Endre Makai

On the principal frequency of a convex membrane and related problems

Czechoslovak Mathematical Journal, Vol. 9 (1959), No. 1, 66–70

Persistent URL: http://dml.cz/dmlcz/100341

ON THE PRINCIPAL FREQUENCY OF A CONVEX MEMBRANE AND RELATED PROBLEMS

E. MAKAI, Budapest

(Received October 8, 1957)

An upper bound of the principal frequency of a convex membrane is established. Neglecting a factor depending only on purely physical quantities it contains only the area and perimeter of the membrane.

1. In his book Patterns of Plausible Inference[1]) G. PÓLYA presents many numerical data about membranes of diverse shapes, giving the length L of their perimeters, their areas A and the pitches Λ of their principal tones (principal frequency of their vibrations if the boundary is fixed, or the minimum of the quotient

$$[\iint_D (\nabla u)^2 \, d\sigma / \iint_D u^2 \, d\sigma]^{\frac{1}{2}} \tag{1}$$

where u may be any continuous function with piecewise continuous first derivates vanishing on the boundary B of D, that is on the boundary of the membrane). From these data it is possible to calculate the quantity c defined by the equation $\Lambda = c \,.\, L/A$ and it is found that in every instance quoted there c is a number between 1 and about 1.3. Other data show that if ε is an arbitrarily little positive quantity c may be as high as $\pi/2 - \varepsilon$. (In the case if a very elongated rectangle.)

Now we will prove the following

Theorem 1. *Let D be a convex plane domain of area A, perimeter L. Then the square root of the least eigenvalue Λ^2 of the differential equation* $\Delta u + \lambda u = 0$ (Δ is the two dimensional Laplace operator, u vanishing on the boundary of D) *or in other words the minimum Λ of the quotient* (1) (u continuous, its first derivatives piecewise continuous in the interior of D, u vanishing on the boundary) *satisfies the inequality $\Lambda \leqq \sqrt{3} L/A$.*

This statement will be proved if we can find a function u satisfying the conditions of the variational problem, for which the expression (1) is not greater

[1]) Princeton University Press 1954, pp. 9 and 11.

than $\sqrt{3}L/A$. It will be seen that such a function is the point function $d(P)$ which is the distance of the point P from that point of the boundary B which is nearest to P:

$$d(P) = \min_{Q \in B} \overline{PQ} \ .$$

2. Clearly it is enough to prove the statement for the case when the boundary is a polygon.[2]) The sides of the polygon (and their length) will be denoted by $a_1, a_2, \ldots, a_n$. This polygon can be divided into domains $D_1, D_2, \ldots, D_n$ where the interior of D_i consists of the set of those points of D which are nearer to the side a_i then to any point of any other side of the polygon. D_i is included in the triangular domain bounded by the side a_i and the two bisectors of the angles formed by a_i and the two adjacent sides.

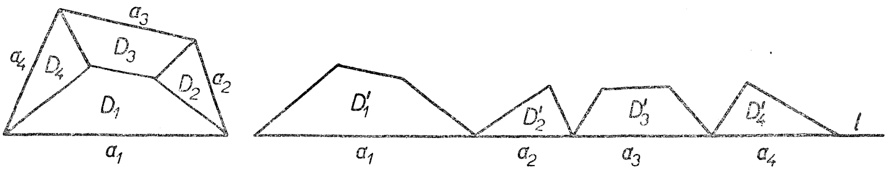

Fig. 1

The function $d(P)$ vanishes on the boundary, it is continuous and its first derivatives are continuous in the interior of the domains D_i, so it is an admissible function from the point of view of the variational problem. Moreover in the interior of D_i, $(\nabla d)^2 = 1$ and so $\iint_D (\nabla d)^2 \, d\sigma = A$.

The integral $\iint_{D_i} d^2 \, d\sigma$ too has a simple meaning. It represents the axial moment of inertia of the domain D_i with respect to the side a_i of the polygon.

Imagine now the domain D being cut up along the lines separating the subdomains D_i. Let these domains be rearranged so, that the sides a_i lie on a common straight line l. (See fig. 1). After the rearrangement the domain D_i will be called D_i'. We imagine that the domains D_i' are all on one side of the line l and the „bases" a_i cover a coherent piece of length L on the line l. It is clear that the domains D_i' are not overlapping.

The integral $\iint_D d^2 \, d\sigma$ may now be interpreted as the sum of the axial moments of inertia of the domains D_i' with respect to l. This is certainly greater than the axial moment of inertia of a rectangle of base L, area A with respect to its side L:

$$\iint_D d^2 \, d\sigma > \tfrac{1}{3} \frac{A^3}{L^2} \, .$$

[2]) Cf. Courant-Hilbert: Methoden der math. Physik vol. *1*, second ed., pp. 365—366.

We conclude that

$$\Lambda^2 \leqq \frac{\iint_D (\nabla d)^2 \, \mathrm{d}\sigma}{\iint_D d^2 \, \mathrm{d}\sigma} < \frac{A}{\frac{1}{3}\frac{A^3}{L^2}} = 3\left(\frac{L}{A}\right)^2 .\ ^{3)}$$

3. G. Pólya and G. Szegö[4]) gave another upper estimate of Λ. According to this

$$\Lambda^2 \leqq j^2 B_a (2A)^{-1} \tag{2}$$

when the domain is star-shaped with respect to some interior point a. Here $j = 2.40\ldots$ is the first positive root of the Bessel function $J_0(x)$ and the quantity B_a is equal to $\int_B h^{-1} \, \mathrm{d}s$. In this formula h denotes the length of the perpendicular drawn from a to the tangent at a variable point of B where $\mathrm{d}s$ is the line element and the integral is extended over the whole boundary B of D.

