOSLAV JIŘINA, Praha

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In the paper new sufficient conditions for finite and complete additivity of conditional probabilities are given.

1. Introduction. Several methods have been used in studying the regularity of conditional probabilities, i. e. the fact that, under certain assumptions, conditional probabilities behave like ordinary probability measures (see [1], [2], [3]). For these methods it is essential that the whole σ-algebra be separable. The method of the present paper makes possible the restriction of the assumption of separability to the σ-algebra of conditions. The method depends on the construction of the conditional probability as the limit of a sequence of conditional probabilities which are trivially regular.

In the whole paper, a basic set X, a σ-algebra $\boldsymbol{S}$ of subsets of X and a probability measure π on $\boldsymbol{S}$ are assumed given. The least σ-algebra generated by a system $\boldsymbol{T}$ of subsets of X will be denoted by $\sigma(\boldsymbol{T})$. If $\boldsymbol{T} \subset \boldsymbol{S}$ is a σ-algebra, then $\boldsymbol{N}(\boldsymbol{T})$ will denote the system of all subsets of π-nullsets from $\boldsymbol{T}$, i. e. $M \in \boldsymbol{N}(\boldsymbol{T})$ if and only if there exists a $N \in \boldsymbol{T}$ such that $M \subset N$ and $\pi(N) = 0$. We shall write $\overline{\boldsymbol{T}} = \sigma(\boldsymbol{T} \cup \boldsymbol{N}(\boldsymbol{T}))$ (the completion of $\boldsymbol{T}$).

If $\boldsymbol{T}$ is a σ-algebra, $\boldsymbol{T} \subset \boldsymbol{S}$, f a bounded function on X and $B \in \boldsymbol{T}$, then we shall denote the upper and the lower integral on B with respect to the measure π restricted to the domain $\boldsymbol{T}$ by $(\boldsymbol{T})\overline{\int_B} f(x)\,\mathrm{d}\pi(x)$ and $(\boldsymbol{T})\underline{\int_B} f(x)\,\mathrm{d}\pi(x)$ respectively. More precisely,

$$(\boldsymbol{T})\overline{\int_B} f(x)\,\mathrm{d}\pi(x) = \inf \sum_{i=1}^{n} \pi(E_i) \sup_{x \in E_i} f(x)$$

where the inf is taken over all the finite disjoint and $\boldsymbol{T}$-measurable partitions of B, and similarly for the lower integral. If the upper and lower integrals coincide, we shall denote their common value by $(\boldsymbol{T})\int_B f(x)\,\mathrm{d}\pi(x)$. This is always the case if f is $\overline{\boldsymbol{T}}$-measurable.

Let $\boldsymbol{T}$ be a σ-algebra, $\boldsymbol{T} \subset \boldsymbol{S}$. There exist functions $\pi(.,.)$ on the domain $X \times \boldsymbol{S}$ such that

$$\pi(., A) \text{ is } \overline{\boldsymbol{T}}\text{-measurable for all } A \in \boldsymbol{S}\,, \tag{1.1}$$

$$(\boldsymbol{T})\int_B \pi(x, A)\,\mathrm{d}\pi(x) = \pi(A \cap B) \text{ for all } A \in \boldsymbol{S} \text{ and } B \in \boldsymbol{T}\,. \tag{1.2}$$

The class of all $\pi(.,.)$ which satisfy (1.1) and (1.2) will be called the conditional probability with respect to $\mathbf{T}$[1]) and will be denoted by $\Pi(\mathbf{T})$. The conditional probability $\Pi(\mathbf{T})$ will be called regular if there exists a $\pi(.,.) \in \Pi(\mathbf{T})$ such that $\pi(x,.)$ is a probability measure on $\mathbf{S}$ for every $x \in X$. $\Pi(\mathbf{T})$ will be called semi-regular if there exists $\pi(.,.) \in \Pi(\mathbf{T})$ such that for every $x \in X$

$$\pi(x,.) \text{ is non-negative and finitely additive}, \tag{1.3}$$

$$\pi(x, X) = 1 . \tag{1.4}$$

Let $\mathbf{T}_n$ $(n = 0, 1, \ldots)$ be σ-algebras, $\mathbf{T}_n \subset \mathbf{S}$. We shall say that the conditional probabilities $\Pi(\mathbf{T}_n)$ converge almost surely to $\Pi(\mathbf{T}_0)$, if there exist $\pi_n(.,.) \in \Pi(\mathbf{T}_n)$ $(n = 0, 1, \ldots)$ such that for every $A \in \mathbf{S}$

$$\{x \in X\colon \pi_n(x, A) \not\to \pi_0(x, A)\} \in \mathbf{N}(\mathbf{T}) . \tag{1.5}$$

Evidently, the validity of (1.5) does not depend on the special choice of $\pi_n(.,.) \in \Pi(\mathbf{T}_n)$.

