Czechoslovak Mathematical Journal

Ladislav Rieger

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Czechoslovak Mathematical Journal, Vol. 9 (1959), No. 1, 1–49

Persistent URL: http://dml.cz/dmlcz/100338

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ЧЕХОСЛОВАЦКИЙ МАТЕМАТИЧЕСКИЙ ЖУРНАЛ
Математический институт Чехословацкой Академии наук
Т. *9* (*84*) ПРАГА 24. III. 1959 г., No 1

A CONTRIBUTION TO GÖDEL'S AXIOMATIC SET THEORY, II

(*Basic notions and application of the theory of dyadic rings of the set theoretical type*)

LADISLAV RIEGER, Praha

(Received February 22, 1958)

The present free continuation of the paper [I][1]) (under the same title) deals with a new kind of "arithmetically" constructed models of the axiomatic set theory of Gödel (see [G]), called dyadic models. As a first application of these methods, we obtain two particular nonnormal models of all Gödel's axioms sub A—sub E of [G] except the axiom C 1 (of infinity), this axiom being replaced by its contrary non C 1 (the s. c. axiom of finity), and of the following properties: In both models, the set of "finite ordinals" of the model is of power $\aleph_1$ and the set of "classes" of the model is also of power $\aleph_1$ in the first — and of power $2^{\aleph_1}$ in the second case.

Contents: 1. Introductory remarks. 2. Dyadic rings and their pseudoperfect immediate extensions. 3. Dyadic rings of the set theoretical type (*s-t*-rings). 4. Skolemian extensions of countable *s-t*-rings.

1. Introductory remarks

The present paper is a free continuation of the author's paper [I][2]) (this Journal, under the same title, see the literature at the end).[3]) The knowledge of the §§ 1 and 2 of [I] is very recommended for a detailed understanding but not necessary for a global one; the same is true of the fundamental treatise [G] of K. Gödel because of its close relation to the present paper. (See the introduction to [I] as remaining valid for the present paper.)

[1]) See the literature at the end.

[2]) I take this opportunity of correcting some omissions in the paper [I]; a list of corrections is added at end of the present paper.

[3]) The main results, though in an imperfect form, have been communicated by the author at the session of November 26th, 1956 of the Mathematical Society of Prague. The elaborated theory has been the subject of a lecture of the author held in the winter semester of the school year of 1957—8 at the University of Prague.

It has been announced in [I] that we shall study some of the s. c. incomplete index models (see § 3 of [I] for the notion) of the axiomatic set theory of [G]. This indeed remains our essential task; and the present particular kind of our models of the theory of finite sets, called the dyadic models, serves this purpose.

Nevertheless the author hopes to have made the paper selfcontained (in reducing its dependence on [I] to a minimum) and, first of all, he wishes to call the attention of the interested readers to the "arithmetical" method rather then to the present set theoretical result itself.

Let us introduce the reader to the relatively simple main idea of this method, i. e. of the method of dyadic (algebraic) rings and of their arithmetic, explaining them on the basis of some elementary facts in Hensel's p-adic numbers and p-adic (exponential) valuations, with $p = 2$. (Comp. e. g. [W I], § 73 or [Sch], Chap. 1.)

Let us recall the well known needed basic facts.

The integer n is called the 2-adic (= dyadic) exponential value of the integer $m \neq 0$, written $W(m) = n$, if $m = 2^n(2k + 1)$. Setting $W(p/q) = W(p) - W(q)$ for any rational $x = p/q$ we extend the dyadic valuation to the field F of rationals. We have

$$W(x + y) \geqq \min (W(x), W(y)) , \quad W(x \,.\, y) = W(x) + W(y)$$

and $W(x)$ is defined whenever $x \neq 0$. Now we metrize F by the distance function $\varrho(x, y) = a^{-W(x-y)}$ requiring in addition that $\varrho(x, x) = 0$, where a is any real constant greater than 1. F then becomes a continuous field, and we can form (in the usual way, i. e. using fundamental Cauchy sequences) its perfect completion, i. e. the s. c. field $\overline{F}$ of Hensel's 2-adic (= dyadic) numbers. Now it can be proved that every Hensel's dyadic number x possesses an unique normal "dyadic" expansion of the form $x = \sum_{i=k}^{\infty} c_i 2^i$ with a fixed integer k (positive or not) and with $c_i = 0$ or $c_i = 1$. (The expansion converges in the sense of our metric ϱ, of course.) This dyadic expansion specializes to the usual finite dyadic expansion if (and only if) x is a nonnegative integer; in this case $k \geqq 0$, of course. But the expansion remains infinite for negative integers; e. g. $-1 = 1 + 2 + 2^2 + 2^3 + \ldots$ (in the sense of the dyadic convergence), though k is also nonnegative.

Now, Hensel's dyadic integral numbers are defined as dyadic numbers with nonnegative k; these integral dyadic numbers form an integrity domain including the ring of integers[4]) and included in the s. c. valuation ring (of the dyadic valuation of $\overline{F}$), i. e. in the ring of dyadic numbers whose dyadic value (as extended from F to $\overline{F}$) is nonnegative.

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[4]) But also containing some rationals, e. g. $1 + 2^2 + 2^4 + \ldots + 2^{2k} + \ldots = \frac{1}{1 - 2^2} = -\frac{1}{3}$ in the sense of the dyadic convergence.

And finally, take the dyadic integral numbers for "classes", the nonnegative integers for "sets" and define the "to belong" relation as follows: $x \epsilon^* y$ if x is a nonnegative integer and 2^x with $c_x = 1$ occurs in the dyadic expansion of the integral dyadic number y.

It can be proved (comp. the general theorem I of § 2) that this interpretation of the primitive notions of Gödel's theory of finite sets is correct (see § 1 of [I]) in the sense that all the axioms of [G] except C 1 (the axiom of infinity) are valid in our model; instead of C 1, its contrary non C 1 (to be called the axiom of finity) is satisfied. (Note that e. g. the integer -1 indeed is the "universal class" of the model.)

We obtain in this way the simplest dyadic model, the s. c. essentially normal dyadic model of the s. c. axiomatic theory of finite sets of Gödel. The expression "essentially normal" here means that the relation ϵ^* is isomorphic to the partialized basic relation ϵ in the following sense: Put in the known manner (see A. Mostowski [M II]) $f'\emptyset = \emptyset$, $f'(n + 1) = \mathbf{P}(f'n)$[5]) $(n \epsilon \omega_0)$.

Then the field of the relation ϵ is to be reduced to the set $\mathbf{P}(f''\omega_0)$ and its domain is to be reduced to the set $f''\omega_0$; this is the s. c. normal (inner) model of the theory of finite sets of Gödel — and the isomorphism between the so partialized ϵ and ϵ^* can be easily constructed recursively by theorem 3 of Mostowski [M II].

Our task now is to generalize the construction of the mentioned particular dyadic model in order to obtain new essentially nonnormal and, moreover, uncountable models of the axiomatic theory of finite sets of Gödel. (The question of the existence of such models for the general set theory has been answered in the negative.)

For this purpose, we shall exhibit some basic arithmetical properties of the dyadic valuation of the ring of integers as well as of the overring of Hensel's dyadic integral numbers, which suffice to define the "to belong" relation analogously to the just mentioned ϵ^*-relation — in a purely arithmetical way. We thus get the "axiomatically" characterised notion of the s. c. $s - t$-rings, i. e. of the s. c. dyadic rings of the set theoretical type, as well as their s. c. pseudoperfect and weakly pseudoperfect immediate extension (as an instance of essentially known notions of the General valuation theory). The elaboration of some basic arithmetics of the mentioned rings is necessary in order to reach the main theorem I, which gives the desired general kind of s. c. dyadic models (of the axiomatic theory of finite sets). This is the main content of § 2 and § 3. In § 4, we shall give a constructive extension method for the countable $s - t$-rings as based on an idea due to T. Skolem (see [Sk]).

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[5]) ‚$\mathbf{P}$' means the potency set, $f'x$ is the value of f at x, $f''x$ is the set of values of f at x, $\emptyset$ is the void set.

Starting from the $s - t$-ring of integers we construct an uncountable ω_1-sequence of successively extended countable $s - t$-rings, the set sum of which gives us the final result, in view of the §§ 2 and 3. It may be emphasized that the basic set theory serving to our purpose is Gödel's axiomatic set theory of [G] with the strong axiom of choice E; the "universal" choice function introduced by Gödel in his axiom E makes possible several steps of our construction hardly performable in any other correct way. Though in view of the main result of [G], our consistency supposition essentially is that of the consistency of Gödel's axioms sub A — sub C only, i. e. as modest as possible. (See also [I].)

Concerning the relationship of our result to the known numerous results on the s. c. noncategoricity of the arithmetics of positive integers, we shall limit ourselves to some basic remarks, first of all in order to prevent possible misunderstanding.

The existence of s. c. nonnormal models of the formalized arithmetics of integers (in the sense of the "reine Zahlentheorie", see e. g. the system Z_μ of D. Hilbert-P. Bernays, [H-B] already can be deduced from the fundamental incompleteness theorem of Gödel (of 1931) — in view of a strong form of the Skolem-Löwenheim theorem; though these nonnormal models are meant in the absolute (obvious intuitive) sense (i. e., they are based on the absolute notion of the whole set of intuitive integers as well as on a certain part of the intuitive set theory). This fact has been realized by Gödel himself, in his review of the important paper of Skolem [Sk] of 1934, in Zbl. f. Math. *10*, 2 (1934). In [Sk] (as well as in a previous paper [Sk′] of 1933, not availlable to the author), Skolem independently of Gödel indeed constructed a "concrete" absolute nonnormal model for any formalized elementary consistent theory of integers (no matter whether recursively axiomatisable or not), as based on countably many individual variables and constants and on countably many functors and predicate constants, the binary predicate of the ordering taken as one of the primitives. It might be assumed that Skolem's construction can be carefully reformulated in any sufficiently powerful formalized axiomatic set theory, in order to provide us with a nonnormal model (of the mentioned formalized type of the theories of integers) in the strict syntactical sense (see § 1 of [I] for the notion.). Nevertheless, it seems that the result would remain unsatisfactory even after this (perhaps unessential) improvement, in that Peano's strong intuitive induction principle (requiring the smallest natural in every nonvoid class of integers) would not be satisfied — nor formulated at all; this requires a further primitive notion of the membership-relation and further variables for number classes, in a suitable formalization. It is to be noted that Skolem itself asserted in [Sk], though without proof, as it has been emphasized by Gödel in the just mentioned review of [Sk], that his methods and results extend to the more complicated cases when "higher" (class) variables are formally introduced. But so far as the author is informed, no proof of this assertion has been published till now. (See e. g. the report of Mostowski and collaborators on the present state of the foundation of mathematics, presented to the Polish Congress of Mathematicians of 1953 in Warsaw, see [M III].) — Thus the final result of the present paper may also be considered as the positive solution of this problem.

One remark may be added concerning the metamathematical tools used. The fact that the used axiomatic set theory of [G] is based on a finite number of axioms enables us to reach our result without the metamathematical use of the unlimited sequence of intuitive integers. In fact, all the needed metamathematical recursive definitions and arguments can be limited by a suitable integers constant number of steps, not surpassing e. g. the integer 50 — and thus is finitary in the strict sense, of course, within the two-valued customary logic.

As in the previous paper [I], we freely use the symbols and notions of [G].[6]) But for the sake of brevity and readability we mainly use the halfformal instead of the strictly formalized statements though (the author hopes) the manner of their full formalization as well as that of the corresponding proofs may be obvious enough. For the same reason, some relatively easy proofs have been traced only or omitted entirely; their selection is, of course, a somewhat subjective one. Nevertheless the author hopes that all the really decisive lemmas are proved *in extenso.*

Every definition merely consisting of an abbreviation or in the recalling of an essentially known notion are named conventions (and enumerated separately by latin numerals). More important lemmas possess a title (in addition to their latin numberings) in order to exhibit their meaning in a rough intuitive manner.

2. Dyadic rings and their immediate ring extensions

Definition I. A *discretely ordered ring* $\mathfrak{R} = \langle RF_1F_2F_3\rangle$ *with unit* is an ordered quadruple consisting of a nonvoid set R (of the elements of $\mathfrak{R}$) followed by a function F_1 on $R \times R$ into R (of the addition of $\mathfrak{R}$), this followed by a second function F_2 on $R \times R$ into R (of the multiplication of $\mathfrak{R}$) and this followed by a third function F_3 on R into R (of the signum of $\mathfrak{R}$) — such that the subsequent requirements ("axioms") (I)—(XII) are satisfied. (General quantifiers are often omitted if not necessary. All the quantifiers are meant in the relativized sense, with respect to the set R.)

[6]) With possibly slight typographical changes, but in the original quotation; see [I], § 2.

Writing $z = x + y$ instead of $z = F'_1\langle x, y\rangle$ we require

(I) $$x + (y + z) = (x + y) + z\,,$$

(II) $$x + y = y + x\,,$$

(III) $$(x)(y)\,\exists\, z(x + z = y)\,.$$

Lemma I (On the ring zero). *With any* $\mathfrak{R}$ *satisfying* (I)—(III) *we get* [7])

$$\exists z!(x)(x + z = x)\,.$$

Usual proof; see e. g. [W I].

Convention I. Let ‚$\Phi(z, \ldots)$' be any normal p. f. with the ‚z' free and ranging over R. Then the (also normal) p. f. $\Phi(z, \ldots)\,.\,(x)(x + z = x)$ will be abbreviated by the p. f. $\Phi(O_{\mathfrak{R}}, \ldots)$ — to be taken normal also. (For the term "normal p. f.", see [G] Chap. II.) E. g. the normal p. f. $u = O_{\mathfrak{R}}$ is the abbreviated p. f. $(z = u)(x)(x + z = x)$. The sign $O_{\mathfrak{R}}$ is indeed a normal term and becomes a values sing of an uniquely determined function defined on the class (as existing by M 2 of [G]) of the $\mathfrak{R}'s$ (to be defined by (I)—(XII)) into the universal class V. Hence $O_{\overline{\mathfrak{R}}}$ is a constant set whenever $\overline{\mathfrak{R}}$ is. If there can be no confusion, the subscripts $\mathfrak{R}$, $\overline{\mathfrak{R}}, \ldots$ will often be omitted. (Comp. also conv. V below.)

Lemma II. (On substraction.) *With* (I)—(III) *we get*

$$(x)(y)\,\exists\, z!(x + z = y)\,.$$

Usual proof; see e. g. [W I].

Lemma III and Convention II. *To every* $\mathfrak{R}$ *with* (I)—(III) *there is exactly one function, say* ${}^{\mathfrak{R}}F_5$ *on* $R \times R$ *into* R, *such that we have the identity* $x + {}^{\mathfrak{R}}F'\langle xy\rangle = y$.

We shall write $z = y - z$ instead of $z = {}^{\mathfrak{R}}F'_5\langle x, y\rangle$ and $z = -x$ instead of $z = O_{\mathfrak{R}} - x$ if there is no danger of confusion.

‚${}^{\mathfrak{R}}F'_5$' can be taken for a normal term, with consequences as in conv. I.[8])

The proof is obvious, in view of lemma II and of M 1, M 2, M 5, 5.18 and 5.19 of [G]. Analogous obvious remarks may often be omitted in the sequel.

Writing $xy = z$ instead of $z = F'_2\langle xy\rangle$ we further require

(IV) $$x(yz) = (xy)z\,,$$

(V) $$xy = yx\,,$$

(VI) $$\exists z(x)(xz = x)\,,$$

(VII) $$(x + y)\,z = xz + yz\,.$$

Lemma IV. (On the ring unit.) *With any* $\mathfrak{R}$ *satisfying to* (I)—(VII) *we get*

$$\exists z!(xz = x)\,.$$

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[7]) As usual ‚$\exists\, z!\,\Phi(z, \ldots)$' (with ‚$z$' free in ‚$\Phi(z, \ldots)$') means "there is exactly one z such that $\Phi(z, \ldots)$", abbreviating the p. f. $\exists\, z(\Phi(z, \ldots)\,.\,(u)(\Phi(u, \ldots) \supset \Phi(z, \ldots)))$.

[8]) The index $\mathfrak{R}$ will often be omitted in the sequel.

Proof. Suppose $(x)(z_1 x = x)$, $(y)(z_2 y = y)$ in the sense of (VI). Then $z_1 z_2 = z_2$, $z_2 z_1 = z_1$ and hence $z_2 = z_1$ by (V).

Convention III. Let $\Phi(z, \ldots)$ be a normal p. f. with the free variable ranging over R.

Then the normal p. f. $\Phi(z, \ldots)(x)(xz = x)$ will be abbreviated to the normal p. f. $\Phi(1_{\mathfrak{R}}, \ldots)$. The same as for ,$0_{\mathfrak{R}}$' now holds for ,$1_{\mathfrak{R}}$', as for a normal term (see conv. I).

Concerning the ordering of $\mathfrak{R}$ in question, we continue the definition I with the following requirements (VIII)—(XII) on F_3.

(VIII) There is $1_{\mathfrak{R}} \neq -1_{\mathfrak{R}}$[9]) and $F_3'' R = \{0_{\mathfrak{R}} 1_{\mathfrak{R}} - 1_{\mathfrak{R}}\}$. Writing (for more convenience) $z = \mathrm{sg}(x)$ instead of $z = F_3' x$ (if there is no danger of confusion) we finally require

(IX) $$\mathrm{sg}\,(x) = 0 \supset x = 0\,,$$

(X) $$\mathrm{sg}\,(x) = \mathrm{sg}\,(y) \supset \mathrm{sg}\,(x + y) = \mathrm{sg}\,(x)\,,$$

(XI) $$\mathrm{sg}\,(xy) = \mathrm{sg}\,(x)\,\mathrm{sg}\,(y)\,,$$

(XII) $$\mathrm{sg}\,(x - y) \neq \mathrm{sg}\,(y - x + 1)\,.$$

Lemma V. (On the discrete ordering relation of $\mathfrak{R}$) and Convention IV. *To every* $\mathfrak{R}$ *with* (I)—(XII) *there is exactly one relation, say* $U_{\mathfrak{R}}$, *such that* $U_{\mathfrak{R}} \subset R \times R$ *and* $\langle xy \rangle \in U_{\mathfrak{R}} \equiv \mathrm{sg}\,(y - x) = 1$.

The same remark as in conv. I and II now *mutatis mutandis* holds for the normal term $U_{\mathfrak{R}}$ (and need not be stated explicitely.)

Writing for more convenience $x < y$ or also $y > x$ instead of $\langle xy \rangle \in U_{\mathfrak{R}}$ and $x \leqq y$ or also $y \geqq x$ instead of $x < y \vee x = y$ we have

a) $\sim (x < x)$, b) $\sim (x < y) \supset y \leqq x$, c) $(x < y)(y < z) \supset x < z$,

d) $x < y \supset x + z < y + z$, e) $(x < y)(0 < z) \supset xz < yz$,

f) $(\mathrm{sg}\,(x) = 1 \equiv 0 < x)\,(\mathrm{sg}\,(x) = -1 \equiv x < 0)\,(\mathrm{sg}\,(x) = 0 \equiv x = 0)$,

g) $x \leqq y < x + 1 \supset x = y$.

Write also $x \in R^{<}$, $x \in R^{>}$ instead of $0 < x$, $x < 0$ resp., and $x \in R^{\leqq}$, $x \in R^{\geqq}$ instead of $0 \leqq x$, $x \leqq 0$ resp.

Proof. (VIII) and (IX) give $\mathrm{sg}\,(0) = 0$ whence a) and b) follow. (X) gives c) and d) and e), the last in view of f), this being itself a consequence of (VIII) and (IX). g) almost immediately follows from (XII) (in view of the just proved statements.)

Lemma VI (On the absence of divisors of zero.) *Assuming* (I)—(XII) *in an* $\mathfrak{R}$ *we have*

$$xy = 0 \supset x = 0 \vee y = 0\,.$$

The usual proof may be omitted.

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[9]) $x \neq y \equiv \sim (x = y)$ of course.