The inequality (2) seems to be a sharper estimate than that of Theorem 1. However in special cases Theorem 1 may be refined so that it yields estimates hardly differing from that of formula (2). So if D is a convex polygon into which a circle can be inscribed in the elementary sense and a is the centre of the inscribed circle, ϱ is its radius, then from (2)

$$\Lambda \leqq j\sqrt{\frac{L}{2\varrho A}} = j\sqrt{\frac{L^2}{2\varrho L A}} = j\sqrt{\frac{L^2}{4A^2}} \doteq 1.20\,\frac{L}{A} .$$

On the other hand in this example the expression (1) can be computed explicitly if we put $u = d(P)$. For now each of the domains D_i is a triangle and

$$\sum \iint_{D_i} d^2 \, \mathrm{d}\sigma = \sum \frac{1}{2}\,\varrho a_i \cdot \frac{\varrho^2}{6} = A \cdot \frac{\varrho^2}{6} .$$

From this

$$\Lambda^2 \leqq \frac{\iint_D (\nabla d)^2 \, \mathrm{d}\sigma}{\iint_D d^2 \, \mathrm{d}\sigma} = \frac{6}{\varrho^2} = \frac{6}{4}\frac{L^2}{A^2} \doteq \left(1.22\,\frac{L}{A}\right)^2 . \tag{3}$$

4. The function $d(P)$ may be used also for calculating a lower estimate of the *torsional rigidity* P of a prism with cross section D. We use the term torsional rigidity in accordance with Pólya and Szegö[5]) namely that P is the maximum of

---

[3]) In exactly the same way one may prove the three dimensional analogy of this theorem:

Theorem 2. *The least eigenvalue* Λ^2 *of the differential equation* $\Delta u + \lambda u = 0$ (Δ is the three dimensional Laplace operator, u vanishing on the surface S of a convex body of volume V), *or the minimum of* $(\iiint_V (\nabla u)^2 \, \mathrm{d}V / \iiint_V u^2 \, \mathrm{d}V)^{\frac{1}{2}}$ (u continuous, its first derivatives piecewise continuous in the interior of V, u vanishing on the surface) *satisfies the inequality* $\Lambda \leqq \sqrt{3}S/V$.

[4]) Pólya-Szegö: Isoperimetric inequalities in Mathematical Physics, Princeton University Press, 1951, pp. 14—15 and 91—94.

[5]) Pólya-Szegö, l. c. p. 87.

$$4\left(\iint_D u \, \mathrm{d}\sigma\right)^2 \Big/ \iint_D (\nabla u)^2 \, \mathrm{d}\sigma \tag{4}$$

where u is subjected to the same conditions as in (1).

Now in the case treated above $\iint_D d(P)\,\mathrm{d}\sigma$ is greater than the momentum of a rectangle with base L, area A, with respect to its side L:

$$\iint_D d(P)\,\mathrm{d}\sigma > A \cdot \frac{A}{2L}$$

and from this follows

Theorem 3. *The torsional rigidity of a prism with convex cross section is not less than A^3/L^2, where L is the length of the perimeter and A is the area of the cross section.*

5. It is easy to obtain a rough *lover estimate for the principal frequency Λ* of a convex membrane in the form $\Lambda > \gamma \dfrac{L}{A}$ with the help of Pólya-Szégö's inclusion lemma.[6]) This lemma states that about any convex domain D of area A one can circumscribe a rectangle R with sides a, b having an area A_R such that $A_R \leqq 2A$. Now if L_R is the perimeter and Λ_R the principal frequency belonging to R, then it is well known that $\Lambda_R \leqq \Lambda$. On the other hand

$$\Lambda_R = \pi\sqrt{a^{-2} + b^{-2}} > 2(a^{-1}b^{-1}) = \frac{L_R}{A_R}$$

furthermore $L_R > L$ [7]) and so for any convex domain

$$\Lambda \geqq \Lambda_R > \frac{L_R}{A_R} = \frac{1}{2} \cdot \frac{L}{A} \, .$$

Резюме

ОБ ОСНОВНОЙ ЧАСТОТЕ КОЛЕБАНИЙ ВЫПУКЛОЙ МЕМБРАНЫ И О РОДСТВЕННЫХ ЗАДАЧАХ

Э. МАКАИ (E. Makai), Будапешт

(Поступило в редакцию 8/X 1957 г.)

Рассмотрим функционал (1), определенный на множестве функций $u(P)$, непрерывных в выпуклой плоской области D, имеющих там кусочно-непрерывные частные производные первого порядка и принимающих на

---

[6])Pólya-Szegö, l. c. p. 109.

[7]) See e. g. Rouché-Comberousse: Traité de Géométrie, 7th ed. vol. 1, p. 26, Gauthier-Villars, Paris, 1900.

границе B области значение нуль. Пользуясь функцией $u(P) = \min\limits_{Q \in B} \overline{PQ}$, автор доказывает теорему:

Минимум Λ функционала (1) *не превышает величины* $\sqrt{3}L/A$, *где A — площадь области D, B — длина ее границы. Если D — многоугольник, в который можно вписать окружность, то эту оценку можно улучшить* (*неравенство* (3)).

Аналогично доказывается, что функционал, образующий правую часть (4), не принимает значений, меньших A^3/L^2. Отсюда следуют ограничения для т. наз. основной частоты Λ мембраны, натянутой на D, и для т. наз. жесткости при кручении P стержня поперечного сечения D.