2. Semi-regular conditional probabilities. We first prove a limit theorem.

2.1. *Let $\mathbf{T}_n$ $(n = 0, 1, \ldots)$ be σ-algebras, $\mathbf{T}_n \subset \mathbf{S}$, let $\Pi(\mathbf{T}_n)$ be semi-regular for $n \geqq 1$, and suppose that $\Pi(\mathbf{T}_n)$ converges almost surely to $\Pi(\mathbf{T}_0)$. Then $\Pi(\mathbf{T}_0)$ is semi-regular.*

Proof. There exists, for every $n > 0$, $\pi_n(.,.) \in \Pi(\mathbf{T}_n)$ which satisfies (1.3) and (1.4), and we can always choose $\pi_0(.,.) \in \Pi(\mathbf{T}_0)$ in such a way that $\pi_0(x, X) = 1$ for all $x \in X$. According to the assumption we have

$$N(A) = \{x \in X\colon \pi_n(x, A) \not\to \pi_0(x, A)\} \in \mathbf{N}(\mathbf{T}) \text{ for every } A \in \mathbf{S} .$$

For every $x \in X$ we shall denote the system of all $A \in \mathbf{S}$ for which $\pi_n(x, A) \to \pi_0(x, A)$ by $\mathbf{M}_1(x)$; let us write $\mathbf{M}_2(x) = \mathbf{S} - \mathbf{M}_1(x)$. Since (1.3) and (1.4) hold for $\pi_n(.,.)$ with $n > 0$, $\pi_n(x,.)$ is a partial measure on $\mathbf{M}_1(x)$ for every $x \in X$ (in the sense of [4], def. 1.6 — see also [4], theorem 1.9 (II)). It follows that $\pi_0(x,.)$, as the limit of $\pi_n(x,.)$ on $\mathbf{M}_1(x)$, is a partial measure on $\mathbf{M}_1(x)$ also. But then, for each $x \in X$, there exists a finitely additive function $\lambda(x,.)$ on $\mathbf{S}$ such that $\lambda(x, A) = \pi_0(x, A)$ for every $A \in \mathbf{M}_1(x)$ (see [4], theorem 1.21), and clearly $\lambda(x, X) = \pi_0(x, X) = 1$. Further, we have $N_1(A) = \{x \in X\colon \lambda(x, A) \neq \pi_0(x, A)\} \subset N(A)$. From this it follows that $\lambda(.,.) \in \Pi(\mathbf{T})$, and consequently $\Pi(\mathbf{T}_0)$ is semiregular.

The following theorem is an easy consequence of 2.1.

2.2. *Suppose there exists a countable basis for the σ-algebra $\mathbf{T}$, $\mathbf{T} \subset \mathbf{S}$. Then the conditional probability $\Pi(\mathbf{T})$ is semi-regular.*

[1]) The assumption (1.1) is somewhat less strict than that of [1], where measurability with respect to the non-completed σ-algebra is required; this weaker definition appears essential for the validity of the theorems of the present paper.

Proof. Let $\{E_1, E_2, \ldots\}$ be the countable basis for $\boldsymbol{T}$ and set $\boldsymbol{T}_n = \sigma(\{E_1, \ldots, E_n\})$. The conditional probabilities $\Pi(\boldsymbol{T}_n)$ are semi-regular, and it is well known that $\Pi(\boldsymbol{T}_n)$ converge almost surely to $\Pi(\boldsymbol{T})$ (see [5], Chap. VII. § 8). The semi-regularity of $\Pi(\boldsymbol{T})$ now follows from 2.1.

3. Regular conditional probabilities. There exists a well-known example due to Dieudonné (see e. g. [6], sec. 48, ex. 4) of a conditional probability with respect to a σ-algebra with a countable basis which is not regular. According to 2.2, this conditional probability is semi-regular and so we have an example of a semi-regular conditional probability which is not regular. The example shows at the same time that the theorem 2.1 does not hold if the world ,,semi-regular" is replaced by ,,regular". In fact, the conditional probability in this example is not regular, although it is — for the same reasons as in the proof of 2.1 — a limit of regular probabilities.