Lemma VII and convention V (On the "natural elements" of $\mathfrak{R}$). *With any given* $\mathfrak{R}$ *satisfying* (I)—(XII) *there is an unique function, say* $f_{\mathfrak{R}}$, *on* ω_0 *into* R *with*

$$f'_{\mathfrak{R}}\emptyset = 0_{\mathfrak{R}}\,, \quad f'_{\mathfrak{R}}(n+1) = f'_{\mathfrak{R}}n + 1_{\mathfrak{R}}\,.$$

Then the subset $f''_{\mathfrak{R}}\omega_0$ of R is said to be the set of "natural elements" of $\mathfrak{R}$.

Concerning ‚$f_{\mathfrak{R}}$' as a normal term, the same remark *mutatis mutandis* holds as in the conv. II.

Every $f_{\mathfrak{R}}$ *is a one-to-one function.*

We write $1_{\mathfrak{R}} + 1_{\mathfrak{R}} = 2_{\mathfrak{R}}$, $2_{\mathfrak{R}} + 1_{\mathfrak{R}} = 3_{\mathfrak{R}}$, ... sometimes omitting the subscript $\mathfrak{R}$ if there is no danger of ambiguities (comp. the following remark).

Proof. That $f_{\mathfrak{R}}$ is one-to-one may be proved by an obvious induction based on the lemma V c) and f). The rest is clear.

Remark. Lemma VII enables us to replace the discretely ordered subring of natural elements of any given discretely ordered ring by the isomorphic ring of integers — and to transform the given discretely ordered ring into an isomorphic ring having the ring of integers as a subring with respect to the discrete ordering also, in an obvious sense.

In order to avoid possible misunderstandings we assume this transformation has been performed automatically in every discretely ordered ring to be treated in the sequel, if nothing else is explicitely required.

Now, pass to the important and less known notion of the dyadicity of discretely ordered rings.

Definition II. Let $\mathfrak{D} = \langle RF_1F_2F_3F_4\rangle$ be an ordered quintuple such that $\mathfrak{R} = \langle RF_1F_2F_3\rangle$ is a discretely ordered ring and F_4 is a fixed function on the (nonvoid) set $R^{\leqq}$ (of nonnegative elements of $\mathfrak{R}$) into $R^{\leqq}$. Then $\mathfrak{D}$ is called a *dyadic ring*-and F_4 is its *dyadic exponentiation* — if the following requirements (d I)—(d V) are satisfied:

(d I) $F'_4 1 = 2\,.$

Writing for more convenience, $F'_4 x = 2^x$ if $x \geqq 0$, we require further:

(d II) $$2^x \,.\, 2^y = 2^{x+y}\,,$$

(d III) $$x < 2^x\,,$$

(d IV) $$(x)(y)\,\exists\, q\,\exists\, r((x \in R)(y \in R^{\leqq}) \supset (x = 2^y q + r)(0 \leqq r < 2^y))\,.$$

(The meaning of (d I)—(d III) requires no comments. (d IV) is the Euclidean property for division with remainder by potencies of 2.)

In order to clarify the sense of the requirement (d IV) and because of the formulation of the last requirement (d V), we state the

Lemma IX and convention VI. *To every* $\mathfrak{D}$ *with* (d I)—(d IV) *of the just stated part of the definition* II, *there are exactly two functions, say* F_6 *and* F_6^*, *on* $R \times R^{\leqq}$ *into* R such *that* (1) $0 \leqq F_6^{*\prime}\langle xy\rangle < 2^y$, (2) $x = 2^y F'_6\langle xy\rangle + F_6^{\prime *}\langle xy\rangle$

for every $x \in R$, $y \in R^{\leqq}$. *Conversely, if there are two such functions satisfying* (1) *and* (2) *then the* $\mathfrak{D}$ *with* (d I)—(d III) *satisfies* (d IV) *also.*

Concerning ‚F_6' ‚F_6^*' as normal terms (uniquely depending on the normal term $\mathfrak{D}$) the same remark holds as in the preceding analogous conventions. — We shall write $q = \left[\frac{x}{2^y}\right]$ instead of $q = F_6'\langle xy\rangle$ if there is no danger of ambiguitities.

Proof. In view of the metatheorem M 2 and by 5.18, 5.19 of [G], there is exactly one relation, say ${}^{\mathfrak{D}}D$, so that ${}^{\mathfrak{D}}D \subset (R \times R^{\leqq}) \times (R \times R^{\leqq})$ and

$$\langle\langle qr\rangle\langle xy\rangle\rangle \in {}^{\mathfrak{D}}D \equiv (x = 2^y q + r)(0 \leqq r < 2^y)\,.$$

By (d IV), we have[10] $\mathbf{D}({}^{\mathfrak{D}}D) = R \times R^{\leqq}$, so that ${}^{\mathfrak{D}}D$ is nonvoid. But ${}^{\mathfrak{D}}D$, moreover is a function on $R \times R^{\leqq}$ into $R \times R^{\leqq}$. Indeed, without loss of generality, suppose $x = 2^y q_1 + r_1 = 2^y q_2 + r_2$ and $0 \leqq r_2 \leqq r_1 < 2^y$. Then $0 \leqq$ $\leqq r_1 - r_2 < 2^y$ i. e.

$$0 \leqq (x - 2^y q_1) - (x - 2^y q_2) = 2^y(q_2 - q_1) < 2^y\,,$$

whence $0 \leqq q_2 - q_1 < 1$ by lemma V e), which gives $q_2 - q_1 = 0$ by the same lemma, sub g). Therefore $q_1 = q_2$ and consequently $r_1 = r_2$ also. Therefore F_6, F_6^* are given by M 5 of [G] thus: let $F_6'\langle x, y\rangle$, $F_6'^*\langle x, y\rangle$ be the first, resp. the second member of the ordered pair $\langle qr\rangle = {}^{\mathfrak{D}}D'\langle xy\rangle$. Since the converse assertion is almost obvious, the lemma is proved.

Remark. We can consider the symbol $\left[\frac{x}{2^y}\right]$ as denoting the *integral part of the quotient* $\frac{x}{2^y}$ in the ordered quotient field ${}^{\mathfrak{D}}\mathfrak{Q}$ of $\mathfrak{D}$ (in the usual sense of algebra) since we easily observe that $\left[\frac{x}{2^y}\right] = z$ is indeed the greatest "integral" element (i. e. $z \in R$) of $\mathfrak{Q}$ not surpassing the quotient $\frac{x}{2^y}$; this fact will be tacitly used in some auxiliary considerations in the sequel. Now, let us complete the convention VI as follows:

The element $r = x - 2^y\left[\frac{x}{2^y}\right] = F_6^{*\prime}\langle xy\rangle$ is called the *smallest nonnegative remainder* of x modulo 2^y. The symbol $[2^x] = \left[\frac{2^x}{2^0}\right]$[11]) may and will be used in such a way that $[2^x] = 2^x$ if $x \geqq 0$ and $[2^x] = \left[\frac{1}{2^{-x}}\right] = 0$ if $x < 0$, in accordance

[10]) $\mathbf{D}(x)$ is the domain of x.

[11]) See lemma X.

with the eventual immerging of $\mathfrak{D}$ in its ordered quotient field ${}^{\mathfrak{D}}\mathfrak{Q}$. Note that if $|x| = x \operatorname{sg}(x)$ then $[2^{-|x|}] = 1$ if and only if $x = 0$, i. e. $[2^{-|x|}] = 0$ if $x \neq 0$.

We now complete the definition II by the requirement

$$\text{(d V)}\quad (x)\,\exists\, y\left(\left((y \geqq 0)((x \neq 0)) \supset \left(\left(\left[\frac{x}{2^y}\right]2^y = x\right)\left(\left[\frac{x}{2^{y+1}}\right]2^{y+1} < x\right)\right)\right)\right).$$

This last requirement means that every nonzero element of a dyadic ring is divisible (without remainder) by a certain maximal potency of 2.

Remark. It is obvious that the ring of interges (of our basic axiomatic set theory) is a dyadic ring. Less obvious are examples of dyadic rings of a quite different nature, with an uncountable power of the set of elements. (See § 4.)

Lemma X. *In every dyadic ring we have* $2^0 = 1$.

Proof. By (d II) we get $2^0 \,.\, 2^0 = 2^0$ whence either $2^0 = 0$ or $2^0 = 1$ by lemma VI. The first is impossible, for it would imply $2^1 = 2^{1+0} = 2^1 \,.\, 0 = 0$, contrary to (d I).

Lemma XI. *In every dyadic ring we have*

$$\text{a) } x > 0 \supset 2^x > 1\,, \quad \text{b) } 0 < x < y \supset 2^x < 2^y\,.$$

Proof of a). By (d III). Of b): If $0 < x < y$ then $2^y = 2^{y-x} \,.\, 2^x$ with $y - x \geqq \geqq 1$, i. e. with $2^{y-x} > 1$ (by (d II) and the already proved a)), whence $2^x < 2^y$ by lemma Vb).

Lemma XII and Convention VII. (On the dyadic valuation.) *Every nonzero element of a dyadic ring can be writen in the form* $x = 2^p(2q + 1)$ *with* $p, q, p \geqq 0$ *uniquely determined by* x.

To every dyadic ring $\mathfrak{D}$ *there is exactly one function, say* F_7, *on* $R \dot{-} \{0\}$[12]) *into* $R^{\leqq}$ *so that* $F_7'x = p$ *in the just writen expression for* x.

We state the following characteristic properties of F_7:

$$\text{(I)}\quad \left[\frac{x}{2^{F_7'x}}\right]2^{F_7'x} = x \quad \text{whenever } x \neq 0,$$

$$\text{(II)}\quad \left[\frac{x}{2^{F_7'x+1}}\right]2^{F_7'x} < x \quad \text{for every } x \neq 0.$$

Conversely, if there is a function F_7 *satisfying* (I) *and* (II), *then the* $\mathfrak{D}$ *in question with* (d I)—(d IV) *satisfies* (V) *also, i. e. is an dyadic ring.*

Concerning ‚F_7' as a normal term, the same remark holds as in conv. II. For more convenience write $p = W(x)$ instead of $p = F_7'x$ if $x \neq 0$ when no confusion menaces. W (as a function on $R \dot{-} \{0\}$ into $R^{\leqq}$) is the s. c. *dyadic valuation* of $\mathfrak{D}$ and $W(x)$ is the s. c. dyadic value of $x \neq 0$.

In view of the definition II and of the requirements (d IV) and (d V) of the definition II, the proof is obvious and can be omitted.

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[12]) ‚$\dot{-}$' means the set-difference.

Remark and Convention VIII (comp. conv. VI). It may be noted for further purposes, that the notions max (.,.), and min (.,.) can be defined by means of the already introduced operations only, i. e. they are s. c. elementary functions of $\mathfrak{D}$, in the sense of the convention XX below. E. g. we can write

$$\text{a) } \max(x, y) = x \operatorname{sg}([2^{x-y}]) + y \operatorname{sg}([2^{y-x-2}]) ,$$

$$\text{b) } \min(x, y) = -\max(-x, -y) .$$

Lemma XIII (On properties of the dyadic valuation). *The dyadic valuation W of the conv.* VII *of any dyadic ring $\mathfrak{D}$ satisfies to the following conditions*:
a) $W(x) = W(-x) = W(|x|)$, b) $W(x + y) \geqq \min(W(x), W(y))$ *and moreover* c) $W(x + y) \geqq W(x) + 1$ *if* $W(x) = W(y)$, d) $W(x + y) = \min(W(x), W(y))$ *if* $W(x) \neq W(y)$. e) $W(xy) = W(x) + W(y)$ (*we assume nonzero variables in W everywhere*).

On account of (d I)—(d V) the proofs are almost immediate and formally do not differ from those of the special case of $\mathfrak{D}$ = the ring of integers; hence they can be omitted.

Remark. Let ${}^{\mathfrak{D}}\mathfrak{Q}$ be the ordered quotient field of the given dyadic ring $\mathfrak{D}$ (${}^{\mathfrak{D}}\mathfrak{Q}$ taken for an ordered overring of $\mathfrak{D}$, the usual formal details omitted) and extend W to the set of all the nonzero elements of ${}^{\mathfrak{D}}\mathfrak{Q}$ by $W(x/y) = W(x) - W(y)$; then W is a s. c. discrete valuation of ${}^{\mathfrak{D}}\mathfrak{Q}$ in the known general sense due to W. Krull (see [Kr], [W I] and [Sch] and compare with the convention XIV below), with the discretely ordered value group as identical with the additive group of $\mathfrak{D}$. The property c) of the lemma XIII is exceptional in that it is not fulfilled e. g. by any p-adic valuation of integers or of rationals with $p \neq 2$.

In order to proceed to the main definition IV of § 2 we need the important

Convention VIII (On valuation congruence systems.)

Let $\mathfrak{D} = \langle RF_1F_2F_3F_4\rangle$ be a dyadic ring. Let $\underline{r}$ be a function on a certain subset $R_{\underline{r}}$ of the set $R^{\leqq}$, into $R^{\leqq}$ and such that

1. $(z) \exists z^*(z \in R_{\underline{r}} \supset (z^* \in R_{\underline{r}})(z^* > z))$ (in words: $R_{\underline{r}}$ has no greatest element) — and

2. $2^z \left[\dfrac{\underline{r}'z^* - \underline{r}'z}{2^z}\right] = \underline{r}'z^* - \underline{r}'z$ if $z \leqq z^*$, $z \in R_{\underline{r}}$, $z^* \in R_{\underline{r}}$ (i. e. $\underline{r}'z^* - \underline{r}'z$ is divisible by 2^z under the stated suppositions). Usually, we write in 2

$$\underline{r}'z \equiv \underline{r}'z^* \pmod{2^{z^*}} \quad (\text{if } 0 \leqq z^* < z,\ z \in R_{\underline{r}},\ z^* \in R_{\underline{r}}) .$$

Then $\underline{r}$ is called a *compatible system of valuation congruences*, in short: a *congruence system*. Then the value $\underline{r}'z$ is said to be a *member of* $\underline{r}$ (corresponding to z). $\underline{r}$ is said to be *normal* if

$$\text{a) } 0 \leqq \underline{r}'z < 2^z \quad \text{for every } z \in R_{\underline{r}} ,$$

$$\text{b) } 0 \in R_{\underline{r}} , \quad \text{c) } ((x < y < z)(x \in R_{\underline{r}})(z \in R_{\underline{r}})) \supset y \in R_{\underline{r}}$$

($R_{\underline{r}}$ contains zero and is "convex"). $\underline{r}$ is called *complete* if $R_{\underline{r}} = R^{\leqq}$. $\underline{r}$ is called *cofinal* if to every $z > 0$ there is a $z^* \epsilon R_{\underline{r}}$ with $z^* > z$.

Lemma XIV and Convention IX. *Let $\mathfrak{D}$ be a dyadic ring. To every $x \epsilon R$ define the function $\underline{r}_x$ on $R^{\leqq}$ into $R^{\leqq}$ by the equation*

$$\underline{r}'_x z = x - \left[\frac{x}{2^z}\right] 2^z$$

(in view of the def. II) *for any $z \epsilon R^{\leqq}$.*

Then $\underline{r}_x$ is a normal complete congruence system, the s. c. *remainder system* of the given x and the member $\underline{r}'_x z$ is the s. c. remainder of x by 2^z; especially, *there is $\underline{r}'z = 0$ for every $z \geqq 0$* (the s. c. zero congruence system) *if and only if $\underline{r} = \underline{r}_0$. There is $\underline{r}'_x 0 = 0$ for every $x \epsilon R$.*

The obvious proof based on the lemma IX may be omitted.

Lemma XV and Convention X (On the normalization of congruence system). *Let $\underline{r}$ be a congruence system* (in the sense of the conv. VIII) *defined on the set $R_{\underline{r}}$ ($R_{\underline{r}} \subseteq R^{\leqq}$). Then the member $\underline{r}'_{\underline{r}'z} z^*$* (the smallest nonnegative remainder of $\underline{r}'z$ by 2^{z^*}, see conv. IX) *does not depend on z whenever $z^* < z$. Now, given $\underline{r}$, we can define the function $\overset{*}{\underline{r}}$ by the equation $\overset{*}{\underline{r}}'z^* = \underline{r}'_{\underline{r}'z} z^*$, with arbitrary z, $z > z^*$, $z \epsilon R_{\underline{r}}$; $\overset{*}{\underline{r}}$ thus is a function on the set $R^*_{\underline{r}}$ of all the z^* to which there is a $z \epsilon R_{\underline{r}}$ with $z > z^*$, onto the set $R^{\leqq}$.*

Then $\overset{*}{\underline{r}}$ is a normal congruence system (on $R^*_{\underline{r}}$) and will be called *the normalized $\underline{r}$.*

$\overset{*}{\underline{r}}$ is complete if and only if $\underline{r}$ is cofinal.

Proof. Since $\underline{r}'z_1 \equiv \underline{r}'z_2 \pmod{2^{z_1}}$ so that $\underline{r}'z_1 \equiv \underline{r}'z_2 \pmod{2^{z^*}}$ for $0 \leqq z^* \leqq \leqq z_1 \leqq z_2$, $z_1 \epsilon R_{\underline{r}}$, $z_2 \epsilon R_{\underline{r}}$, hence indeed $\underline{r}'_{\underline{r}'z_1} z^* = \underline{r}'_{\underline{r}'z_2} z^* = \overset{*}{\underline{r}}'z^*$ (= the common smallest nonnegative remainder modulo 2^{z^*} of both the $\underline{r}'z_1$ and $\underline{r}'z_2$), whenever $0 \leqq z^* \leqq z_1 \leqq z_2$, $z_1 \epsilon R_{\underline{r}}$, $z_2 \epsilon R_{\underline{r}}$. More precisely, in order to avoid logical ambiguities, we define the (normal) term $\overset{*}{\underline{r}}'z^*$ as being equal to the term $\underline{r}'_{\underline{r}'\bar{z}} z^*$ where $\bar{z}$ is the "marked"[13]) element of the nonvoid set of all the $z \epsilon R_{\underline{r}}$ with $z > z^*$, uniquely determined. The remaining reasoning (including the use od M 5 of [G]) is now obvious, in view of the lemma XIV.

Lemma XVI and Convention XI. *Given a complete normal congruence system $\underline{r}$, there is an unique x with $x \geqq 0$, $\underline{r} = \underline{r}_x$ if and only if there is a $\bar{z}$ with $\bar{z} \geqq 0$ and such that $\underline{r}'\bar{z} = \underline{r}'z$ for every $z \geqq \bar{z}$.*

We say in this last case that $\underline{r}$ is *a stationary congruence system.*

Proof. The condition of stationarity is necessary, because any $\underline{r}_x$ ($x \geqq 0$) with the $\bar{z} = x$ of the lemma is stationary, in view of the requirement (d IV), as is easily seen ($\underline{r}'_x z = x$ for $z \geqq x$ because of $x < 2^x$).

[13]) "Marked" in the sense of the axiom of choice E.

The condition is sufficient, because the unique ultimately common member $\underline{r}'\bar{z} = \underline{r}'z$ (for all the $z \geqq \bar{z}'$) of a stationary normal complete congruence system can be taken for the desired unique $x \geqq 0$, in view of the lemma XV.

Remark. There exist, of course, $x \in R$ with a nonstationary $\underline{r}_x$. E. g., $-1 \in R$ has the nonstationary $\underline{r}_{-1}$, for we observe that $\left[\frac{-1}{2^z}\right] = -1$ for every $z > 0$ by the def. II (see also the remark after lemma XIII), whence

$$-1 = \left[\frac{-1}{2^z}\right] 2^z + \underline{r}'_{-1}z = -2^z + \underline{r}'_{-1}z = \left[\frac{-1}{2^{z+1}}\right] 2^{z+1} + \underline{r}'_{-1}(z+1) =$$
$$= -2^{z+1} + \underline{r}'_{-1}(z+1)\,,$$

so that $\underline{r}'_{-1}(z+1) - \underline{r}'_{-1}z = 2^z$ for every $z \geqq 0$.

Corollary. *In general, it is easy to prove that in any dyadic ring* $\mathfrak{D}$ *an* r_x *is nonstationary if and only if* $x < 0$.

Lemma XVII (On the "characteristical function" of a normal congruence system). *To every normal*[14]) *congruence system (of any dyadic ring* $\mathfrak{D}$*)* $\underline{r}$ *and to every element* z *of* $\mathfrak{D}$ *there is an uniquely determined function value* $C_z^{\underline{r}}$ *with* $C_z^{\underline{r}} = 0 \vee C_z^{\underline{r}} = 1$ *so that* $\underline{r}'(z+1) - \underline{r}'z = C_z^{\underline{r}} \,.\, 2^z$ *whenever* $z \in R_{\underline{r}}$.

Proof. Prove the following auxiliary identity (with x arbitrary, $z \geqq 0$):

$$\left[\frac{x}{2^{z+1}}\right] = \left[\frac{1}{2}\left[\frac{x}{2^z}\right]\right]. \qquad (*)$$

Indeed, if $\left[\frac{x}{2^z}\right] = 2q$, then $x = 2q \,.\, 2^z + r$, $0 \leqq r < 2^z$, i. e. $x = q \,.\, 2^{z+1} + r$, $0 \leqq r < 2^{z+1}$, whence

$$q = \left[\frac{x}{2^{z+1}}\right] = \frac{1}{2}\left[\frac{x}{2^z}\right] = \left[\frac{1}{2}\left[\frac{x}{2^z}\right]\right].$$

If however $\left[\frac{x}{2^z}\right] = 2q + 1$, then $x = (2q+1)\,2^z + r$, $0 \leqq r < 2^z$, i. e. $x = q \,.\, 2^{z+1} + 2^z + r$, $0 < 2^z + r < 2^{z+1}$, whence

$$q = \left[\frac{x}{2^{z+1}}\right] = \left[\frac{1}{2}\,(2q+1)\right] = \left[\frac{1}{2}\left[\frac{x}{2^z}\right]\right]$$

by lemma IX. Hence the identity (*) is proved.

Now, assume $z > 0$, $z \in R_{\underline{r}}$, $z^* \in R_{\underline{r}}$, $z^* > z$; thus we may write

$$x = \underline{r}'z^* = 2^{z+1}\left[\frac{x}{2^{z+1}}\right] + \underline{r}'(z+1)\,, \quad 0 \leqq \underline{r}'(z+1) < 2^{z+1}\,,$$

$$x = \underline{r}'z^* = 2^z\left[\frac{x}{2^z}\right] + \underline{r}'z\,, \quad 0 \leqq \underline{r}'z < 2^z$$

in view of the lemmas XIV and XV.

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14) But not necessarily complete!

Hence we get

$$\underline{r}'(z+1) - \underline{r}'z = 2^z \left(\left[\frac{x}{2^z} \right] - 2 \left[\frac{x}{2^{z+1}} \right] \right) = 2^z C_z^{\underline{r}} .$$

But we have $2 \left[\frac{x}{2^{z+1}} \right] = 2 \left[\frac{1}{2} \left[\frac{x}{2^z} \right] \right]$ by the identity (*). Hence indeed the term $C_z^{\underline{r}} = \left[\frac{x}{2^z} \right] - 2 \left[\frac{1}{2} \left[\frac{x}{2^z} \right] \right]$ by lemma IX and conv. VI cannot admit other values than 0 or 1, these values being uniquely determined in $\mathfrak{D}$ by the equation

$$C_z^{\underline{r}} = \left[\frac{1}{2^z} (\underline{r}'(z+1) - \underline{r}'z) \right] = \left[\frac{x}{2^z} \right] - 2 \left[\frac{x}{2^{z+1}} \right],$$

whenever $z \in R_{\underline{r}}$, of course. The lemma is thus proved.

Convention XII. Let $\underline{R}$ be the set of the normal congruence systems of a given dyadic ring $\mathfrak{D} = \langle RF_1F_2F_3F_4 \rangle$. Then let ${}^{\mathfrak{D}}C$ denote the function on $\underline{R} \times \underline{R}$ into the set $\{0\ 1\}$ defined as follows

$$ {}^{\mathfrak{D}}C' \langle \underline{r}x \rangle = C_z^{\underline{r}} \quad \text{if} \quad x \geqq 0 ,$$

$$ {}^{\mathfrak{D}}C' \langle \underline{r}y \rangle = 0 \quad \text{if} \quad y < 0 .$$

Concerning ‚${}^{\mathfrak{D}}C$' as a normal term the same remark holds as in conv. II.

Assuming $\underline{r} \in \underline{R}$ fixed, we obtain a function $C^{\underline{r}}$ on $\underline{R}$ into $\{0\ 1\}$, the so called *characteristic function* of the given normal congruence system $\underline{r}$, with $C^{\underline{r}}{}'x = {}^{\mathfrak{D}}C' \langle \underline{r}x \rangle$.

If $\underline{r} = \underline{r}_x$ then we will often write ‚C_z^x' instead of ‚$C_z^{\underline{r}_x}$'.

Remark. Of course, $C_z^x = \left[\frac{x}{2^z} \right] - 2 \left[\frac{x}{2^{z+1}} \right]$ holds (by the proof of the preceding lemma.)

Lemma XVIII. *Let _ be a normal congruence system in any dyadic ring $\mathfrak{D}$. Then $\underline{r}'z_1 \leqq \underline{r}'z_2$ whenever $z_1 < z_2$, $z_1 \in R_{\underline{r}}$ $z_2 \in R_{\underline{r}}$.*

Proof. If $z_1 < z_2$, $z_1 \in R_{\underline{r}}$, $z_2 \in R_{\underline{r}}$ then $\underline{r}'z_1 \equiv \underline{r}'z_2 \pmod{2^{z_1}}$ and by the normality of $\underline{r}$, $\underline{r}'z_1$ is the smallest nonnegative remainder modulo 2^{z_1} of itself, whence indeed $\underline{r}'z_1 \leqq \underline{r}'z_2$.

Lemma XIX and Convention XIII (On the dyadic valuation of congruence systems). *Let $\underline{r}$ be a nonzero normal congruence system. Then $W(\underline{r}'z_1) = W(\underline{r}'z_2)$ whenever $0 < z_1 < z_2$, $z_1 \in R_{\underline{r}}$, $z_2 \in R_{\underline{r}}$, and with $\underline{r}'z_1 \neq 0$, of course.*

Hence we can write $\overline{W}(\underline{r}) = W(\underline{r}'\bar{z})$ with the "marked" $z = \bar{z}$ (in the sense of the axiom of choice E*), such that $z \in R_{\underline{r}}$, $\underline{r}'z \neq 0$, thus defining the dyadic valuation $\overline{W}$ as a uniquely determined function on the set $\underline{R} \dot{-} \{r_0\}$ of all the normal nonzero congruence systems of the given dyadic ring $\mathfrak{D}$.*

Especially, we have $\overline{W}(\underline{r}_x) = W(x)$ if $x \neq 0$, $x \in R$.

Proof. Suppose $0 < z_1 < z_2$, $z_1 \in R_{\underline{r}}$, $z_2 \in R_{\underline{r}}$, $\underline{r}'z_1 > 0$. Then $\underline{r}'z_2 > 0$ by lemma XVIII, so that we may write $\underline{r}'z_1 = 2^{p_1}(2q_1 + 1)$, $\underline{r}'z_2 = 2^{p}(2q_2 + 1)$ in view of the lemma XII, with the uniquely determined $p_1 = W(r'z_1)$, $p_2 = W(\underline{r}'z_2)$. Now $p_1 \neq p_2$ imply

$$W(\underline{r}'z_1 - \underline{r}'z_2) = \min(p_1, p_2) = p$$

by lemma XIIIb). But since $\underline{r}'z_1 \equiv \underline{r}'z_2 \pmod{2^{z_1}}$, hence we observe $p \geqq z_1$ by lemma XII. Now clearly $\underline{r}'z_1 \equiv 0 \pmod{2^p}$ and the more so $\underline{r}'z_1 \equiv 0 \pmod{2^{z_1}}$, i. e. $\underline{r}'z_1 \equiv 0$ by the normality of $\underline{r}$. This would contradict the supposition. Therefore indeed $W(\underline{r}'z_1) = W(\underline{r}'z_2)$ if $z_1 > z$, $z_2 > z$, $\underline{r}'z \neq 0$.

In order to prove $\overline{W}(\underline{r}_x) = W(x)$ write $x = p^{W(x)}(2q + 1)$. Hence $\underline{r}_x'(W(x)) = 0$, but $\underline{r}_x'(W(x) + 1) \neq 0$ by lemma XIV and by (d V). Therefore

$$W(\underline{r}_x'(W(x) + 1)) = W(\underline{r}_x'z) = \overline{W}(\underline{r}_x)$$

for every z with $z \geqq W(x) + 1$, on account of the proof already given. Now, observe $x \geqq W(x) + 1$ by (d II) and $\underline{r}_x'x = x$ as it is easy to see by (d II) also. Hence indeed $\overline{W}(\underline{r}_x) = W(x)$.

Remark. It will be convenient to extend the preceding lemma and convention to general congruence systems in dyadic rings.

Indeed, let $\underline{r}$ be any (not necessarily normal) congruence system defined on the set $R_{\underline{r}}$ ($R_{\underline{r}} \subseteq R^{\leqq}$) and let $\overset{*}{\underline{r}}$ be the result of the normalization of $\underline{r}$ in the sense of the lemma XV, $\overset{*}{\underline{r}}$ being defined on the "convex completion" R^* of the set $R_{\underline{r}} \dotplus \{0\}$ (See conv. XI.) Assume $\underline{r}$ is a nonzero congruence system, i. e. $\underline{r}'z \equiv 0 \pmod{2^z}$ for every $z \in R_{\underline{r}}$ is not true.

Then of course $W(\underline{r}'z) = W(\overset{*}{\underline{r}}'z)$ whenever $0 \neq \underline{r}'z = \overset{*}{\underline{r}}'z$, $z \in R_{\underline{r}}$; but in the contrary case of $\underline{r}'z \neq \overset{*}{\underline{r}}'z$ with a $z \in R_{\underline{r}}$ write $r'z = \overset{*}{\underline{r}}'z + 2^{z^*}(2q + 1)$ with the uniquely determined $z^*, z^* \geqq z$, $q \in R$ (provided $z \in R_{\underline{r}}$), whence $W(\underline{r}'z) = \min(W(\overset{*}{\underline{r}}'z), z^*) = W(\overset{*}{\underline{r}}'z)$ too (in view of the lemma XIIIb) and of the already proved lemma.) It is therefore clear that we can put $\overline{W}(\overset{*}{\underline{r}}) = \overline{W}(\underline{r})$ = the ultimately common dyadic value of every $\underline{r}'z$ with a sufficiently great $z \in R_{\underline{r}}$, no matter whether $\underline{r}$ is normal or not.

Now, the use of the auxiliary $\bar{z}$ of the preceding lemma XIX can be replaced by the use of a more suitable $\underline{z}$ as the smallest $z \geqq 0$ with $\underline{r}'z \neq 0$, in view of the following useful strenghtening of the lemma in question, as well as of the subsequent remark to this lemma.

Corrollary to the lemma XIX. *Let $\underline{r}$ be a nonzero congruence system of $\mathfrak{D}$. Then $\underline{z} = \overline{W}(\underline{r}) + 1$ is exactly the smallest z such that $z \geqq 0$ and $\underline{r}'z \neq 0$, $z \in R_{\underline{r}}$ — as well as $\overline{W}(\underline{r}) = \overline{W}(\underline{r}'z)$.*

Proof. Let $z \in R_{\underline{r}}$ be an arbitrarily chosen z with $\underline{r}'z \neq 0$. Then we can write (in view of the proof of the preceding lemma and of the subsequent remark)

$$\underline{r}'z = 2^{\overline{W}(\underline{r})}(2q + 1) = 2^{\overline{W}(\underline{r})+1}q + 2^{\overline{W}(\underline{r})} \equiv 2^{\overline{W}(\underline{r})} (\mathrm{mod}\ 2^{\overline{W}(\underline{r})+1}) .$$

Therefore $\underline{r}'(\overline{W}(\underline{r}) + 1)$ as the smallest nonnegative remainder of $\underline{r}'z$ $(\mathrm{mod}\ 2^{\overline{W}(\underline{r})+1})$ indeed is $2^{\overline{W}(\underline{r})} \neq 0$. On the other hand, clearly $\underline{r}'z \equiv 0\ (\mathrm{mod} 2^{\overline{W}(\underline{r})})$ for every $z \in R_{\underline{r}}$ with $\underline{r}'z \neq 0$, q. e. d.

Convention XIV. (The ring with valuation and its extension in general.)

a) An ordered quadruple $\mathfrak{R} = \langle RF_1F_2V\rangle$ is said to be a *ring with the valuation* V if

(i) $\langle RF_1F_2\rangle$ is a commutative ring without divisors of zero — in the sense of (I)—(VII) of the def. I — obvious details omitted.

(ii) V is a function defined on the set $R \dot{-} \{0\}$ (of elements of R except the zero) onto the set A of elements of a commutative (additively written) simply ordered semigroup, say $\mathfrak{A} = \langle A \oplus <\rangle$, with cancellation and such that if $x + y \neq 0$, $x \neq 0$, $y \neq 0$, then

$$V(x + y) \geqq \min (V(x), V(y)) , \quad V(xy) = V(x) \oplus V(y) .$$

As a rule, $\mathfrak{R}$ will have the unit; then $\mathfrak{A}$ is assumed to have the zero as a lowest element.

Remark. It is easy to see that the quotient field of $\mathfrak{R}$ possesses the valuation W (in the obvious sense of the valuation theory) if we set $W(x/y) = V(x) \ominus V(y)$ (provided $x \neq 0 \neq y$) in extending the value semigroup $\mathfrak{A}$ to the ordered commutative value group, say $\tilde{\mathfrak{A}}$, in the obvious way.

b) A ring with valuation $\mathfrak{R}_2 = \langle R_2F_{12}F_{22}V_2\rangle$[15]) is *an extension* of the ring with valuation $\mathfrak{R}_1 = \langle R_1F_{11}F_{21}V_1\rangle$ if $R_1 \subseteq R_2$, $F_{11} \subseteq F_{12}$, $F_{21} \subseteq F_{22}$, $V_1 \subseteq V_2$ and the ordered value semigroup $\mathfrak{A}_1$ of $\mathfrak{R}_1$ is an ordered subsemigroup of the value semigroup $\mathfrak{A}_2$ of $\mathfrak{R}_2$, in an obvious sense (with the inclusion of the orderings $<_1 \subseteq <_2$ of $\mathfrak{A}_1$, $\mathfrak{A}_2$ respectively). Denoting by $\mathfrak{P}_1$, $\mathfrak{P}_2$ the valuation maximal ideals of $\mathfrak{R}_1$, $\mathfrak{R}_2$ resp., i. e. the ideals consisting of elements with positive values (provided both the rings possess units) define further:

c) $\mathfrak{R}_2$ is *an immediate extension* of $\mathfrak{R}_1$ if $\mathfrak{A}_1 = \mathfrak{A}_2$ and the s. c. valuation residue class fields $\mathfrak{R}_1/\mathfrak{P}_1$, $\mathfrak{R}_2/\mathfrak{P}_2$ are isomorphic.

Remark. Passing from rings with unit to their quotient fields as well as from value semigroups to value groups we convert the terms already introduced into the usual terms of valuation theory; compare e. g. [Sch], especially p. 36, def. 8 of chap. 2, concerning the notion of immediate extension.

In the case of dyadic rings and their dyadic valuations, the value semigroups are the additive semigroups of nonnegative elements, the valuation

[15]) With the addition F_1 and multiplication F_2 as functions on $R \times R$ into R.

maximal ideals consist of odd elements and the valuation remainder class fields are prime fields of characteristic 2.

Convention XV (On complete normal congruence systems of a dyadic ring). Let $\mathfrak{D} = \langle RF_1F_2F_3F_4\rangle$ be a dyadic ring. Denote by $\overline{\mathfrak{R}}$ the ordered quadruple $\langle \overline{R}\overline{F}_1\overline{F}_2\overline{W}\rangle$ defined (uniquely by $\mathfrak{D}$) as follows:

a) $\overline{R}$ is the set of all the complete normal congruence systems of $\mathfrak{D}$ and $\overline{F}_1$ as well as $\overline{F}_2$ are functions on $\overline{R} \times \overline{R}$ into $\overline{R}$.

b) Let ${}^1\underline{r} \in \overline{R}$, ${}^2\underline{r} \in \overline{R}$. Denote by $\overline{F}_1'\langle {}^1\underline{r}{}^2\underline{r}\rangle = {}^+\overset{*}{\underline{r}}$ the normal complete congruence system which is the result of the normalization of the complete congruence system ${}^+\underline{r}$ with ${}^+\underline{r}'z = {}^1\underline{r}'z + {}^2\underline{r}'z$, in these sense of the conv. XI. — Write ${}^+\overset{*}{\underline{r}} = {}^1\underline{r} + {}^2\underline{r}$ instead of ${}^+\overset{*}{\underline{r}} = \overline{F}_1'\langle {}^1\underline{r}{}^2\underline{r}\rangle$, if there is no danger of ambiguities. Further,

c) let $\overline{F}_2'\langle {}^1\underline{r}{}^2\underline{r}\rangle = {}^\cdot\overset{*}{\underline{r}}$ denote the normal complete congruence system, which is the result of the normalization of the complete congruence system ${}^\cdot\underline{r}$ with ${}^\cdot\underline{r}'z = {}^1\underline{r}'z \,.\, {}^2\underline{r}'z$. — Write ${}^\cdot\overset{*}{\underline{r}} = {}^1\underline{r} \,.\, {}^2\underline{r}$ instead of ${}^\cdot\overset{*}{\underline{r}} = \overline{F}_2'\langle {}^1\underline{r}{}^2\underline{r}\rangle$.

d) Finally, let $\overline{W}$ be the function given in the convention XIII.

Lemma XX. *The quadruple $\overline{\mathfrak{R}} = \langle \overline{R}\overline{F}_1\overline{F}_2\overline{W}\rangle$ of the preceding convention is a ring with unit and with the valuation $\overline{W}$ (and with the value semigroup* (see conv. XIV) *$\mathfrak{A}$ formed of the nonnegative elements of $\mathfrak{D} = \langle RF_1F_2F_3F_4\rangle$). Moreover, the correspondence of any $x \in R$ to the $\underline{r}_x \in \overline{R}$* (which clearly is one-to-one by the lemma XVI) *is a value-preserving isomorphism of the ring $\langle RF_1F_2W\rangle$ onto the subring of $\overline{\mathfrak{R}}$ formed of the remainder systems $\underline{r}_x$ of elements x of $\mathfrak{D}$ (i. e. there is $W(x) = \overline{W}(\underline{r}_x)$).*

Proof. Using basic properties of congruences, the correctness of convention XV and the fullfilling of the ring postulates (I)—(VII) are easy to see, in view of the lemma XV. The unit of $\overline{\mathfrak{R}}$ is, of course, the remainder system $\underline{r}_1$ of the unit 1 of $\mathfrak{D}$. In order to prove the absence of divisors of zero in $\overline{\mathfrak{R}}$, suppose in contrary that we have ${}^1\underline{r}'z_1 \neq 0 \neq {}^2\underline{r}'z_2$ with suitable fixed ${}^1\underline{r}$, ${}^2\underline{r}$ in $\overline{R}$, z_1, z_2 in $R^<$, but there is ${}^1\underline{r}'z \,.\, {}^2\underline{r}'z \equiv 0 \pmod{2^z}$ for every $z \in R^{\leqq}$.