All known theorems on regular probabilities involve assumptions of a topological character. We shall use similar assumptions, but — to be able to cover several special cases — we shall formulate them in an abstract manner, without supposing that X is a topological space.

A system $\boldsymbol{C}$ of subsets of X will be called a C-system, if the following four conditions are satisfied.

C 1. $\emptyset \in \boldsymbol{C}$.

C 2. $\boldsymbol{C}$ is finitely additive, i. e. $C_i \in \boldsymbol{C}$ $(i = 1, \ldots, n)$ implies $\bigcup_{i=1}^{n} C_i \in \boldsymbol{C}$.

C 3. $\boldsymbol{C}$ is countably compact, i. e. if $C_i \in \boldsymbol{C}$ and $\bigcap_{i=1}^{n} C_i \neq \emptyset$ for all n, then $\bigcap_{i=0}^{\infty} C_i \neq \emptyset$.

C 4. For each n and arbitrary $C_i \in \boldsymbol{C}$ $(i = 0, 1, \ldots, n)$ such that $\bigcap_{i=1}^{n} C_i = \emptyset$ there exist $D_i \in \boldsymbol{C}$ $(i = 1, \ldots, n)$ such that $C_0 = \bigcup_{i=1}^{n} D_i$, $C_i \cap D_i = \emptyset$ $(i = 1, \ldots, \ldots, n)$.

Let $\boldsymbol{C}$ be a C-system and let us write

$$\boldsymbol{G}_0 = \{X - C : C \in \boldsymbol{C}\}, \quad \boldsymbol{G} = \{\bigcup_{i=1}^{\infty} G_i : G_i \in \boldsymbol{G}_0 \ (i = 1, 2, \ldots)\}.$$

We then have

G 1. If $G \in \boldsymbol{G}$, $C \in \boldsymbol{C}$, then $G \cap (X - C) \in \boldsymbol{G}$.

G 2. If $C \in \boldsymbol{C}$, $G_i \in \boldsymbol{G}$ and $C \subset \bigcup_{i=1}^{\infty} G_i$, then there exists an n such that $C \subset \bigcup_{i=1}^{n} G_i$.

G 3. For every n it is true that if $G_i \in \boldsymbol{G}$ $(i = 1, 2, \ldots, n)$ and $C \in \boldsymbol{C}$ are such that $C \subset \bigcup_{i=1}^{n} G_i$, then there exist $D_i \in \boldsymbol{C}$ such that $D_i \subset G_i$, $C = \bigcup_{i=1}^{n} D_i$.

The probability measure π on $\mathbf{C}$ will be called compact with respect to $\mathbf{C}$, if $\mathbf{C} \subset \mathbf{S}$ and $\pi(E) = \sup\limits_{C \subset E,\, C \in \mathbf{C}} \pi(C)$ for every $E \in \mathbf{S}$.

The following theorem will be useful in the sequel.

3.1. *Let $\mathbf{C}$ be a C-system with $X \in \mathbf{C}$ and let a monotone, finitely subadditive and finitely additive set function λ on $\mathbf{C}$ be given such that $\lambda(\emptyset) = 0$. Then there exists a measure μ on $\sigma(\mathbf{C})$ such that*

$$\mu(X) = \lambda(X)\,, \tag{3.1}$$

$$\mu(C) \geqq \lambda(C) \quad \text{for all} \quad C \in \mathbf{C}\,, \tag{3.2}$$

$$\mu(X - C) \geqq \lambda(D) \quad \text{for all} \quad C,\, D \in \mathbf{C},\, C \cap D = \emptyset\,. \tag{3.3}$$

Proof. The proof will be sketched only, because 3.1 is an easy generalisation of the known theorem on the extension of a content to a measure. Let $\mathbf{G}$ have the same meaning as above and let us define

$$\lambda_*(G) = \sup_{C \subset G,\, C \in \mathbf{C}} \lambda(C) \tag{3.4}$$

for all $G \in \mathbf{G}$ and $\mu^*(E) = \inf\limits_{G \supset E,\, G \in \mathbf{G}} \lambda_*(G)$ for all $E \subset X$. Clearly

$$\mu^*(X) = \lambda_*(X) = \lambda(X)\,, \tag{3.5}$$

$$\mu^*(G) = \lambda^*(G) \quad \text{for} \quad G \in \mathbf{G}\,, \tag{3.6}$$

$$\mu^*(C) \geqq \lambda(C) \quad \text{for} \quad C \in \mathbf{C}\,. \tag{3.7}$$

Using G 2 and G 3 we can prove that μ^* is an outer measure by the same method as in the proofs of theorems 1 and 2 of [6], section 53. G 1 then implies the measurability with respect to μ^* of all sets from $\sigma(\mathbf{C})$ (see [6], section 56, proof of theorems 4 and 5), and we may set $\mu(E) = \mu^*(E)$ for all $E \in \sigma(\mathbf{C})$. The relations (3.1) (3.3) follow from (3.4)—(3.7).