From this supposition we infer $W({}^1\underline{r}'z \,.\, {}^2\underline{r}'z) \geqq z$ whenever ${}^1\underline{r}'z \,.\, {}^2\underline{r}'z \neq 0$. Now ${}^1\underline{r}'z \neq 0 \neq {}^2\underline{r}'z$ if $z \geqq \max(z_1, z_2)$ by lemma XVIII, whence ${}^1\underline{r}'z \,.\, {}^2\underline{r}'z \neq 0$ holds if $z \geqq \max(z_1, z_2)$, by lemma VI. — And finally, by lemma XIX we have $W(\underline{r}'z) = W({}^1\underline{r}'z_1)$, $W({}^2\underline{r}'z) = W({}^2\underline{r}'z_2)$ whenever $z \geqq \max(z_1, z_2)$, whence, by lemmas XII and XIII, we obtain

$$W({}^1\underline{r}'z \,.\, {}^2\underline{r}'z) = W({}^1\underline{r}'z_1) + W({}^2\underline{r}'z_2) \geqq z$$

for every $z \geqq \max(z_1, z_2)$. This contradiction excludes the existence of divisors of the zero in $\overline{\mathfrak{R}}$.

Since the valuation postulates follow by lemma XIX and the last statement is an easy consequence of the corresponding definitions, the lemma is proved.

Lemma XXI. *Let $\mathfrak{D}$ be a dyadic ring, $\overline{\mathfrak{R}}$ the corresponding ring (with valuation) of complete normal congruence systems of* $\mathfrak{D}$. (See conv. XVI.) *Let us replace every remainder system $\underline{r}_x$ (of any element x of $\mathfrak{D}$) by its counterimage $x \in R$ in the (value preserving) isomorphism of the lemma* XX; *let us, moreover, perform the obvious change of the functions $\overline{F}_1$, $\overline{F}_2$, $\overline{W}$ into the corresponding functions, say $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{W}$, in the sense of the mentioned replacements.*

Then the ring ${}^{\mathfrak{D}}\mathfrak{R} = \langle \tilde{R}\tilde{F}_1\tilde{F}_2\tilde{W}\rangle$, whose elements $\tilde{x} \in \tilde{R}$ are either elements of $\mathfrak{D}$ or are complete normal congruence systems $\underline{r} \in \overline{R}$ different from all the remainder systems $\underline{r}_x$ (of elements of $\mathfrak{D}$), (see conv. XIV) *is a ring with the valuation $\tilde{W}$ and is an immediate extension of the given dyadic ring* (see conv. XIV) $\mathfrak{D}$. *The value semigroup of ${}^{\mathfrak{D}}\mathfrak{R}$ consists of nonnegative elements of $\mathfrak{D}$ (in the sense of addition and ordening of $\mathfrak{D}$), ${}^{\mathfrak{D}}\mathfrak{R}$ is uniquely determined by $\mathfrak{D}$ and ‚${}^{\mathfrak{D}}\mathfrak{R}$' is to be taken for a normal term; the (often repeated) remark to the conv.* II *applies.*

In view of the preceding lemma, the proof is almost obvious.

It will be necessary to generalize conventions VI and XIII, in the

Convention XVI. Let $\mathfrak{R} = \langle RF_1F_2V\rangle$ be a ring (with unit and) with the valuation V (in the sense of the conv. XIV).

(i) Let $\overline{\underline{r}}$ be a function defined on a subset $A_{\overline{\underline{r}}}$ of the set A of elements of the value semigroup $\mathfrak{A} = \langle A \oplus \prec\rangle$ (of $\mathfrak{R}$) into the set R, and such that $A_{\overline{\underline{r}}}$ has no last element and the following compatibility condition holds:

To every $z \in A_{\overline{\underline{r}}}$ there is a $\overline{z} \in A_{\overline{\underline{r}}}$ such that either $\overline{\underline{r}}'z_1 = \overline{\underline{r}}'z_2$ or $V(\overline{\underline{r}}'z_1 - \overline{\underline{r}}'z_2) \geqq \geqq z$ whenever $z_1 \geqq \overline{z}$, $z_2 \geqq \overline{z}$, $z_1 \in A_{\overline{\underline{r}}}$, $z_2 \in A_{\overline{\underline{r}}}$.

Then, quite generally, $\overline{\underline{r}}$ is called a *compatible system of valuation congruences on* $A_{\overline{\underline{r}}}$.

(ii) An $\overline{\underline{r}}$ sub (i) is said to be *cofinal* if $A_{\overline{\underline{r}}}$ is cofinal to A (in the sense of the ordering $\prec$ of the value semigroup $\mathfrak{A}$ in question).

(iii) An element $y \in R$ is called a *solution of the system* $\overline{\underline{r}}$ (of (i)) in $\mathfrak{R}$ if to every $z \in A_{\overline{\underline{r}}}$ there is a $\overline{z} \in A_{\overline{\underline{r}}}$ such that either $\overline{\underline{r}}'z^* = y$ or $V(\overline{\underline{r}}'z^* - y) \geqq z$ whenever $z^* \geqq \overline{z}$, $z^* \in A_{\overline{\underline{r}}}$.

(iv) The ring $\mathfrak{R} = \langle RF_1F_2V\rangle$ (with unit and with the valuation V) is called *pseudoperfect* if every cofinal compatible system of valuation congruences $\overline{\underline{r}}$ of $\mathfrak{R}$ has a solution in $\mathfrak{R}$.

Lemma XXII. (A descriptive characterization of ${}^{\mathfrak{D}}\mathfrak{R}$ as the immediate pseudoperfect ring extension of $\mathfrak{D}$.) *Let $\mathfrak{D}$ be a dyadic ring, ${}^{\mathfrak{D}}\mathfrak{R}$ its ring extension of the lemma* XXI.

Then (i) *${}^{\mathfrak{D}}\mathfrak{R}$ is a pseudoperfect ring with the valuation $\tilde{W}$ (and with unit* 1) *and ${}^{\mathfrak{D}}\mathfrak{R}$ is an immediate extension of $\mathfrak{D}$ in the sense of the conv.* XIVc);

(ii) *every element of ${}^{\mathfrak{D}}\mathfrak{R}$ is a solution of exactly one normal complete congruence system of $\mathfrak{D}$, and conversely, every normal complete congruence system of $\mathfrak{D}$ possesses exactly one solution in ${}^{\mathfrak{D}}\mathfrak{R}$;*

(iii) *if $\mathfrak{R}^*$ is another pseudoperfect immediate ring extension of the given* *dyadic* $\mathfrak{D}$ *with the properties* (i), (ii), *then there is a value-preserving isomorphism of* ${}^{\mathfrak{D}}\mathfrak{R}$ *and* $\mathfrak{R}^*$ *consisting in the one-to-one correspondence of the unique solutions of the same normal complete congruence system of* $\mathfrak{D}$ *in* ${}^{\mathfrak{D}}\mathfrak{R}$ *and* $\mathfrak{R}^*$ *respectively.*

(iv) *If the set of elements of* $\mathfrak{D}$ *is of the power* $\aleph_\alpha$, *then the set of the elements of* ${}^{\mathfrak{D}}\mathfrak{R}$ *is of the power* $2^{\aleph_\alpha}$.

Proof. Ad (i): For more convenience prove the pseudoperfectness of the ring $\overline{\mathfrak{R}}$ (of conv. XV) instead of the ring ${}^{\mathfrak{D}}\mathfrak{R}$ itself, in view of the value preserving isomorphism between $\overline{\mathfrak{R}}$ and ${}^{\mathfrak{D}}\mathfrak{R}$ (see lemma XX).

Thus let $\bar{\underline{r}}$ be a nonzero cofinal compatible system of valuation congruences of $\overline{\mathfrak{R}}$. By conv. XVI, there is a subset $R_{\bar{\underline{r}}}$ of $R^{\leqq}$ cofinal to $R^{\leqq}$ and such that 1. $\bar{\underline{r}}$ is a function defined on $R_{\bar{\underline{r}}}$ into $\overline{R}$ (i. e. any member $\bar{\underline{r}}'z = \underline{r}$ is a complete normal congruence system of $\mathfrak{D}$), 2. denoting $\bar{\underline{r}}'z_1 = {}^1\underline{r}$, $\bar{\underline{r}}'z_2 = {}^2\underline{r}$, we have $\overline{W}({}^1\underline{r} - {}^2\underline{r}) \geqq z$ unless ${}^1\underline{r} = {}^2\underline{r}$, whenever $z_1 \geqq \bar{z}$, $z_2 \geqq \bar{z}$, where $\bar{z}$ is a suitable element of $R_{\bar{\underline{r}}}$; on account of the axiom of choice E, this $\bar{z}$ can be assumed to be uniquely given to any chosen $z \in R_{\bar{\underline{r}}}$ and, moreover, we can assume that $\bar{z}_1 \leqq \bar{z}_2$ if $z_1 \leqq z_2$ and that $\bar{z} \leqq z$.

In view of the lemma XIX and conv. XIII as well as by the corresponding corrollary, the already stated requirement 2. means that — under the stated conditions concerning z_1, z_2 and with the given $z \in R^{\leqq}$ — the complete difference congruence system ${}^1\underline{r} - {}^2\underline{r}$ (of $\mathfrak{D}$) has zero members $({}^1\underline{r} - {}^2\underline{r})'\,z^* = 0$ for every $z^* < z$, $z^* \in R^{\leqq}$. Therefore the members ${}^1\underline{r}'z^* = (\bar{\underline{r}}'z_1)'\,z^*$, ${}^2\underline{r}'z^* = (\bar{\underline{r}}'z_2)'\,z^*$ are equal for every $z^* < z$, $z^* \in R^{\leqq}$, provided $z_1 \in R_{\bar{\underline{r}}}$, $z_2 \in R_{\bar{\underline{r}}}$, $z_1 \geqq \bar{z}$, $z_2 \geqq \bar{z} \in R_{\bar{\underline{r}}}$.

Now define the "diagonal system" $\underline{r}$ (of $\bar{\underline{r}}$) thus:

If $z^* \in R_{\bar{\underline{r}}}$ then put $\underline{r}'z^* = (\bar{\underline{r}}'(\overline{z^* + 1}))'\,z^*$.

Prove that $\underline{r}$ is a cofinal compatible system of valuation congruences of $\mathfrak{D}$.

The cofinality of $\underline{r}$ being clear, suppose $0 \leqq z_1^* < z_2^*$, $z_1^* \in R_{\bar{\underline{r}}}$, $z_2^* \in R_{\bar{\underline{r}}}$ and observe that

$$(\bar{\underline{r}}'(\overline{z_1^* + 1}))'\,z_1^* = (\bar{\underline{r}}'(\overline{z_2^* + 1}))'\,z_1^*\,,$$

as well that

$$(\bar{\underline{r}}'(\overline{z_2^* + 1}))'\,z_1^* \equiv (\underline{r}'(\overline{z_2^* + 1}))'\,z_2^* \pmod{2^{z_1*}}\,.$$

It follows that $\underline{r}'z_1^* \equiv \underline{r}'z_2^* \pmod{2^{z_1*}}$, i. e. the desired compatibility condition for $\underline{r}$.

Finally, let $\overset{*}{\underline{r}}$ be the result of the normalization of $\underline{r}$. Prove that this normal complete congruence system $\overset{*}{\underline{r}}$ of $\mathfrak{D}$ as an element of $\overline{\mathfrak{R}}$ is a solution of $\underline{r}$ (in the sense of conv. XVI (iii)).

Indeed, let $z \in R_{\bar{\underline{r}}}$ be arbitrarily choosen. Then for every $z^* \geqq \bar{z}$ (of conv. XVI (iii)) either $\bar{\underline{r}}'z^* = \overset{*}{\underline{r}}$ or $W(\bar{\underline{r}}'z^* - \overset{*}{\underline{r}}) \geqq z$, because the members $(\bar{\underline{r}}'z^*)'\,\tilde{z}$, $\overset{*}{\underline{r}}'\tilde{z}$ of the normal complete congruence systems $\bar{\underline{r}}'z^*$ and $\overset{*}{\underline{r}}$ (of $\mathfrak{D}$) are equal at least for any $\tilde{z} < z$ by the already given definition of $\overset{*}{\underline{r}}$. Thus the pseudoperfectness of $\overline{\mathfrak{R}}$ as well as that of ${}^{\mathfrak{D}}\mathfrak{R}$ is proved.

Ad (ii): If an element $\tilde{x}$ of ${}^{\mathfrak{D}}\mathfrak{R}$ lies in R, then it is the unique solution of the complete remainder system of itself. If $\tilde{x}$ does not lie in R then it is itself a normal complete congruence system of $\mathfrak{D}$ and trivially a solution of itself. Hence we have only to show that a) a normal complete congruence system, say $\underline{r} \in \tilde{R} \dot{-} R$, of $\mathfrak{D}$ ($\underline{r} \neq \underline{r}_x$ for every $x \in R$) cannot have two different solutions, say ${}^1\underline{r} \neq {}^2\underline{r}$ in ${}^{\mathfrak{D}}\mathfrak{R}({}^1\underline{r}, {}^2\underline{r} \in \tilde{R} \dot{-} R)$, and b) two different normal complete congruence systems of $\mathfrak{D}$, say ${}^1\underline{r}$ and ${}^2\underline{r}$, cannot have a common solution, say $\underline{r} \in \tilde{R} \dot{-} R$.

For a), observe that $\tilde{W}(\underline{r}'z^* - {}^1\underline{r}) \geqq z$, $\tilde{W}(\underline{r}'z^* - {}^2\underline{r}) \geqq z$ for every z (with a suitable z^*) implies

$$\tilde{W}({}^1\underline{r} - {}^2\underline{r}) = \tilde{W}((\underline{r}'z^* - {}^2\underline{r}) - (\underline{r}'z^* - {}^1\underline{r})) \geqq z \quad \text{if} \quad {}^1\underline{r} \neq {}^2\underline{r} ;$$

this impossible result shows that indeed ${}^1\underline{r} = {}^2\underline{r}$.

For b) observe that in the contrary case the elements ${}^1\underline{r}$, ${}^2\underline{r}$ of ${}^{\mathfrak{D}}\mathfrak{R}$ would be in fact two different solutions of the normal complete congruence system $\underline{r}$ (of $\mathfrak{D}$), which contradicts the already proved a).

The assertions (iii) of the lemma now is almost obvious. The assertion (iv) (on the powers) follows by the usual diagonal process. Hence the lemma is proved.

Remarks: A) In the case of $\mathfrak{D}$ = ring of integers, the ring ${}^{\mathfrak{D}}\mathfrak{R}$ converts into the ring of Hensel's integral 2-adic numbers; hence ${}^{\mathfrak{D}}\mathfrak{R}$ is a generalization of this ring.

B) It is to be emphasized that the already stated result concerns complete, or more generally, cofinal congruence systems (of dyadic rings) only.

C) Let us add some remarks for readers familiar with basic notions of the general theory of valuations of algebraic fields concerning the relation of the already stated relatively simple notions and result to the basic ones of this theory.

First, let us explain in what sense our notion of (compatible) congruence system and the known notion of compatible sequence of elements of the valuation ring of an algebraic field with valuation are equivalent.

By [Sch] def. 11 of chap. 2, a well ordered (infinite) sequence $\{A_\tau\}_{\tau<\lambda}$ (with a limit ordinal λ) of elements of the valuation ring $\mathfrak{O}$ of a field $\mathfrak{Q}$ with the valuation W is called *compatible* if there is an infinite well-ordered corresponding sequence $\{\mathfrak{A}_\tau\}_{\tau<\lambda}$ of ideals of $\mathfrak{O}$ such that

(i) $\mathfrak{A}_\tau \subsetneqq \mathfrak{A}_\sigma$, (ii) $a_\tau \equiv a_\sigma \pmod{\mathfrak{A}_\sigma}$ for $\sigma < \tau < \lambda$.

In [Sch] an element x of $\mathfrak{O}$ is called a *solution* of the corresponding compatible system (as given by $\{a_\tau\}_{\tau<\lambda}$ if $x \equiv a_\tau \pmod{\mathfrak{A}_\tau}$ for every $\tau < \lambda$). One can also say that x is a *pseudolimit* of $\{a_\tau\}_{\tau<\lambda}$ (see [Sch], def. 15 of chap. 2), and that $\{a_\tau\}_{\tau<\lambda}$ is *pseudoconvergent* (see [Sch] def. 10 of chap. 2 and p. 40) to the *pseudolimit* x.

Now assume $\mathfrak{Q}$ is the quotient field of a dyadic ring $\mathfrak{D}$, $\mathfrak{O}$ is the corresponding valuation ring in the sense of the dyadic valuation W of $\mathfrak{Q}$, that is, $\mathfrak{O}$ is the ring of all the quotients x/y ($y \neq 0$, $x \in R$, $y \in R$) with $W(x/y) \geqq 0$ i. e. with $W(x) \geqq W(y)$.

Then clearly $\mathfrak{O}$ is an immediate ring extension (in the sense of conv. XIVc) of $\mathfrak{D}$.

Consider a congruence system $\underline{r}$ of $\mathfrak{D}$ defined on a subset $R_{\underline{r}}$ of the set $R^{\leqq}$ of all the nonnegative elements of $\mathfrak{D}$ ($R_{\underline{r}}$ has no greatest element, see conv. VIII). Let $\{t_\tau\}_{\tau<\lambda}$ be a transfinite sequence of elements of $R_{\underline{r}}$ cofinal with $R_{\underline{r}}$, i. e. to every $z \in R_{\underline{r}}$ there is a $\tau < \lambda$ with $z_\tau \geqq z$, $z_\tau \in R_{\underline{r}}$.

Then, as is not difficult to prove, the sequence $\{\underline{r}'t_\tau\}_{\tau<\lambda}$ is a compatible sequence of elements of $\mathfrak{O}$ and, moreover, of $\mathfrak{D}$ — in the already recalled sense of [Sch], if $\mathfrak{A}_\tau$ is the principal ideal of $\mathfrak{O}$ generated by the potency 2^{z_τ}, i. e. the ideal of all the elements u of $\mathfrak{O}$ such that $W(u) \geqq z_\tau$.

Further. it is easy to prove that any pseudolimit of $\{\underline{r}'z_\tau\}_{\tau<\lambda}$, i. e. any solution of the congruence system $x \equiv \underline{r}'z_\tau \pmod{\mathfrak{A}_\tau}$ in any immediate extension (see conv. XVIv) of $\mathfrak{D}$ is a solution of the congruence system $\underline{r}$ in the sense of our conv. XVI (iii) — and also conversely, any solution of $\underline{r}$ in any immediate extension of $\mathfrak{D}$ in our sense is a pseudolimit of $\{\underline{r}'z_\tau\}_{\tau<\lambda}$ in an immediate extension of $\mathfrak{Q}$, in the sense of [Sch]. Such a solution (pseudolimit) is unique if $R_{\underline{r}}$ (and $\{z_\tau\}_{\tau<\lambda}$) are cofinal to $R^{\leqq}$; otherwise, one congruence system (one mentioned compatible sequence) can have many solutions (= pseudolimits).

Conversely, let us consider a compatible sequence $\{a_\tau\}_{\tau<\lambda}$ of elements of a dyadic ring $\mathfrak{D}$ (as a subring of the valuation ring $\mathfrak{O}$ of the quotient field $\mathfrak{Q}$ of $\mathfrak{D}$ with the dyadic valuation); let $\{\mathfrak{A}_\tau\}_{\tau<\lambda}$ be the corresponding decreasing sequence of the valuation ideals in the already recalled sense of [Sch]. Then, as is not difficult to prove, in our case of dyadic rings, an ideal $\mathfrak{A}_\tau$ of the ring $\mathfrak{O}$ must be either a principal ideal generated by a potency of 2, say of 2^{z_τ} (with an unique $z_\tau \geqq 0$) or a s. c. limit ideal of such ideals, i. e. a set product of a decreasing transfinite sequence, say $\{\mathfrak{A}_{\tau\varrho}\}_{\varrho<{}^{\tau}\varrho}$, of such principal ideals generated by the potencies, say $2^{z_{\tau\varrho}}$, with $z_{\tau\varrho} < z_{\tau\varrho^*}$ whenever $\varrho < \varrho^* < {}^{\tau}\varrho$.

Let us distinguish two cases: Case (a): the $\tau^* = \tau_\varkappa$ $(\varkappa < \bar{\varkappa})$ with $\mathfrak{A}_{\tau^*} = (2^{z_{\tau^*}})$[16]) form a transfinite subsequence cofinal with the given sequence $\{\mathfrak{A}_\tau\}_{\tau<\lambda}$.

Case (b): from some $\tau = \bar{\tau}$ up, for every τ with $\bar{\tau} \leqq \tau < \lambda$ every $\mathfrak{A}_\tau$ is a limit ideal.

In the first case (a), let us form the set $R_{\underline{r}}$ as a subset of R consisting of all the $z_{\tau\varkappa}$ $(\varkappa < \bar{\varkappa})$.

In the second case (b), to any $\tau \geqq \bar{\tau}$ choose a $\varrho = \varrho_\tau$ so that $\mathfrak{A}_\tau \supseteqq \mathfrak{A}_{\tau+1\,\varrho_\tau+1} \supseteqq \mathfrak{A}_{\tau+1}$; this is possible. Then let $R_{\underline{r}}$ consist of all the $z_{\tau\varrho_\tau}$.

Now it is not difficult to prove that, letting the element $a_{\tau\varkappa}$ be $\underline{r}'z$ if $z = z_{\tau\varkappa} \in R_{\underline{r}}$ in the first — and the element a_τ be $\underline{r}'z$ if $z = z_{\tau\varrho_\tau} \in R_{\underline{r}}$ in the second case, we always obtain $\underline{r}$ as a compatible system of valuation congruences of $\mathfrak{D}$ in our sense, with the same solutions as $\{a_\tau\}_{\tau<\lambda}$ has.

Hence the equivalence relation between our congruence systems in dyadic rings, and of the compatible (pseudoconvergent) transfinite sequences of elements of such rings in the usual sense of valuation theory, may be clear. (Comp. also the remark 2 on p. 48 of [Sch], concerning the last part of our arguments.) Of course, one could work with our congruence systems instead of pseudoconvergent sequences (in an obvious general sense) in the general theory of valuation of algebraic fields also. But this generalization and replacement of the usual tools would hardly be useful in general valuation theory; though congruence systems in our sense seem to be especially suitable in the case of dyadic rings, because e. g. of the possibility of normalization of congruence systems (in the sense of lemma XV, as made possible by the close connection of the dyadic valuation and of the ordering in dyadic rings).

[16]) (2^z) means the principal ideal of 2^z.

Now it is easy to see how we could obtain our pseudoperfect ring extension ${}^{\mathfrak{D}}\mathfrak{R}$ of a dyadic ring $\mathfrak{D}$ by the general theory of valuations. Indeed, the ring ${}^{\mathfrak{D}}\mathfrak{R}$ *is nothing other than the ring of all the pseudolimits of pseudoconvergent transfinite sequences* (of elements of $\mathfrak{D}$), *of the s. c. zero breadth*,[17]) these limits being taken from one of the s. c. maximal immediate extensions of the quotient field $\mathfrak{Q}$ of the given $\mathfrak{D}$ (with respect to the dyadic valuation of $\mathfrak{Q}$). See theorem 2 of chap. 2. and the definitions 15 and 16, as well as the lemma 17 of [Sch]. Of course, the use of the very general and relatively complicated notion of the maximal immediate extension (of a field with valuation) is avoided on passing directly to the pseudoperfect ring extension ${}^{\mathfrak{D}}\mathfrak{R}$ of $\mathfrak{D}$.

Returning to our task, let us state a decisive

Lemma XXIII. (On "extensionality".) *In any dyadic ring* $\mathfrak{D} = \langle RF_1F_2F_3F_4\rangle$, *if* $x \neq y$, $x \in R^{\leqq}$, $y \in R^{\leqq}$ *then* $C^x_{W(x-y)} \neq C^y_{W(x-y)}$. In other terms: *If* $C^x_z = C^y_z$ *for every* $z > 0$ *then* $x = y$.

Proof. I. Prove first that either $\left[\frac{x}{2^z}\right] - \left[\frac{y}{2^z}\right] = \left[\frac{x-y}{2^z}\right]$ or $\left[\frac{y}{2^z}\right] - \left[\frac{x}{2^z}\right] = \left[\frac{y-x}{2^z}\right]$, for an arbitrary $z \in R^{\leqq}$.

Indeed, $\left[\frac{x}{2^z}\right] + u = \frac{x}{2^z}$, $\left[\frac{y}{2^z}\right] + v = \frac{y}{2^z}$ hold in the quotient field $\mathfrak{Q}$ of $\mathfrak{D}$, with uniquely determined elements u, v of $\mathfrak{Q}$ and with $0 \leqq u < 1$, $0 \leqq v < 1$. Hence if $v \leqq u$, then $\left[\frac{x-y}{2^z}\right] = \left[\frac{y}{2^z} - \frac{x}{2^z}\right] = \left[\left[\frac{x}{2^z}\right] - \left[\frac{y}{2^z}\right] + u - v\right] = \left[\frac{x}{2^z}\right] - \left[\frac{y}{2^z}\right]$ because of $0 \leqq u - v < 1$.

If however $u < v$, then clearly $\left[\frac{y-x}{2^z}\right] = \left[\frac{y}{2^z}\right] - \left[\frac{y}{2^z}\right]$ by the same argument (with $0 \leqq v - u < 1$ instead of $0 \leqq u - v < 1$).

II. Supposing $x \neq y$, let us now write (by conv. XII and lemma XVII)

$$\left.\begin{aligned} C^x_{W(x-y)} &= \left[\frac{x}{2^{W(x-y)}}\right] - 2\left[\frac{y}{2^{W(x-y)+1}}\right], \\ C^y_{W(x-y)} &= \left[\frac{x}{2^{W(x-y)}}\right] - 2\left[\frac{x}{2^{W(x-y)+1}}\right]. \end{aligned}\right\} \qquad (*)$$

If there were $C^x_{W(x-y)} = C^y_{W(x-y)}$, then by the already stated I and using the equality

$$\left[\frac{x}{2^{W(x-y)}}\right] - \left[\frac{y}{2^{W(x-y)}}\right] = 2\left(\left[\frac{y}{2^{W(x-y)+1}}\right] - \left[\frac{x}{2^{W(x-y)+1}}\right]\right)$$

(as resulting from (*)), we would obtain exactly one of the following equalities: either

[17]) The *breadth* of a pseudoconvergent sequence $\{a_\sigma\}_\sigma$ is the ideal of all the elements with values greater than every of the values $V(a_\sigma - a_{\sigma^*})$ (assuming $a_\sigma - a_{\sigma^*} \neq 0$, of course). See [Sch] p. 48.

a) $\left[\frac{x-y}{2^{W(x-y)}}\right] = 2\left[\frac{x-y}{2^{W(x-y)+1}}\right]$ or b) $\left[\frac{x-y}{2^{W(x-y)}}\right] = -2\left[\frac{y-x}{2^{W(x-y)+1}}\right]$,

or c) $-\left[\frac{y-x}{2^{W(x-y)}}\right] = 2\left[\frac{x-y}{2^{W(x-y)+1}}\right]$ or d) $-\left[\frac{y-x}{2^{W(x-y)}}\right] = -2\left[\frac{y-x}{2^{W(x-y)+1}}\right]$.

But since $\left[\frac{x-y}{2^{W(x-y)}}\right] = \frac{x-y}{2^{W(x-y)}} = 2q+1$ and $W(x-y) = W(y-x)$ (with an uniquely determined $q \in R$), we get a contradiction in all the possible cases a), b), c), d). Therefore the lemma is proved.

Corollary to the lemma XXIII. a) *Let us replace the nonnegative x, y in the lemma* XXIII *by the normal complete congruence systems $^1\underline{r}$, $^2\underline{r}$ of $\mathfrak{D}$ (as elements of the ring $\overline{\mathfrak{R}}$ of the lemma* XX). *Then the lemma* XXIII *remains valid*; *i. e., $^1\underline{r} \neq {}^1\underline{r}$ implies*

$$C^{^1\underline{r}}_{\overline{W}(^1\underline{r}-^2\underline{r})} \neq C^{^2\underline{r}}_{\overline{W}(^1\underline{r}-^2\underline{r})} .$$

(*Indeed, it suffices to use the lemma* XXIV *for $x = {}^1\underline{r}'z$, $y = {}^2\underline{r}'z$ with a sufficiently great $z \geqq 0$, in view of the lemma* XX.)

b) *As a special case of* a), *the lemma* XXIII *now extends to arbitrary elements x, y of $\mathfrak{D}$ (in using $\underline{r}_x = {}^1\underline{r}$, $\underline{r}_y = {}^2\underline{r}$).*

Remark. It turns out later that the (extended) lemma XXIII will ensure the axiom of extensionality A 3 in our dyadic models.

Lemma XXIV. a) *If $x \geqq 0$, then there is $C^{2^x-1}_z = 1$ whenever $0 \leqq z < x$, and $C^{2^x-1}_z = 0$ whenever $z \geqq x$.*

b) *If $2^z > y > 0$, then $C^y_z = 0$; and especially, if $z \geqq y > 0$, then $C^y_z = 0$ (for $2^z \geqq 2^y > y$).*

c) *$C^{\underline{r}}_z = 0$ whenever $z < \overline{W}(\underline{r})$; and especially $C^x_z = 0$ whenever $z < W(x)$.*

d) *$C^{\underline{r}}_{\overline{W}(\underline{r})} = 1$ as well as (especially) $C^y_{W(y)} = 1$.*

Proof. (Comp. lemma XVII and its proof.) a) If $0 \leqq z < x$, i.e. $x = z - 1 \geqq \geqq 0$, then $C^{2^x-1}_z = \left[\frac{2^x-1}{2^z}\right] - 2\left[\frac{2^x-1}{2^{z+1}}\right] = \left[2^{x-z} - \frac{1}{2^z}\right] - 2\left[2^{x-z-1} - \frac{1}{2^{z+1}}\right] = = 2^{x-z} - 1 - 2(2^{x-z-1} - 1) = 1$. If $0 < x \leqq z$, then $0 \leqq 2^x - 1 < 2^z < < 2^{z+1}$, whence $C^{2^x-1}_z = \left[\frac{2^x-1}{2^z}\right] - 2\left[\frac{2^x-1}{2^{z+1}}\right] = 0 - 0 = 0$.

b) If $2^z > y > 0$, then $C^y_z = \left[\frac{y}{2^z}\right] - 2\left[\frac{y}{2^{z+1}}\right] = 0 - 0 = 0$.

c) If $z < \overline{W}(\underline{r})$, then $C^{\underline{r}}_z = C^{\underline{r}'z^*}_z$ with a $z^* > \overline{W}(\underline{r})$ such that $\overline{W}(\underline{r}) = = \overline{W}(\underline{r}'z^*) = p$, $\underline{r}'z^* = 2^p(2q+1)$. Then

$$C^{\underline{r}'z^*}_z = \left[\frac{2^p(2q+1)}{2^z}\right] - 2\left[\frac{2^p(2q+1)}{2^{z+1}}\right] = 2^{p-z}(2q+1) - 2[2^{p-z}q + 2^{p-z-1}].$$

But since $p - z - 1 \geqq 0$, we have $[2^{p-z}q + 2^{p-z-1}] = 2^{p-z}q + 2^{p-z-1}$, whence indeed $C^{\underline{r}'z^*}_z = C^{\underline{r}}_z = 0$.

d) If $z = W(r)$, then now $p = z$ and

$$C_z^r = C_z^{r'z*} = \left[\frac{2^p(2q+1)}{2^p}\right] - 2\left[\frac{2^p(2q+1)}{2^{p+1}}\right] =$$
$$= 2q + 1 - 2\left[q + \frac{1}{2}\right] = 2q + 1 - 2q = 1 .$$

Let us add some more notions and lemmas needed in the sequal.

Lemma XXV and Convention XVII (On the integral part of a dyadic logarithm). *Let x be a positive element of a dyadic ring $\mathfrak{D}$. Assume there is a nonnegative z such that* $\left[\frac{x}{2}\right] < 2^z \leq x$.

Then such a z is uniquely determined by the x — and will be called the integral part of the dyadic logarithm of x, in symbols $z =$ Log (x).

Remark. There is Log $(2^z) = z = W(2^z)$, $W(x) =$ Log (x) if and only if $x = 2^z$, and $W(x) \leq$ Log (x) in general, supposing Log (x) exists, of course.

Proof of the lemma. Suppose the contrary, that $0 \leq z_1 < z_2$, $[\frac{1}{2}x] < 2^{z_1} < < 2^{z_2} \leq x$. Multiplying by 2 we get $2[\frac{1}{2}x] < 2^{z_1+1} < 2^{z_2+1} \leq 2x$. In case $2[\frac{1}{2}x] = x$ (x is "odd") we thus have $2^{z_2} \leq x < 2^{z_1+1} < 2^{z_2+1} \leq 2x$, whence $z_2 < z_1 + 1 < z_2 + 1$, which is impossible.

In the remaining case $2[\frac{1}{2}x] + 1 = x$ (is "even"), we have $2^{z_2} < x$ (i. e. $2^{z_2} \leq x - 1$) and $x - 1 < 2^{z_1+1} < 2^{z_2+1} \leq 2x$, whence $z_2 < z_1 + 1 < z_2 + + 1$ again. Therefore the lemma is proved.

Lemma XXVI. *There is* $C^x_{\mathrm{Log}(x)} = 1$ *and* $C_z^x = 0$ *whenever* $\bar{z} >$ Log (x) (*assuming* $x > 0$ *and* Log (x) *exists*).

Proof. Write $z =$ Log (x). Then $C_z^x = \left[\frac{x}{2^z}\right] - 2\left[\frac{x}{2^{z+1}}\right]$ and $\left[\frac{x}{2^{z+1}}\right] = 0$, $\left[\frac{x}{2^z}\right] \neq 0$ (by conv. XVII), whence $C_z^x = 1$; but if of $\bar{z} > z$, then $\left[\frac{x}{2^{\bar{z}}}\right] = 0$ too, whence $C_{\bar{z}}^x = 0$.

Lemma XXVII. a) *There is $x = x^* + 2^{\mathrm{Log}(x)}$ with $0 \leq x^* < 2^{\mathrm{Log}(x)}$ (x^* uniquely determined by x), provided $x > 0$ and* Log (x) *exists; hence always $x \geq \geq 2^{\mathrm{Log}(x)}$.*

b) *If $0 < x \leq y$, then* Log $(x) \leq$ Log (y) *provided both* Log (x), Log (y) *exist.*

c) *There is* Log (x) + Log $(y) \leq$ Log $(xy) \leq$ Log (x) + Log $(y) + 1$, *provided $x, y > 0$, if all the* Log *in question exist.*

d) *If $C_z^x \leq C_z^y$ for every positive z then $x \leq 2y$, provided $x > 0$, $y > 0$, if* Log (x), Log (y) *exist.* (Caution: we cannot assert $((z)\ C_z^x \leq C_z^y) \supset x \leq y$ here, as in the dyadic ring of integers; compare however with theorem I in § 3).

Proof. Ad a): If there were $x^* = x - 2^{\mathrm{Log}(x)} \geqq 2^{\mathrm{Log}(x)}$, then we would have $x \geqq 2^{\mathrm{Log}(x)} + 2^{\mathrm{Log}(x)} = 2^{\mathrm{Log}(x)+1}$ in contradiction to the definition of Log (conv. XVII). Since $x^* = x - 2^{\mathrm{Log}(x)} \geqq 0$ by definition of Log, a) is proved.

Ad b): If there were Log $(x) >$ Log (y), then we would obtain (since $[\frac{1}{2}x] \leqq [\frac{1}{2}y]$) $[\frac{1}{2}x] \leqq [\frac{1}{2}y] < 2^{\mathrm{Log}(y)} < 2^{\mathrm{Log}(x)} \leqq x$ in contradiction to the lemma XXV.

Ad c): In view of a) and b) we can write

$$2^{\mathrm{Log}(x)+\mathrm{Log}(y)} = 2^{\mathrm{Log}(x)} \cdot 2^{\mathrm{Log}(y)} \leqq xy = (x^* + 2^{\mathrm{Log}(x)})(y^* + 2^{\mathrm{Log}(y)}) =$$
$$= x^*y^* + x^*2^{\mathrm{Log}(y)} + y^*2^{\mathrm{Log}(x)} + 2^{\mathrm{Log}(x)+\mathrm{Log}(y)} < 2^{\mathrm{Log}(x)+\mathrm{Log}(y)} +$$
$$+ 2^{\mathrm{Log}(x)+\mathrm{Log}(y)} + 2^{\mathrm{Log}(x)+\mathrm{Log}(y)} + 2^{\mathrm{Log}(x)+\mathrm{Log}(y)} = 2^2 \cdot 2^{\mathrm{Log}(x)+\mathrm{Log}(y)} = 2^{\mathrm{Log}(x)+\mathrm{Log}(y)+2}.$$

Therefore on account of b) we indeed obtain

$$\mathrm{Log}\,(x) + \mathrm{Log}\,(y) \leqq \mathrm{Log}\,(xy) < \mathrm{Log}\,(x) + \mathrm{Log}\,(y) + 2\,, \quad \text{q. e. d.}$$

Ad d): If there were $x > 2y$, then we would have Log $(x) \geqq$ Log $2y \geqq$ $\geqq$ Log (y) + Log 2 = Log $(y) + 1$ because of c), but since Log $(x) \leqq$ Log (y) by supposition (in regard to the corollary of Lemma XXV), contradiction.

Definition III (The logarithmicity of a dyadic ring). A dyadic ring $\mathfrak{D}$ every positive element x of which possesses the Log (x) is called *logarithmic*.

Remark. Of course, the ring of integers is a logarithmic dyadic ring. Perhaps not every dyadic ring is logarithmic, though only such rings are important for the present purposes.

3. Dyadic rings of the set theoretical type (s-t-rings) and dyadic models of the Gödel's axiomatic theory of finite sets.

In the sequel ‚$\mathfrak{D}$' continues to mean a dyadic ring. If $C_z^x = 1$ $(x \geqq 0,\ y \geqq 0)$ then we often say that the *"set" z is an "element" of the "set" x*. In order to prepare the construction of dyadic models (of Gödel's theory of finite sets) we shall often use the following normal terms introduced by the

Convention XVIII (In accordance with [G], definitions 1.1, 1.11, 1.14, 1.15). We abbreviate: (for every x, y as elements of a dyadic ring)

a) $\{xy\}_* = [2^x] + [2^y]\,\mathrm{sg}\,(|x - y|)$ (the *"pair"* ("non-ordered")),

c) $\langle xy\rangle_* = \{\{x\}_*\,\{xy\}_*\}_*$ (the *"ordered pair"*),

d) $\langle x\rangle_* = x$ (the *"ordered* 1*-tuple"*),

e) $\langle x_1x_2 \ldots x_n\rangle_* = \langle x_1\,\langle x_2 \ldots x_n\rangle_*\rangle_*$ (the *"ordered n-tuple"*, by induction for $n = 2, 3, \ldots$).

Caution. The integers $n = 2, 3, \ldots$ $(n \in \omega_0)$ are to be taken in the *relative* sense of our basic set theory (as finite ordinals). But see (β) of the proof of thm. I!