We can now prove the theorem on the regularity property.

3.2. *Suppose that the following two assumptions hold for $\mathbf{C} \subset \mathbf{S}$,*

$$\mathbf{C} \textit{ is a C-system}\,, \tag{3.8}$$

$$C \cap \mathbf{S} = \sigma(\{D:\ D \subset C,\ D \in \mathbf{C}\}) \textit{ for all } C \in \mathbf{C}\,, \tag{3.9}$$

and let the probability measure π be compact with respect to $\mathbf{C}$. Then for an arbitrary σ-algebra $\mathbf{T} \subset \mathbf{S}$ the conditional probability $\Pi(\mathbf{T})$ is regular if and only if it is semi-regular.

Note. (3.9) is always satisfied if $\mathbf{S} = \sigma(\mathbf{C})$. Both (3.8) and (3.9) are satisfied e. g. in the following two cases:

a) X is a Hausdorff space, $\mathbf{C}$ the system of all compact sets of X and $\mathbf{S} = \sigma(\mathbf{C})$.

b) X is an arbitrary topological space, $\mathbf{C}$ the system of all countably compact sets which are representable in the form $\{x \in X: f(x) = 0\}$ where f is a continuous function on X and $\mathbf{S} = \sigma(\mathbf{C})$.[2])

All assumptions of the theorem (including the compactness of π) are satisfied if X is a locally compact Hausdorff space, $\mathbf{C}$ the system of all compact sets which are G_δ and $\mathbf{S}$ the system of all Baire sets, i. e. $\mathbf{S} = \sigma(\mathbf{C})$.

Proof of 3.2. It is sufficient to prove that semi-regularity implies regularity. Suppose (1.3) and (1.4) hold for some $\pi_0(.,.) \in \Pi(\mathbf{T})$. Since π is compact, there exist $C_n \in \mathbf{C}$ such that $C_n \subset C_{n+1}$ and $\pi(\bigcup_{n=1}^{\infty} C_n) = 1$. Let us write $\mathbf{C}_n = \{C \in \mathbf{C} : C \subset C_n\}$, $\mathbf{S}_n = \sigma(C_n)$. For every n, $x \in X$ and $C \in \mathbf{C}_n$, we define $\lambda_n(x, C) = \pi_0(x, C)$. Since $\pi_0(.,.) \in \Pi(\mathbf{T})$, $\lambda_n(., C)$ is $\bar{\mathbf{T}}$-measurable and

$$(\mathbf{T}) \int_B \lambda_n(x, C) \, \mathrm{d}\pi(x) = \pi(B \cap C) \quad \text{for all} \quad C \in \mathbf{C}_n\,, \quad B \in \mathbf{T}\,. \tag{3.10}$$

From the additivity of $\pi_0(x, .)$ on $\mathbf{S}$ it follows that, for every $x \in X$, $\lambda_n(x, .)$ satisfies all assumptions of the theorem 3.1 if we replace X and $\mathbf{C}$ by C_n and $\mathbf{C}_n$ respectively. Consequently there exists for every $x \in X$ a measure $\mu_n(x, .)$ on $\mathbf{S}_n$ such that

$$\mu_n(x, C_n) = \lambda_n(x, C_n)\,, \tag{3.11}$$

$$\mu_n(x, C) \geqq \lambda_n(x, C) \quad \text{if} \quad C \in \mathbf{C}_n\,, \tag{3.12}$$

$$\mu_n(x, C_n - C) \geqq \lambda_n(x, D) \quad \text{if} \quad C, D \in \mathbf{C}_n\,, \quad D \subset C_n - C\,. \tag{3.13}$$