Lemma XXVIII (In accordance with [G], thm. 1.13, 1.16). *In every dyadic ring there is*

a) $\{xy\}_* = \{yx\}_*$; b) $\{x\}_* = 2^x$ *if* $x \geqq 0$;

b) $\{x\}_* = \{xx\}_*$,

c) $\{x_1 y_1\}_* = \{x_2 y_2\}_* \equiv \{x_1 y_1\} = \{x_2 y_2\}$ *if* $x_1 \geqq 0$, $y_1 \geqq 0$, $i = 1, 2$;

d) $\{xy\}_* = 0 \equiv x < 0$, $y < 0$; e) $x \geqq 0, y \geqq 0, x = y \equiv W(\{xy\}_*) = x$;

f) $\{xy\}_* = 2^x + 2^y$ *if* $x \geqq 0$, $y \geqq 0$, $x \neq y$;

g) $\langle xy\rangle_* = 2^{2^x} + 2^{2^x + 2^y}$ *if* $x \geqq 0$, $y \geqq 0$, $x \neq y$; *and* $\langle xy\rangle_* = \langle xx\rangle_* = 2^{2^x}$ *if* $x \geqq 0$, $y \geqq 0$, $x = y$; *and* $\langle xy\rangle_* \neq 0$;

h) $\langle x_1 y_1\rangle_* = \langle x_2 y_2\rangle_* \equiv (x_1 = x_2)(y_1 = y_2)$ *provided* $x_i \geqq 0$, $y_i \geqq 0$ $(i = 1, 2))$;

i) $\{xy\}_* > \max(x, y)$ *and* $\langle xy\rangle_* > \max(x, y)$ *if* $x \geqq 0$, $y \geqq 0$;

j) $\{x_1 y\}_* < \{x_2 y\}_*$ *and* $\langle x_1 y\rangle_* < \langle x_2 y\rangle_*$ *if* $y \geqq 0$, $0 \leqq x_1 < x_2$, $\{x y_1\}_* < \{x y_2\}_*$ *and* $\langle x y_1\rangle_* < \langle x y_2\rangle_*$ *if* $x \geqq 0$, $0 \leqq y_1 < y_2$.

The easy proofs may be omitted. The lemma will often be used tacitly.

Let us further introduce the following normal terms and notions:

Convention XIX. a) For every $u \in R$ set ${}^1u = x$ if $u = \langle xy\rangle_*$, $0 \leqq x$, $0 \leqq y$ — and ${}^1u = -1$ otherwise; if $u = \langle xy\rangle_*$ then we call ${}^1u = x$ the "*first member*" of the "ordered pair" $\langle xy\rangle_*$ (provided $x \geqq 0$, $y \geqq 0$, thus especially ${}^1\langle xx\rangle_* = x$ if $x \geqq 0$);

b) For every $u \in R$ set ${}^2u = y$ if $u = \langle xy\rangle_*$ provided $0 \leqq x$, $0 \leqq y$, $x \neq y$ — and set ${}^2u = {}^1u$ otherwise; if $u = \langle xy\rangle_*$, then call ${}^2u = y$ the "*second member*" of the "ordered pair" $u = \langle xy\rangle_*$ whenever $x \geqq 0$, $y \geqq 0$.

Now we are able to state our main notion of the set theoretical dyadic ring, called in short *s-t*-ring. — For the convenience of readers, let us introduce the definition of this notion by the following explicative remarks:

The definitory requirements (s I)—(s VII) are essentially axioms of the group B (of [G] in an "arithmetical" formulation (the class-complement axiom B 3 together with the class-product axiom B 2 replaced by a certain "set-difference axiom" (s II).) But our "axioms" have only to do with "sets" instead of "classes" of the s. c. dyadic model (to be constructed). In order to formulate the arithmetical axiom (s III) equivalent to the domain axiom B 4 of [G], we need two new primitive notions[18a]) (operations) D_1 and D_2, in addition to those of the dyadic ring of definition II; we formulate (s III) as an identity requirement for D_1, D_2. The last "axiom" (s VIII) represents a weakened ("concrete") form of the choice axiom E of [G]. And finally, the singular initial "axiom" (s $\emptyset$) will ensure, first the needed strong existence metatheorems (M 1)**—(M 6)** of [G] (analogous to M 1—M 6 in the model)

[18a]) During the printing. I happened to define D_1 as a secondary operation-and to avoid D_2 entirely. This will be shown in a next paper, as well as the avoidance of (s VIII).

and it also will make possible the verification of non C 1, C 2, C 3, C 4 as we shall see in the proof of theorem I. Each of the definitory requirements has a name and is followed by a comment in order to indicate its sense intuitively; further, conventions are added everywhere in order to introduce the corresponding defined operation (as a function given by a normal term, on the basis of M 5 of [G]).

Definition IV (The set theoretical dyadic ring). Let $\mathfrak{S} = \langle RF_1F_2F_3F_4D_1D_2\rangle$ be an ordered seventuple. We say that $\mathfrak{S}$ is an *s-t-ring* (*set theoretical dyadic ring*), whenever.

a) $\langle RF_1F_2F_3F_4\rangle$ is a dyadic ring in the sense of the def. II,

b) the following requirements (s $\emptyset$), (s 1)—(s VIII) are satisfied. (We use the already introduced symbols without recalling their definitions, hence the knowledge of the preceding § 2 is necessary. Let us emphasize, once for all, that the individual variables as well as their quantifications are all meant to be relativized to the set $R^{\geqq}$ of nonnegative elements (the s. c. "sets") of the dyadic ring in question — if nothing other is said explicitely.)

(s $\emptyset$) (of the "successor relation"):

$$(x)\ \exists\ y(z)(v)(C^y_z = C^x_z\ [2^{-|v-{}^1z|}][2^{-|v+1-{}^2z|}])\ .$$

(In words: to every "set" x there is a "set" y formed exactly of all the "ordered pairs" $z = \langle v\ v + 1\rangle_*$ which are "elements" of x).

Remark to (s $\emptyset$). The y in (s $\emptyset$) is uniquely determined by the given x in view of the decisive lemma XXIII. Therefore on the basis of (s $\emptyset$), there is a unique function, say F_8, on $R^{\geqq}$ into $R^{\geqq}$ (we write in a normal term equation $F_8'\ x = y$)[18b]) determined by the identity

$$C^{F_8'x}_z = C^x_z\,[2^{-|v-{}^1z|}][2^{-|v+1-{}^2z|}] \quad \text{for every } x,\ z,\ v \geqq 0\ .$$

(s I) (on the "to belong" relation):

$$(x)\ \exists\ y(z)(u)(v)(C^y_z = C^x_z\ C^u_v\ [2^{-|v-{}^1z|}][2^{-|u-{}^2z|}])\ .$$

(In words: To every "set" x there is a "set" y formed exactly of all the "ordered pairs" $z = \langle vu\rangle_*$ which are "elements" of x and such that the "first member" is an "element" of the "second member".)

Remark to (s I). By a convention and in the sense analogous to the preceding one (with (s I) instead of (s $\emptyset$)) we introduce the normal term F_9 denoting the unique function on $R^{\geqq}$ into $R^{\geqq}$ which satisfies the identity (for $x,\ z,\ u,\ v \geqq 0$)

$$C^{F_9'x}_z = C^x_z\ C^u_v\ [2^{-|v-{}^1z|}][2^{-|u-{}^2z|}]\ .$$

(s II) (on the "set difference"):

$$(x)(y)\ \exists z(u)(C^z_u = [2^{C^x_u - C^y_u - 1}])\ .$$

18b) ‚F_8' is itself a normal term dependent on the normal term ‚$\mathfrak{S}$'. Analogously later.

(In words: To arbitrary "sets" x, y there is a "set" z so that u is an "element" for z if and only if u is an "element" of x and u is not an "element" of y.)

Remark to (s II). By a convention analogous to the preceding ones, we introduce the normal term F_{10} as denoting the unique function on $R^{\leqq} \times R^{\leqq}$ into $R^{\leqq}$ such that $C_u^{F'_{10}\langle xy\rangle} = [2^{C_u^x - C_u^y - 1}]$ identically in $x, y, u \geqq 0$. $F'_{10}\langle x, y\rangle$ is called the "set difference" between x and y (in $\mathfrak{S}$). — Write also for more convenience $z = x \mathbin{\dot{-}_*} y$ instead of $z = F'_{10}\langle xy\rangle$.

(s III) (the "axiom of the domain"): D_1 is a function on $R^{\leqq}$ in $R^{\leqq}$, D_2 is a function on $R^{\leqq} \times R^{\leqq}$ into $R^{\leqq}$, D_1 may be called the first, D_2 the second of the s. c. domain-operations; D_1 and D_2 satisfy the following identity (in $x, u, w \geqq 0$):

$$C_u^{\boldsymbol{D}_1{}'x}(C^u_{\langle u\,\boldsymbol{D}_2{}'\langle xu\rangle\rangle_*} - 1) + C^x_{\langle uw\rangle_*}(C_u^{\boldsymbol{D}_1{}'x} - 1) = 0\,.$$

Remarks to (s III). a) This non-intuitive arithmetical formulation of the axiom B 4 (of the domain, with respect to "sets") is to be understood as follows:

The immediate transcription of B 4 for "sets" is

$$(x)\ \exists y(u)(C_u^y = 1 \equiv \exists v(C^x_{\langle uv\rangle_*} = 1))\,. \tag{*}$$

(In words: To every "set" x there is a "set" y se that u is an "element" of y if and only if u is the "first member" of an "ordered pair" $\langle uv\rangle_*$ being an "element" of x.)

In order to avoid existential quantifiers by the well known device due to Skolem (see [Sk]), we transform (by an easy logical and arithmetical adaptation) this p. f. into an equivalent (on the basis of the requirements of def. II) prenex normal form

$$(x)\ \exists y(u)\ \exists v(w)(C_u^y(C^x_{\langle uv\rangle_*} - 1) + C^x_{\langle uw\rangle_*}(C_u^y - 1) = 0)\,. \tag{**}$$

This form clearly is itself equivalent to the existence of the operations D_1 and D_2 satisfying the identity (s III), on the basis of the axiom of choice E.

b) Returning (in our basic set theory) from the assumed identity sub (s III) to its equivalent form (*), we may and shall introduce, by a convention analogous to the preceding ones, the uniquely determined function, say F_{11}, on $R^{\leqq}$ into $R^{\leqq}$, such that the equivalence

$$C_u^{F'_{11}x} = 1 \equiv \exists v(C^x_{\langle uv\rangle_*} = 1)$$

is true for every $x \geqq 0$, $u \geqq 0$. Write also $y = \mathbf{D}'_* x$ instead of $y = F'_{11}x$. $\mathbf{D}'_* x$ is called the "domain" of x (in $\mathfrak{S}$).

(s IV) (on the "direct product"):

$$(x)(y)\ \exists z(u)(v)(w)(C_w^z = C_u^x\, C_v^y\, [2^{-|u - {}^1w|}][2^{-|v - {}^2w|}])\,.$$

(In words: To arbitrary "sets" x, y there is a "set" z so that z has for its "elements" w exactly all the "ordered" pairs $\langle uv\rangle_* = w$ with the "first member" u an "element" of x and the "second member" v an "element" of y.)

Remark to (s IV). By a convention analogous to the preceding ones, we introduce the unique function, say F_{12}, on $R^{\leqq} \times R^{\leqq}$ into $R^{\leqq}$, such that

$$C_w^{F'_{12}\langle xy\rangle} = C_u^x C_u^y [2^{-|u-{}^1w|}][2^{-|v-{}^2w|}]$$

identically in $x \geqq 0$, $y \geqq 0$, $u \geqq 0$, $v \geqq 0$, $w \geqq 0$. Write also $z = x \overset{\times}{*} y$ instead of $z = F'_{12}\langle xy\rangle$. $x \overset{\times}{*} y$ is called the "direct product" of x by y.

The following three requirements "on conversions" in the "ordered pairs" and "ordered triples" (see conv. XVIIIe) as corresponding to the remaining axioms of the group B, can now be stated in short with the corresponding conventions as follows:

(s V) (on the "first conversion"):

$$(x)\,\exists\, y(z)(u)(v)(C_z^y = C_{\langle vu\rangle_*}^x[2^{-|u-{}^1z|}]]2^{-|v-{}^2z|}]) .$$

The corresponding function F_{13} on $R^{\leqq}$ into $R^{\leqq}$ satisfies

$$C_z^{F'_{13}x} = C_{\langle vu\rangle_*}^x[2^{-|u-{}^1z|}][2^{-|v-{}^2z|}]$$

identically in $x \geqq 0$, $z \geqq 0$, $u \geqq 0$, $v \geqq 0$. Write also $y = \mathbf{C}nv_{*1}(x)$ instead of $y = F'_{13}x$ and call y the "first conversion" of x (in $\mathfrak{S}$).

(s VI) (on the "second conversion"):

$$(x)\,\exists\, y(z)(u)(v)(w)(C_z^y = C_{\langle uvw\rangle_*}^x[2^{-|u-{}^1({}^2z)|}][2^{-|v-{}^1({}^1z)|}][2^{-|w-{}^2({}^2z)|}]) .$$

The corresponding function F_{14} on $R^{\leqq}$ into $R^{\leqq}$ satisfies

$$C_z^{F'_{14}x} = C_{\langle uvw\rangle_*}^x[2^{-|u-{}^1({}^2z)|}][2^{-|v-{}^1({}^1z)|}][2^{-|w-{}^2({}^2z)|}]$$

identically in $x \geqq 0$, $z \geqq 0$, $u \geqq 0$, $v \geqq 0$, $w \geqq 0$. Write also $y = \mathbf{C}nv_{*2}(x)$ instead of $y = F'_{14}x$, and call y the "second conversion" of x (in $\mathfrak{S}$).

(s VII) (on the "third conversion"):

$$(x)\,\exists\, y(z)(u)(v)(w)(C_z^y = C_{\langle uvw\rangle_*}^x[2^{-|u-{}^2({}^2z)|}][2^{-|v-{}^1({}^2z)|}][2^{-|w-{}^1({}^1z)|}]) .$$

The corresponding function F_{15} on $R^{\leqq}$ into $R^{\leqq}$ satisfies

$$C_z^{F'_{15}x} = C_{\langle uvw\rangle_*}^x[2^{-|u-{}^2({}^2z)|}][2^{-|v-{}^1({}^2z)|}][2^{-|w-{}^1({}^1z)|}]$$

identically in $x \geqq 0$, $z \geqq 0$, $u \geqq 0$, $v \geqq 0$, $w \geqq 0$. Write also $y = \mathbf{C}nv_{*3}(x)$ instead of $y = F'_{15}x$, and call y the "third conversion" of x (in $\mathfrak{S}$).

Remark. This last "conversion axiom" could perhaps be omitted, according to a result of A. Hajnal and L. Kalmár [H-K]; in any case, it will not do any harm.

The last requirement now is

(s VIII) (the "axiom" of the "dyadic valuation function", i. e. an "axiom of choice")

$$(x)\,\exists\, y(z)(v)(C_z^y = [2^{-|z-\langle W(v)v\rangle_*|}]\, C_z^x) .$$

(In words: To every "set" x there is a "set" y having as "elements" exactly all the "ordered pairs" of the form $z = \langle W(v)\, v\rangle_*$ which are "elements" of the "set" x. — Since $W(v)$ is always an "element" of $v > 0$ (see lemma XXIVd), the connection of (s VIII) with the axiom of choice E of [G] is obvious.

As in the previous cases, we introduce the corresponding function, say F_{16}, on $R^{\leqq}$ into $R^{\leqq}$, such that $C_z^{F'_{16}x} = [2^{-|z-\langle W(v)v\rangle_*|}]\, C_z^x$ identically in $x \geqq 0$, $z \geqq 0$, $v > 0$. — This completes our definition IV.

Remark to the definition IV. *S-t*-rings exist: the ring of integers is such, as it is easy to see. The construction of other *s-t*-rings is one of our main tasks — see § 4 later. — We shall use ‚$\mathfrak{S}$' as a normal variable (possibly with subscripts) for *s-t*-rings.

Now, we need a suitable modification of the notion of the immediate pseudoperfect extension of a dyadic ring, for the case of *s-t*-rings.

Definition V (The weakly pseudoperfect extension of an *s-t*-ring). Let $\mathfrak{S} = \langle RF_1F_2F_3F_4D_1D_2\rangle$ be an *s-t*-ring with the corresponding dyadic ring $\mathfrak{D} = \langle RF_1F_2F_3F_4\rangle$.

Assume $\mathfrak{R}^*$ is a subring of the immediate pseudoperfect extension of ${}^{\mathfrak{D}}\mathfrak{R}$ the dyadic ring $\mathfrak{D}$. Then $\mathfrak{R}^*$ is called *weakly pseudoperfect with respect to* $\mathfrak{S}$ if the following (i)—(v) is true. (See conv. XVI and lemma XXII.)

(i) $\mathfrak{R}^*$ is an extension of $\mathfrak{D}$ (and therefore an immediate extension of $\mathfrak{D}$, see conv. XIV).

(ii) There is an element, say $\langle +1\rangle$, of $\mathfrak{R}^*$, such that $C_x^{\langle +1\rangle} = 1$ if and only if $x = \langle u\, u+1\rangle_*$, $(u \geqq 0)$. $\langle +1\rangle$ is said to be "class" of the "*successor relation*" of $\mathfrak{S}$.

(iii) There is an element, say $\langle \epsilon_*\rangle$, of $\mathfrak{R}^*$, such that $C_x^{\langle \epsilon_*\rangle} = 1$ if and only if $x = \langle uv\rangle_*$ and $C_u^v = 1$. $\langle \epsilon_*\rangle$ is said to be the "class" of the "*to belong relation*" of $\mathfrak{S}$.

(iv) There is an element of $\mathfrak{R}^*$, say W_*, such that $C_x^{W_*} = 1$ if and only if $x = \langle uv\rangle_*$ and $v = W(u)$. W_* is said to be the "*universal* (dyadic) *choice function*" of $\mathfrak{S}$.

(v) Let $\tilde{x}$, $\tilde{y}$ be elements of $\mathfrak{R}^*$. Then there is a) exactly one element of $\mathfrak{R}^*$, say $\tilde{x} \dot{-}_* \tilde{y}$, such that $C_u^{\tilde{x} \dot{-}_* \tilde{y}} = 1 \equiv (C_u^{\tilde{x}} = 1)(C_u^{\tilde{y}} = 0)$;

b) exactly one element, say $\tilde{x} \overset{\times}{*} \tilde{y}$, of $\mathfrak{R}^*$, such that $X_u^{\tilde{x} \times y} = 1$ if and only if $u = \langle u_1u_2\rangle_*$ and $C_{u_1}^{\tilde{x}} = C_{u_2}^{\tilde{y}} = 1$;

c) exactly one element of $\mathfrak{R}^*$, say $\mathbf{D}_*(\tilde{x})$, such that $C_u^{\mathbf{D}_*(\tilde{x})} = 1$ if and only if there is a $z \geqq 0$ with $C_{\langle uz\rangle_*}^{\tilde{x}} = 1$ $(u \geqq 0,\ z \geqq 0)$;

d) exactly one element, say $\mathbf{C}nv_{*1}(\tilde{x})$, of $\mathfrak{R}^*$, such that $C_u^{\mathbf{C}nv_{*1}(\tilde{x})} = 1$ if and only if $u = \langle u_1u_2\rangle_*$, $C_{\langle u_2u_1\rangle_*}^{\tilde{x}} = 1$;

e) exactly one element, say $\mathbf{C}nv_{*2}(\tilde{x})$, of $\mathfrak{R}^*$, such that $C_u^{\mathbf{C}nv_{*2}(\tilde{x})} = 1$ if and only if $u = \langle u_1u_2u_3\rangle_*$, $C_{\langle u_2u_1u_3\rangle_*}^{\tilde{x}} = 1$;

f) exactly one element of $\mathfrak{R}^*$, say $\mathbf{C}nv_{*3}(\tilde{x})$, such that $C_u^{\mathbf{C}nv_{*3}(\tilde{x})} = 1$ if and only if $u = \langle u_1u_2u_3\rangle$, $C_{\langle u_3u_2u_1\rangle_*}^{\tilde{x}} \doteq 1$.

The elements of $\mathfrak{R}^*$ required in (v), a)—f) are respectively called the "*class difference*", the "*direct product of classes*", the "*domain*" of a "*cla*ss" and the "*first, second* and *third conversion*" of a "class"; in general, the elements of a weakly pseudoperfect immediate extension of the dyadic ring of any *s-t*-ring may be called "*classes*".

Lemma XXIX (A condition for the compatibility of valuation congruence systems). *A function* $\underline{r}$ *defined on a convex and zero containing subset* $R_{\underline{r}}$ *of the set* $R^{\leqq}$ *of nonnegative elements of any dyadic ring* $\mathfrak{D}$ *is a normal*[19]) *compatible systems of valuation congruences if and only if the following is true*:

(i) $C_u^{\underline{r}'z_1} = C_u^{\underline{r}'z_2}$ *whenever* $u < z_1 \leqq z_2$, $z_1 \in R_{\underline{r}}$, $z_2 \in R_{\underline{r}}$ (*weakened compatibility*),

(ii) $C_v^{\underline{r}'z} = 0$ *whenever* $z \leqq v$. $z \in R_{\underline{r}}$ (*weakened normality*).

Proof. The necessity of (i) and (ii) follows immediately from the corresponding definitions (see § 2). — In order to prove their sufficiency, first assume without loss of generality) $\underline{r}'z_1 \neq \underline{r}'z_2$, $0 \leqq z_2 < z_2$, $z_1 \in R_{\underline{r}}$, $z_2 \in R_{\underline{r}}$ satisfying (i), (ii). Then $C_{W(\underline{r}'z_2 - \underline{r}'z_1)}^{\underline{r}'z_1} \neq C_{W(\underline{r}'z_2 - \underline{r}'z_1)}^{\underline{r}'z_2}$ by lemma XXIII. In view of (i) and (ii) this is possible only with $C_{W(\underline{r}'z_2 - \underline{r}'z_1)}^{\underline{r}'z_2} = 0$ and $C_{W(\underline{r}'z_2 - \underline{r}'z_1)}^{\underline{r}'z_1} = 1$ as well as with $z_2 > W(\underline{r}'z_2 - \underline{r}'z_1) \geqq z_1$. Therefore $\underline{r}'z_2 - \underline{r}'z_1 \equiv 0 \pmod{2^{z_1}}$ and the more so $\underline{r}'z_2 \equiv \underline{r}'z_1 \pmod{2^u}$ for every u with $0 \leqq u \leqq z_1$; q. e. d.

Lemma XXX (On the minimal weakly pseudoperfect immediate extension of an *s-t*-ring). *Assume the symbols of the definition* V.

To every given s-t-ring $\mathfrak{S}$, *there is a uniquely determined weakly pseudoperfect extension* (*in the sense of the definition* V), *say* $\mathfrak{R}_{\mathfrak{S}}$, *such that* $\mathfrak{R}_{\mathfrak{S}}$ *is a subring of any other immediate and weakly pseudoperfect ring extension* $\mathfrak{R}^*$ *of* $\mathfrak{D}$. *The ring* $\mathfrak{R}_{\mathfrak{S}}$ *is called the minimal weakly pseudoperfect extension of* $\mathfrak{S}$. *The power of the set of elements of* $\mathfrak{R}_{\mathfrak{S}}$ *equals that of the given* $\mathfrak{S}$.

Proof. First, prove the existence and unicity of the "classes" required in (ii), (iii) and (iv) of the definition V — as elements of the pseudoperfect immediate extension ${}^{\mathfrak{D}}\mathfrak{R}$ of $\mathfrak{D}$ (see conv. XV and lemma XXII).

Ad (ii) of the definition *V*: Let us define the function $\langle +1 \rangle$ on $R^{\leqq}$ into $R^{\leqq}$ by the (normal term) equation $F_8'(2^z - 1) = \langle +1 \rangle' z$ (with $z \geqq 0$). — Note that if $0 \leqq z_1 < z_2$, then $(z)(C_z^{2^{z_1}-1} \leqq C_z^{2^{z_1}-1})$ by the lemma XXIVa). Therefore in view of (s $\emptyset$) of definition IV we observe that the condition (i) of the lemma XXIX is fullfilled by the $\underline{r} = \langle +1 \rangle$ (with $R_{\underline{r}} = R^{\leqq}$). The same is true as to the second condition (ii) of lemma XXIX. Hence the existence and unicity of the element of ${}^{\mathfrak{D}}\mathfrak{R}$ required by (ii) of the definition V is clear. Ad (iii) of the definition V. — Put $F_9'(2^z - 1) = \langle \epsilon_* \rangle' z$, use (s I) (instead of (s $\emptyset$)) and argue as before.

Ad (iv) of the definition V. — Put $F_{16}'(2^z - 1) = W_*'z$, use (s VIII) and argue as before.

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19) But not necessarily complete.

Pass to the requirement (v) of the definition V. For the sake of greater formal simplicity, let us work, for a moment, in the ring $\overline{\mathfrak{R}}$ of complete normal congruence systems of $\mathbf{D}$, in view of the value preserving isomorphism between $\overline{\mathfrak{R}}$ and ${}^{\mathfrak{D}}\mathfrak{R}$ (see lemma XX). On account of the lemma XXIX, we have to show that, given normal complete congruence systems r, s, the functions

$$\underline{r} \underset{*}{\dot{+}} \underline{s}\,, \quad \underline{r} \overset{\times}{\underset{*}{}} \underline{s}\,, \quad \mathbf{D}_*(r)\,, \quad \mathbf{Cnv}_{*i}(r) \quad (\text{with } i = 1,2,3)$$ [20])

defined by the following equations:

$$(\underline{r} \underset{*}{\dot{+}} \underline{s})'\, z = F'_{10}\, \langle \underline{r}'z\, \underline{s}'z \rangle\,,$$
$$(\underline{r} \overset{\times}{\underset{*}{}} \underline{s})'\, z = F'_{10}\langle 2^z - 1\, F'_{10}\langle 2^z - 1\, F'_{12}\langle \underline{r}'z\, \underline{s}'z \rangle\rangle\rangle\,,$$
$$(\mathbf{D}_*(\underline{r}))'\, z = F'_{11}\underline{r}'z\,,$$
$$(\mathbf{Cnv}_{*i}(\underline{r}))'\, z = F'_{10}\langle 2^z - 1\, F'_{10}\langle 2^z - 1\, F'_{12+i}\, \underline{r}'z \rangle\rangle \quad (i = 1,\, 2,\, 3)\,,$$

satisfy the conditions (i) and (ii) of the lemma XXIX. This is not difficult (though somewhat lenghty) to prove by the argument just used as based on (s II)—(s VII) of the definition IV — exactly in the same manner as we have based on (s ∅) the first instance. — Now, returning to ${}^{\mathfrak{D}}\mathfrak{R}$ (from $\overline{\mathfrak{R}}$), we conclude the proof in taking the set $R^{\leqq} + \{\langle +1 \rangle\} + \{\langle \epsilon_* \rangle\} + \{W_*\}$ of the s. c. "basic classes" for "basic" elements of the desired ring $\mathfrak{R}_{\mathfrak{S}}$ — and in forming $\mathfrak{R}_{\mathfrak{S}}$ itself by means of the closure in ${}^{\mathfrak{D}}\mathfrak{R}$ of the set of "basic classes" with respect to the operations $\underset{*}{\dot{+}}$, $\overset{\times}{\underset{*}{}}$, $\mathbf{D}_*$, $\mathbf{Cnv}_{*i}$ $(i = 1,\, 2,\, 3)$ in ${}^{\mathfrak{D}}\mathfrak{R}$ — in the well known sense, see e. g. [G], def. 8.7, 8.71, 8.72 and Theorem *8.73. Since the rest of the proof is now immediate, the lemma is proved.

Now, we are able to state our first main

Theorem I (On the dyadic model of Gödel's axiomatic theory of finite sets). *Let* $\mathfrak{S} = \langle RF_1F_2F_3F_4D_1D_2 \rangle$ *be an s-t-ring and* $\mathfrak{R}^* = \langle R^*F_1^*F_2^*W^* \rangle$ *be an immediate weakly pseudoperfect extension of* $\mathfrak{S}$, *in the sense of the definition* V. *Put*

$$\mathbf{Cls}_*(y) \equiv y \in R^*\,, \quad \mathbf{M}_*(x) \equiv x \in R^{\leqq}\,,$$
$$x \in_* y \equiv (C^y_x = 1)(y \in R^*)(x \in R^{\leqq})$$
$$(\text{admitting formally } C^y_x = 0 \text{ if } x \in R^* \dot{-} R^{\leqq})\,.$$

Then $\mathbf{Cls}_*$, $\mathbf{M}_*$, ϵ_* *define (by interpretation) the s. c. dyadic model* $\Delta(\mathfrak{S}, \mathfrak{R}^*)$ *of the axiomatic theory of finite sets of* [G] *as based on the axioms sub* A — *sub* E — *the axiom of infinity* C 1 *replaced by its contrary* ∼C 1 (non C 1), *i. e. by the s. c. axiom of finity.* (Concerning the syntactical notion of interpretation and model, see [I], § 1.)

Proof. (α) Axioms A 1, A 2 clearly are valid (see lemma XXI).

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[20]) $(\underline{r} \overset{\times}{\underset{*}{}} \underline{s})'z$ is the "set product" of $\underline{r}'z \overset{\times}{\underset{*}{}} \underline{s}'z$ by $2^z - 1$, in order to get the normality of $\underline{r} \overset{\times}{\underset{*}{}} \underline{s}$ (analogously later).

The axiom A 3 of extensionality is given by the lemma XXIII and by the corresponding corollary. The pair axiom A 4 obviously is warranted by lemma XXVIII and conv XVIII, with the "pair" $\{xy\}_* = 2^x + 2^y$ if $x \neq y$, resp. with $\{xx\}_* = 2^x$ if $x = y$, of the "sets" x, y. Thus the axioms sub A are verified; note that this does not depend on the s-t-postulates (s $\emptyset$), (s I)—(s VIII).

Let us agree that the class-theoretical and syntactical notions of our model shall be systematically written by the corresponding words in quotes; the symbols of the model shall be denoted by an asterisk. Thus e. g. "proper class" means an element of the set $R^* \dot{-} R^{\leqq}$ of elements of the ring $\mathfrak{R}^*$ which are not nonnegative; note that all the negative elements of $\mathfrak{S}$ are "proper classes" and e. g. -1 is the "universal class" (see the corollary after lemma XVI). The phrase, x is an "element" of the "set" y' and the symbol $(x \epsilon_* y)$ . . $(y \epsilon_* -1)$, mean the same. The "*pf*" means: propositional function of the model (see [I], § 1).

Returning to the verification of the axioms, we observe that the validity of the axiom D (of J. v. NEUMANN) is an immediate consequence of the lemma XXIVd) (and also does not depend on s — t-postulates).

The verification of the axioms of the group B of [G] is now an almost immediate consequence of the lemma XXX and definition V, as it is not difficult to observe. (Here the s — t-postulates (s I) —(s VII) are essential.) Moreover, the same is true concerning the axiom of choice E of [G], with the "proper class" $W_* \epsilon R^*$ as the "universal (dyadic) choice function" of the model.

(β) In order to prove that the axioms C 2, C 3, C 4 of [G] are satisfied (as well as for disproving C 1, i. e. proving non C 1), we need a careful *metamathematical analysis of the validity of the existence metatheorems* M 1—M 6 of [G] — *in the sense of our model*. For the sake of brevity, we shall describe the essential steps of this analysis only; the details may be left to the reader.

The first difficulty to overcome is that the notion of integer to be used in the inductive definition of an "ordered n-tuple" $(n \epsilon \omega_0)$ is a relative (axiomatic) one (of the interpreting axiomatic set theory), whereas the syntactical notion of the propositional function is based on the absolute notion of integer. Nevertheless, this difficulty is unessential for our purpose because we do not need the mentioned model-metatheorems in their (rather obscure) generality, but only use a very limited number of their instances; therefore the number of inductive steps in the proofs of the model-metatheorems to be performed shall be limited e. g. to $n = 1, 2, 3, \ldots, 50$ — and within this limitation (avoiding, in fact, the metamathematical considerations in the obvious manner) the logical difference between the absolute and relative notions of positive integer is irrelevant, and will be disregarded in the sequel.

Now, in view of § 1 of (I), we define the notion of the s. c. "*basic primitive propositional function*" (of the model), in short the "bppf" (comp. [G] Chap. II):

(1) Suppose Π, Γ are any normal terms denoting elements of R^*. Suppose moreover, that firstly: Π is a "set" variable, i. e. a normal (set) variable ranging over $R^{\leqq}$, and secondly: Π does not occur (as a proper subterm) in Γ. Then the pf $\Pi \epsilon_* \Gamma$ is a "bppf".

(2) If φ and ψ are "bppf" then $\sim\varphi$ and $\varphi\psi$ are "bppf" too.

(3) If φ is a "bppf" and Π is a "set" variable ranging over $R^{\leqq}$ then $\exists \Pi\, \varphi$ is a "bppf" also — provided Π does not occur as a proper subterm in any term in φ.

(4) No other pf are "bppf". (So e. g., $x \epsilon_* y + z'$ is a "bppf" ,but ,$x \epsilon_* y + x$' and ,$x + y \epsilon_* z$' are not.)

Note that

a) in the sense of the remark in [G] after thm. 2.8, we assume, without loss of generality, that terms different from "set" variables do not occur as first members of the ϵ_*-relation — so that our notion of the "bppf" exactly corresponds (in the model) to the reduced notion (of [G]) of a ppf. Further note that

b) the caution made in the definition of the "bppf" enables the following model-analogy, say (M 1)*, of the existential metatheorem M 1, on the basis of the already verified axioms of the group B:

Metatheorem (M 1)* (of the model $\Delta(\mathfrak{S}, \mathfrak{R}^*)$): *Let* $\varphi(x_1, \ldots, x_n)$ *be a "bppf" containing no other free "set" variables than the given* ,x_1', ,x_2', … ,x_n'; ($n \epsilon \omega_0$ is less than an absolutely given numerical constant, e. g. 50). *Assume none of the*, ,x_1', ,x_2', …, ,x_n' *occur as proper subterms of any term in* φ.

Then there exists a "class" $A^* \epsilon R^*$ *such that* '$\langle x_1, \ldots, x_n \rangle_* \epsilon_* A^*$' *is equivalent to* φ (on the basis of our interpreting theory, i. e. this equivalence is a consequence of the axioms A—E of [G].).

The inductive proof of (M 1)* on the basis of the already verified axioms sub B and D (of [G]) is exactly the same as that of M 1 of [G], with the only change in possibly replacing the word ,special class' by the word 'term denoting a "class" (as in) the proof of M 3 of [G]); the reader may realize why our caution concerning the free "set" variables is necessary and sufficient in order to reproduce the arguments of the proof of M 1 (and of M 3) of [G].

According to [G], we now introduce the notion of the "basic normality" of concepts of the model, i. e. "*basic normal*" is defined on the ground of "bppf" exactly in the same way as "normal" is defined on the basis of ppf in [G], chap. II. Hereby, we obtain the corresponding model "metatheorems" (M 2)* — (M 6)* concerning "*basic normal propositional functions*" and "*basic normal terms*", as simple consequences of (M 1)*, exactly in the same manner as M 2—M 6 are consequences of M 1 (in [G]).

As an important (for our purpose) consequence, we obtain the existence and unicity of the "identity relation", say I_*, with $I_* \epsilon R^*$ and

$$\langle xy\rangle_* \epsilon_* I_* \equiv (x = y)\,(x\epsilon_* - 1)\,(y\epsilon_* - 1)\,.$$

But thus far we could not have taken even such a simple propositional function as e. g. ,$\exists\, y\,(x = y + 1)$' for a "basic normal propositional function" (of the model), because (as it is not difficult to observe) '$y + 1$' perhaps is not a "basic normal term".[21]) Therefore a suitable extension of the notion of "bppf" (to "*ppf*") (and then of "basic normality" to "*normality*") is desirable. This is possible on account of the requirement (s $\emptyset$) (of the "successor relation"), not used till now, e., we have the special "proper class" $\langle +1\rangle \epsilon R^*$ at our disposal in the model, such that $\langle uv\rangle_* \epsilon_* \langle +1\rangle \equiv u = v + 1$ (assuming $u \geqq 0$, $v \geqq 0$).

Therefore we modify the definition of a "bppf" just given as follows: (i) we admit '$x_1 + 1$', ... '$y_1 + 1$', ... ($x_1, x_2, \ldots, \epsilon R^{\leqq}$, $y_1, y_2, \ldots, \epsilon R^{\leqq}, \ldots$) for the Π and for the Γ in (1); (ii) we allow the quantified "set" variable Π to appear in φ in the form $\Pi + 1 = \Gamma$ also, in (2) of the just stated definition of "bppf". Hereby we have defined the (extended) notion of "ppf", i. e. of the "*primitive propositional function*" (of the model in question).

Replacing now "bppf" by "ppf" and weakening the caution in admitting that the free "set" variables ,x_1' ... ,x_n' may occur in φ in terms ,$x_1 + 1$', ... ,$x_n + 1$', we obtain the extended model "metatheorem", say (M 1)**, from the just stated original (M 1)*.

It is not difficult to observe that in order to prove (M 1)** by the standard induction of [G], we have to complete the beginning of the induction as well as the case 2c) (of [G]) of the inductive step only.

The cases of φ to be performed in addition at the beginninof theg induction are as follows:

a) $x_r + 1 \epsilon_* x_s$, b) $x_r + 1 \epsilon_* x_s + 1$, c) $x_r \epsilon_* x_s + 1$, d) $x_r + 1 \epsilon_* X^*$ (where $X^* \epsilon R^*$) ,
where $1 \leqq r \leqq n$, $1 \leqq s \leqq n$ throughout.

We have to find the corresponding "classes" A^* so that

$$\langle x_1, \ldots, x_n\rangle_* \epsilon_* A^* \equiv \varphi\,.$$

In the first three cases, we have to distinguish the subcases $r < s$, $r = s$ $r > s$. But since the first and third of these subcases are equivalent on the basis of the "first conversion" axiom B 6 (comp. the proof of M 1 in [G]) we can disregard the third subcases. Concerning the subcases $r = s$ of the cases a), b), c), we can disgregard them also, in writing e. g.

$$x_r + 1 \epsilon_* x_r \equiv (y_r + 1 \epsilon_* x_r)(\langle x_r y_r\rangle_* \epsilon_* I_*)$$

[21]) I. e., in general, perhaps cannot be defined by means of ϵ_* alone.

in view of (M 3)* (comp. also the proof of M 1 in [I]). Therefore only the subcases $r < s$ of a), b), c) are essential. Hence suppose φ is ‚$x_r + 1 \epsilon_* x_s$' with $r < s$ (case a).

Write the following obvious equivalences:

$$x_r + 1 \epsilon_* x_s \equiv \exists\, y_r((y_r \epsilon_* x_s) \,.\, (y_r = x_r + 1)) \equiv$$
$$\equiv \exists\, y_r((\langle y_r x_s \rangle_* \epsilon_* \langle \epsilon_* \rangle) \,.\, (\langle y_r x_s \rangle_* \epsilon_* \langle +1 \rangle)) \,.$$

Now, in view of (M 2)*, form the "class" C_1^* such that $\langle y_r z_r x_r x_s \rangle_* \epsilon_* C_1^* \equiv \equiv y_r = z_r$.

Further, form the "converse class" C_2^* of the "class product" $\langle \epsilon_* \rangle \,.\, \langle +1 \rangle$ such that $\langle y_r z_r x_r x_s \rangle_* \epsilon_* C_2^* \equiv (\langle y_r x_s \rangle_* \epsilon_* \langle \epsilon^* \rangle) \,.\, (\langle z_r x_r \rangle_* \epsilon_* \langle +1 \rangle)$.

Finally, let C^* be the "class product" $C_1^* C_2^*$ of C_1^* and C_2^*. Then clearly $x_r + + 1 \epsilon_* x_s \equiv \langle y_r z_r x_r x_s \rangle_* \epsilon_* C^* \equiv \exists\, y_r \exists\, z_r(\langle y_r z_r x_r x_s \rangle_* \epsilon_* C^*)$, whence $x_r + 1 \epsilon_* \epsilon_* x_s \equiv \langle x_r x_s \rangle_* \epsilon_* \mathbf{D}_*(\mathbf{D}_*(C^*))$.

Now, it is almost obvious how to define (along [G], see the proof of M 1) the desired "class" A^* such that $x_r + 1 \epsilon_* x_s \equiv \langle x_1 \ldots x_r \ldots x_s \ldots x_n \rangle_* \epsilon_* A^*$.