From (3.10), (3.12) and (3.13) we deduce

$$(\mathbf{T}) \overline{\int_X} \mu_n(x, C) \, \mathrm{d}\pi(x) \geqq (\mathbf{T}) \int_X \lambda_n(x, C) \, \mathrm{d}\pi(x) = \pi(C) \quad \text{for all} \quad C \in \mathbf{C}_n\,, \tag{3.15}$$

$$(\mathbf{T}) \underline{\int_X} \mu_n(x, C_n - C) \, \mathrm{d}\pi(x) \geqq (\mathbf{T}) \int_X \lambda_n(x, D) \, \mathrm{d}\pi(x) = \pi(D) \tag{3.14}$$

for all $C, D \in \mathbf{C}_n$, $D \subset C_n - C$. Since π is compact, we have by (3.15)

$$(T) \underline{\int_X} \mu_n(x, C_n - C) \, \mathrm{d}\pi(x) \geqq \sup_{D \in \mathbf{C}_n\, D \subset C_n - C} \pi(D) = \pi(C_n - C)\,. \tag{3.16}$$

Suppose that there exists a $C \in \mathbf{C}_n$ for which strong inequality holds in (3.14). Since $\mu_n(x, .)$ is additive, we have by (1.1), (3.14) and (3.16)

$$(\mathbf{T}) \overline{\int_X} \mu_n(x, C_n) \, \mathrm{d}\pi(x) \geqq (\mathbf{T}) \overline{\int_X} \mu_n(x, C) \, \mathrm{d}\pi(x) + (\mathbf{T}) \underline{\int_X} \mu_n(x, C_n - C) \, \mathrm{d}\pi(x) >$$

$$> \pi(C) + \pi(C_n - C) = \pi(C_n)\,.$$

But this contradicts (3.10) and (3.11), and we have

$$(\mathbf{T}) \overline{\int_X} \mu_n(x, C_n) \, \mathrm{d}\pi(x) = (\mathbf{T}) \int_X \lambda_n(x, C) \, \mathrm{d}\pi(x)$$

for all $C \in \mathbf{C}_n$. Hence we deduce by (3.12) that

$$\{x \in X: \mu_n(x, C) \neq \lambda_n(x, C)\} \in \mathbf{N}(\mathbf{T}) \quad \text{for every} \quad C \in \mathbf{C}_n\,.$$

[2]) The example b) was communicated to the author by P. Mandl and this influenced the final formulation of the assumptions C 1—C 4.

This and (3.10) prove that $\mu_n(x, C)$ is $\bar{\mathbf{T}}$-measurable and

$$(\mathbf{T})\int_B \mu_n(x, C)\,\mathrm{d}\pi(x) = \pi(B \cap C)$$

for $C \in \mathbf{C}_n$, $B \in \mathbf{T}$. Since $\mu_n(x, .)$ is a measure on $\mathbf{S}_n$, $\mathbf{S}_n = \sigma(\mathbf{C}_n)$ and $\mathbf{C}_n$ is finitely additive, we may prove by usual methods of measure theory that

$$\mu_n(., A) \quad \text{is } \bar{\mathbf{T}}\text{-measurable for all} \quad A \in \mathbf{S}_n\,, \tag{3.17}$$

$$(\mathbf{T})\int_B \mu_n(x, A)\,\mathrm{d}\pi(x) = \pi(A \cap B) \quad \text{for all} \quad A \in \mathbf{S}_n\,, \quad B \in \mathbf{T}\,. \tag{3.18}$$

From (3.9) we have $\mathbf{S}_n = C_n \cap \mathbf{S}$, and we can consequently define (with $C_0 = \emptyset$)

$$\mu(x, A) = \sum_{n=1}^{\infty} \mu_n(x, A \cap (C_n - C_{n-1}))$$

for all $x \in X$, $A \in \mathbf{S}$. Clearly $\mu(., A)$ is $\bar{\mathbf{T}}$-measurable, $\mu(x, .)$ is a measure on S and it follows by (3.18)

$$(\mathbf{T})\int_B \mu(x, A)\,\mathrm{d}\pi(x) = \sum_{n=1}^{\infty} (\mathbf{T})\int_B \mu_n(x, A \cap (C_n - C_{n-1}))\,\mathrm{d}\pi(x) =$$

$$= \sum_{n=1}^{\infty} \pi(A \cap B \cap (C_n - C_{n-1})) = \pi(A \cap B \cap \bigcup_{n=1}^{\infty} C_n) = \pi(A \cap B)$$

for all $A \in \mathbf{S}$, $B \in \mathbf{T}$. Hence $N = \{x \in X\colon \mu(x, X) \neq 1\} \in \mathbf{N}(\mathbf{T})$ and if we set $\pi(x, A) = \mu(x, A)$ for x non $\in N$ and $\pi(x, A) = \pi(A)$ for $x \in N$, we have $\pi(., .) \in \Pi(\mathbf{T})$ and $\pi(x, .)$ is a probability measure for every $x \in X$.

From 2.2 and 3.2 it follows that

3.3. *If all the assumptions of* 3.2 *are satisfied and if the σ-algebra $\mathbf{T}$ has a countable basis, then $\Pi(\mathbf{T})$ is regular.*

Added in proof (July 8. 1959): Using a pointwise convergence theorem for martingales with a special index set we can prove that the assumption of countable basis for $\mathbf{T}$ in the theorems 2.2 and 3.3 may be dropped. Some details will be published in a short note in one of the subsequent issues of this journal. The theorem on martingales mentioned above has been communicated to the author hy Prof. K. Krickeberg.

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Резюме

РЕГУЛЯРНЫЕ УСЛОВНЫЕ ВЕРОЯТНОСТИ

МИЛОСЛАВ ИРЖИНА (Miloslav Jiřina), Прага

(Поступило в редакцию 5/IX 1958 г.)

Для заданного поля вероятностей $(X, \mathbf{S}, \pi)$ и заданной σ-алгебры $\mathbf{T} \subset \mathbf{S}$ обозначим через $\bar{\mathbf{T}}$ σ-алгебру всех подмножеств X, которые отличаются от множеств из $\mathbf{T}$ на некоторое подмножество π-нулевого множества из $\mathbf{T}$. Условной вероятностью $\Pi(\mathbf{T})$ (относительно σ-алгебры $\mathbf{T}$) мы будем в этой статье называть систему всех функций $\pi(.,.)$, определенных на $X \times \mathbf{S}$, $\bar{\mathbf{T}}$-измеримых относительно x и удовлетворяющих условию $\int_B \pi(x, A)\, \mathrm{d}\pi(x) = \pi(A \cap B)$ для всех $A \in \mathbf{S}$ и $B \in \mathbf{T}$. Условная вероятность $\Pi(\mathbf{T})$ называется полурегулярной, если существует $\pi(.,.) \in \Pi(\mathbf{T})$ такая, что $\pi(x, X) = 1$ для всех x и что $\pi(x, .)$ конечно — аддитивна на $\mathbf{S}$ для всех x. Условная вероятность называется регулярной, если существует $\pi(.,.) \in \Pi(\mathbf{T})$ такая, что $\pi(x, .)$ является вероятностной мерой на $\mathbf{S}$ для всех x.

Главные результаты:

Теорема 2.2. *Если σ-алгебра $\mathbf{T}$ обладает счетным базисом, по $\Pi(\mathbf{T})$ всегда полурегулярна.*

Теорема 3.2. *Пусть система подмножеств $\mathbf{C} \subset \mathbf{S}$ выполняет следующие условия:*

а) *$C \cap \mathbf{S} = \sigma(\{D \in \mathbf{C} : D \subset C\})$ для всякого $C \in \mathbf{C}$.*

б) *$\mathbf{C}$ замкнута относительно строения конечных соединений.*

в) *$\mathbf{C}$ счетно компактна, т. е. для всякой последовательности $C_i \in \mathbf{C}$ такой, что $\bigcap_{i=0}^{n} C_i \neq 0$ (для всех n), имеет место $\bigcap_{i=0}^{\infty} C_i \neq 0$.*

г) *Для всякого n и произвольных $C_i \in \mathbf{C}$ $(i = 0, 1, \ldots, n)$ таких, что $\bigcap_{i=0}^{n} C_i = \emptyset$, существуют $D_i \in \mathbf{C}$ $(i = 1, \ldots, n)$ такие, что $C_0 = \bigcup_{i=1}^{n} D_i$ и $D_i \cap C_i = 0$ $(i = 1, \ldots, n)$.*

д) *Вероятность π компакна осносутельно $\mathbf{C}$, т. е. $\pi(E)$ $\sup_{C \subset E,\, C \in \mathbf{C}} \pi(C)$ для всех $E \in \mathbf{S}$.*

Тогда для любой σ-алгебры $\mathbf{T} \subset \mathbf{S}$ условная вероятность $\Pi(\mathbf{T})$ регулярна тогда и только тогда, когда она полурегулярна.

Теорема 3.3. *Если выполняются все условия предыдущей теоремы и если $\mathbf{T}$ обладает счетным базисом, то условная вероятность $\Pi(\mathbf{T})$ регулярна.*