Once the method has been described in the case a), its repeated use in the remaining cases b), c), d) may be omitted. Therefore let us return to the modification in the inductive step 2c) of the proof of (M 1) in [G] as needed in order to prove (M 1)**. But it is not difficult to see how to reduce, without loss of equivalence, all the terms (in the φ in question) (built up by means of ‚+1') to the form $\Gamma + 1$, where Γ has no subterm of the form $\Pi + 1$, by the introduction of a number of additional auxiliary "set" variables and a corresponding number of equations. Thus, we are able to apply the method just described, replacing the instances of ‚$\Gamma = \Pi + 1$' by ‚$\langle \Gamma \Pi \rangle_* \epsilon_* \langle +1 \rangle$' in order to complete the proof.

Having proved the extended existence metatheorem (M 1)** (for "ppf" of the model), we define the notion of "*normality*" (extending in an obvious manner the previous notion of "basic normality" of concepts of the model) — and we prove the corresponding existence metatheorems (M 2)** — — (M 6)** on the basis of the (M 1)** in exactly the same way as the notion of normality and the methateorems M 2—M 6 are based on M 1 in [G]. — Especially, we see that '$\Pi + 1$', '$\Pi - 1$' are "normal terms" whenever 'Π' is a "normal term" (provided $\Pi > 0$ in the last case).

(γ) After these preparations, we come to an important conclusion: *Every $s - t$-ring is logarithmic* in the sense of the definition III. — Let us prove this fact.

First of all, note that every inequality $u < v$ between "set" variables now can be taken for a "normal pf", in view of the equivalences

$$u \leqq v \equiv (\langle uv \rangle_* \epsilon_* - 1) \,.\, (z)(z \epsilon_* 2^u - 1 \supset z \epsilon_* 2^v - 1) \equiv 2^u - 1 \subseteq_* 2^v - 1 \equiv$$
$$\equiv u R_* v$$

(with the "normal terms", $2^u - 1$, $2^v - 1$) — and with the so defined "relation" R_* (on account of the lemma XXIVa). Second, note that every "subclass" of a "set" is a "set" itself, in view of the lemma XVI, together with lemma XXX.

Now, let $x > 0$ be a "set". Then because of the inequalities $0 < x < 2^x < < 2^{x+1}$ we can define (by (M 2)**) the "set", say y_x, uniquely determined by x, by the equivalence (with the "normal" right hand "pf"):

$$u \epsilon_* y_x \equiv u \epsilon_* (2^{x+1} - 1)(v)(v \epsilon_* x \supset (v < u)) , \tag{1}$$

(because every $2^z = \{z\}_*$ $(z \geqq 0)$ is given by a "normal term"). Moreover, 'y_x' ist to be taken for a "normal term" too, and y_x is "a nonvoid set" for every $x \epsilon R^{\leqq}$ (because clearly e. g. $x \epsilon_* y_x \subseteq_* 2^{x+1} - 1$). Hence by (M 5)**, we have a "function", i. e. a "class" $Y \epsilon R^*$ with $(Y)'_* x = y_x$; and the "compound function" $(W_* Y)_*$ is defined for every $x \epsilon R^{\leqq}$ in the model, with the "value" $(W_* Y)'_* x = W_*(y_x)$ as the smallest "element" of the "set" y_x (smallest in the sense of the ordering of $\mathfrak{S}$). Moreover, $W_*(y_x) > 0$, by the definitory equivalence (1). Hence $W_*(y_x) - 1$ is always a "set". But this "set" can never be an "element" of y_x, though it is, of course, an "element" of the "set" $2^{x+1} - 1$. Therefore, in view of the definition of the "set" y_x by (1), we observe $W_*(y_x) - - 1 \epsilon_* x$ and, moreover, $W_*(y_x) - 1$ is the greatest "element" of x in the sense of the ordering of $\mathfrak{S}$. This means that indeed $W_*(y_x) - 1 = \text{Log}\,(x)$ in view of the lemma XXV and XXVI, q. e. d.

(δ) Now, the verification of the axioms C 2, C 3 and C 4 is relatively easy. In view of the already proved existence "metatheorems" (of the model), to every "set" x we have its "sum class" $\mathbf{S}_*(x)$, its "potency class" $\mathbf{P}_*(x)$ — and with every "function" $F \epsilon R^*$, the "image class" $(F)'_* x$. It suffices to show that the mentioned "classes" indeed are "sets", in constructing suitable "sets" "containing" them as "subclasses".

Ad C 2: By definition (provided $x \epsilon R^{\leqq}$, of course), there is

$$v \epsilon_* \mathbf{S}_*(x) \equiv \exists\, w((v \epsilon_* w)(w \epsilon_* x)) .$$

If $x = 0$ or $x = \{0\}_* = 1$, then clearly $\mathbf{S}_*(x) = 0$.

Hence we can suppose $x > 1$, $w > 0$, $(v \epsilon_* w)(w \epsilon_* x)$, i. e. there is $(v < w) . . (w < x)$. Then (by lemma XXVIIb)) $v \leqq \text{Log}\,(w)$, $w \leqq \text{Log}\,(x)$, whence

$$v \leqq \text{Log}\,(\text{Log}\,(x)) < \text{Log}\,(\text{Log}\,(x)) + 1 .$$

Therefore by lemma XXVIa)

$$\mathbf{S}_*(x) \subseteq_* 2^{\text{Log}(\text{Log}(x)) + 1} - 1 , \quad \text{q. e. d.}$$

Ad C 3: By definition (provided $x \epsilon R^{\leqq}$), there is

$$v \epsilon_* \mathbf{P}_*(x) \equiv v \subseteq_* x .$$

By lemma XXVIId) we have $v \subseteq_* x \supset v \leqq 2x < 2x + 1$, whence $v \subseteq_* x \supset \supset v \epsilon_* 2^{2x+1} - 1$, i. e. $\mathbf{P}_*(x) \subseteq_* 2^{2x+1} - 1$, q. e. d.

Ad C 4: Suppose the "class" $X \in R^*$ is a "function" and x is a "set".

Without loss of generality, we may limit ourselves to x of the form $2^z - 1$ ($z \in R^{\leqq}$), because every "set" x is e. g. a "subset" of the "set" $2^{x+1} - 1$ of this form — and if the "image class" $(X)''_* (2^{x+1} - 1)$ is a "set", the more so $(X)''_* x$, as a "subclass" of the former, also is a "set" (as we know).

Hence assume $x = 2^z - 1$ ($z \geqq 0$) and define the "set" v as follow (on account of the existence „metatheorem" (M 3)**)[22])

$$w \in_* v \equiv (u)((0 \leqq u \leqq x)(\sim \mathbf{M}_*((X)''_*(2^u - 1) \supset w \in_* 2^u - 1)) .$$

(In words: the "set" v is the "set product" of all the "subsets" of x of the form $2^u - 1$, and such that the "X-image class" of $2^u - 1$ is a "proper class".)

Now assume that the "image class" $(X)''_* x$ is a "proper class". Then we have $0 < v \subseteq_* x$. (Note that $0 \in_* v$, of course, if $(X)''_* x \neq 0$, as assumed.) In this case we observe that

$$(0 \leqq z_2 < z_1)(z_1 \in_* v) \supset z_2 \in_* v .$$

whence on account of $0 < v$ we easily conclude that $v = 2^{\mathrm{Log}(v)+1} - 1$, in view of the lemma XXVI, XXIVa). Here, of course, Log $(v) \in_* v$ (by the same lemas).

Further, we have $2^{\mathrm{Log}(v)} - 1 \underset{\neq}{\subset} 2^{\mathrm{Log}(v)+1} - 1 = v$, whence clearly

$$\mathbf{M}_*((X)''_*(2^{\mathrm{Log}(v)} - 1))$$

by the definition of v. But since $2^{\mathrm{Log}(v)+1} - 1 = 2^{\mathrm{Log}(v)} - 1) \dot{\underset{*}{+}} \{\mathrm{Log}\,(v)\}_*$, we easily conclude $\mathbf{M}_*((X)''_* (2^{\mathrm{Log}(v)+1} - 1))$. Therefore Log (v) cannot be an "element" of v (by the definition of v); this is a contradiction, i. e. indeed the "image class" $(X)''_* x$ of the "set" x cannot be a "proper class", q. e. d.

(ε) Concerning the validity of the axiom non C 1 (of finity) in our dyadic model of theorem I, let us first note the following:

The "universal class" $V_* = -1$ of our dyadic model is "well ordered" by the "relation" $R_* = <$ of $\mathfrak{S}$ *and "isomorphic"* (and the more so isomorphic) *with the "class" On_* of "ordinal numbers" of this model* (as "well ordered" by $\in^*$).

Indeed, we define (on acount of (M 2)**) the "ordering relation", say R^*, in $V_* = -1$ of our model by the equivalence $\langle xy \rangle_* \in_* R_* \equiv 2^x - 1 \underset{\neq *}{\subset} 2^y - 1$, whence $\langle xy \rangle_* \in_* R_* \equiv x < y$. But clearly R_* "well orders" V_* since every "class" C(with $C \subseteq_* V_*$) has its dyadic value $\tilde{W}(C)$ for its smallest "element" in the sense of $<$. Now the "isomorphism" (as a "class") of On_* with V_* is given by theorem 7.7.1 of [G], which holds in our model on account of the facts already proved, since the suppositions of this theorem clearly are satisfied. — And finally, we now conclude that every "ordinal number" is indeed

[22]) Note that $\sim\mathbf{M}_*$ is a "normal concept"; see [G], proof of M 2.

"finite" because every "set" x has the Log (x) for its greatest (in the sense of $<$) "element", q. e. d. Thus the proof of our theorem I is complete.

Let us add some remarks concerning the notion of s-t-rings and the corresponding dyadic models.

(i) We easily observe that the requirements (s I)—(s VII) are necessarily satisfied in any dyadic ring which could serve to form a model of Gödelian axiomatic set theory as that of theorem I (no matter whether with or without the axioms C 1 and E); but it appears that (s I)—(s VII) alone can hardly be sufficient to this purpose, since it is in no sense obvious how to ensure the axioms of the group C in this case. Adding the requirement of logarithmicity of $\mathfrak{S}$ we ensure C 2, C 3, but perhaps not C 4. Further, the logarithmicity itself disproves C 1. We thus see how strong is the additional requirement (s $\emptyset$) together as giving (with (s VIII)), the logarithmicity and C 4, i. e. we observe how close is the connection of the "to belong relation" ϵ^* with the "ordering relation" $<$ of the given s-t-ring $\mathfrak{S}$, because of the definability of $<$ in "normal terms" of the model (enabled by (s $\emptyset$)) —and thus, by means of ϵ_* (in the model).

(ii) Unfortunately enough, the important question of whether there is a suitable requirement (other than (s $\emptyset$)) which, being added to (s I)—(s VIII), would ensure all the axioms A — E of [G] including C 1 for ϵ_*, is to be answered in the negative, by an easy argument of the general valuation theory.

(iii) Forming the discretely ordered ring of "integers" in the usual way in our model $\Delta(\mathfrak{S}, \mathfrak{R}^*)$ we do not know whether this ring is "isomorphic" with the original ring $\langle RF_1F_2F_3\rangle$.

(iv) There are two extreme cases of dyadic models $\Delta(\mathfrak{S}, \mathfrak{R}^*)$ given by a certain s-t-ring $\mathfrak{S}$: (1) The case of $\mathfrak{R}^*$ = the minimal weakly pseudoperfect immediate extension of $\mathfrak{S}$, of the lemma XXX, and (2) the case of $\mathfrak{R}^*$ = the (whole) pseudoperfect immediate extension ${}^{\mathfrak{D}}\mathfrak{R}$ (of $\mathfrak{D}$), of the lemma XXII.

In the first case, of $\aleph_\alpha$ is the power of the set of "sets", then the power of the set of "classes" is $\aleph_\alpha$ too, whereas in the second case, this last power is $2^{\aleph_\alpha}$.

4. Skolemian extensions of s-t-rings

Thus far we have had only one example of s-t-ring: the ring of integers of our interpreting theory. Our further main task is to construct an uncountable transfinite ω_1 — sequence of succesively extended s-t-rings — and then to obtain the desired s-t-ring of the first uncountable power $\aleph_1$ as the set sum of this ω_1-sequence.

Lemma XXXI. *Let* $\mathfrak{R} = \langle R + . <\rangle$ *be an ordered ring with* $\overline{\overline{R}} = \aleph_0$ ($\mathfrak{R}$ is countable). Let $\mathfrak{F} = \langle F \oplus \odot \prec\rangle$ *be a. s. c. asymptotically semiordered ring of functions on R into R, i. e., with* $f \in F$, $g \in F$ *we assume*

$$(f \oplus g)'\, x = f'x + g'x\,, \quad (f \odot g)'\, x = f'x\,.\,g'x$$

and $f \prec g \equiv f'x < g'x$ *for every sufficiently great* $x > x_{f,g}$. *Let* $\overline{\overline{F}} = \aleph_0$ ($\mathfrak{F}$ *is countable too*).

Then there exists a subset $P \neq \emptyset$ *of* R *such that the following is true:*

(i) P *is cofinal with* R, *i. e.* $(x) \exists\, y((y \geqq x)(y \in P))$ (*with* $x \in R$, $y \in R$ *of course*).

(ii) *The set* F_P *of all the functions* $f(f \in F)$ *such that* $f'x = 0$ *for every sufficiently great* $x \in P$, *i. e. the* F_P *given by*

$$f \in F_P \equiv \exists\, x(y)((x \leqq y)(y \in P)) \supset f'y = 0)\,,$$

is to be taken for the set of all the elements of a prime ideal $\mathfrak{P}$ *of the ring* $\mathfrak{F}$.

(iii) *Putting* $\hat{f} << \hat{g}$ *for given elements* $\hat{f}, \hat{g}$ (with $f \in \hat{f}$, $g \in \hat{g}$) *of the coset ring* $\mathfrak{F}/\mathfrak{P}$ if $f'x < g'x$ *for every sufficiently great* $x \in P$, *we obtain a simply ordering relation* $<<$ *for the ring* $\mathfrak{F}/\mathfrak{P}$.

(iv) $\hat{f} << \hat{g}$ *holds in* $\mathfrak{F}/\mathfrak{P}$ *whenever* $f < g$ *in* $\mathfrak{F}$, *i. e., the s. c. natural homomorpfic mapping of the ring* $\mathfrak{F}$ *into the ring* $\mathfrak{F}/\mathfrak{P}$ *is order preserving.*

(v) *Let* $\mathfrak{R}$ *be discretely ordered and let* F *contain every constant function as an element. Then* $\mathfrak{F}/\mathfrak{P}$ *is also discretely ordered and, moreover, we can uniquely determine (by* $\mathfrak{R}$ *and by the "marked" sequence* $\{f_n\}_{n \in \omega_0}$ *of all the* $f \in F$) *a simply discretely ordered ring, say* $\tilde{\mathfrak{R}}$, *such that* $\tilde{\mathfrak{R}}$ *is order isomorphic with* $\mathfrak{F}/\mathfrak{P}$ *and* $\mathfrak{R}$ *is an ordered subring of* $\tilde{\mathfrak{R}}$; (,$\tilde{\mathfrak{R}}$' *then is a normal term depending uniquely on* ,$\mathfrak{R}$').

(vi) *If* F *contains a function* g *asymptotically surpassing every constant on a cofinal subset* $\underline{S}$ *of* G (*i. e. if to every* $x \in R$ *there is an* $y_{x,f} \in R$ *so that* $f'\,t > x$ *for every* $t \in \underline{S}$ *with* $t \geqq y_{x,f}$) *then* $\mathfrak{R}$ *is a proper subring of the ring* $\tilde{\mathfrak{R}}$; $\tilde{\mathfrak{R}}$ *then is called the Skolemian extension of* $\mathfrak{R}$.

Proof. α) Assume that the functions $f \in F$ are arranged in a simple sequence $\{f_n\}_{n \in \omega_0}$; for the sake of unicity, $\{f_n\}_{n \in \omega_0}$ may be the "marked" sequence in the sense of the axiom of choice E. For more convenience, we shall write sg $f'x = -1, 0, 1$ respectively, according to whether $f'x < 0$, $f'x = 0$, $f'x > 0$ respectively, in the sense of the definition I of § 2. For the sake of further unicity of choice in the subsequent construction, let us make the following agreement:

Let S denote a cofinal subset of R.

Given the finite subset $e = \{f_1 \dots f_n\}$ of F, we observe that for every $i = 1, 2, \dots, n$

$$S = SR = S((\mathrm{sg} f_i^{-1})''\{-1\} \dotplus (\mathrm{sg}\, f_i^{-1})''\{0\} \dotplus (\mathrm{sg}\, f_i^{-1})''\{1\})$$

with disjoint set summands in R. Writing $S_{i,j} = (\mathrm{sg}\, f_i^{-1})''\{j\})\, S$ (with $i = 1, 2, \dots, n$; $j = -1, 0, 1$), we further observe that

$$S = (S_{1,-1} \dotplus S_{1,0} \dotplus S_{1,1})(S_{2,-1} \dotplus S_{2,0} \dotplus S_{2,1}) \dots \dots (S_{n,-1} \dotplus S_{n,0} \dotplus S_{n,1}) = \sum_{\iota} \prod_{i=1}^{n} S_{i,\iota' i}$$

with the disjoint set products $\prod_{i=1}^{n} S_{i,\iota' i}$ as summands, where the set sum $\sum_{\iota}$ is extended over all the 3^n functions ι on the finite set $\{1, 2, \ldots, n\}$ into the set $\{-1, 0, 1\}$.

Now, clearly at least one of these 3^n disjoint summands of S is cofinal with S (and therefore with R).

Hence we can and will agree that S_e is the "marked" one (in the sense of the axiom of choice E of [G]) of these cofinal set summands of S. Then ,S_e' is a normal term, depending on the normal term e. Writing $S_e = \prod_{i=1}^{n} S_{i,\iota' i}$, we see that with the i fixed, sg $f_i' x = \underline{\iota}' i$ is a constant function on S_e.

Now, set $P_1 = R$ and $P_{n+1} = (P_n)_{\{f_1 \ldots f_n\}}$, by induction $(e = \{f_1 \ldots f_n\})$. Then every P_n is a cofinal (and hence nonvoid) subset of R for every $n \in \omega_0$ and clearly

$$P_1 \supseteq P_2 \supseteq \ldots \supseteq P_n \supseteq \ldots$$

We further see that sg f_i is constant on every P_n with $i \geqq n$. Let $\{y_n\}_{n \in \omega_0}$ be a fixed ("marked") simple cofinal increasing sequence of elements of R, according to the supposition; i. e., to every $x \in R$ there is a $n_x \in \omega_0$ such that $y_m \geqq x$ for every $m \geqq n_x$. Let further x_n be the "marked" element of the non-void set of all the $x \in R$ with $(x \geqq y_n)(x \in P_n)$. Then define the desired P as the set of all these x_n $(n \in \omega_0)$.

This done, the verification of the items of the theorem is now easy.

Ad (i): Clearly P is cofinal with R by definition- and we observe that every sg f $(f \in F)$ is ultimately constant on P, i. e. constant except perhaps on a finite subset of P.

Ad (ii): 1. If $f'x = 0$ for every $x \geqq x_f$, $x \in P$ and $g'x = 0$ for every $x \geqq x_g$, $x \in P$, then $(f \odot g)'\, x = 0$ for every $x \geqq \max(x_f, x_g)$.

2. If $f'x = 0$ for every $x \geqq x_f$, $x \in P$, and if $g \in F$, then $(f \odot g)'\, x = 0$ for every $x \geqq x_f$, $x \in P$ also.

3. Assume $(f \odot g)'\, x = 0$ for every $x \geqq \bar{x}$, $x \in P$. By the construction of P, sg f and sg g are constant for $x \geqq x_f$, $x \in P$ and for $x \geqq x_g$, $x \in P$ respectively. Take $\underline{x} = \max(\bar{x}, x_f, x_g)$. Then $\underline{x} \in P$, $\underline{x} \geqq \bar{x}$, hence $(f \odot g)'\, \underline{x} = 0$, i. e. $f'\underline{x} = 0$ or $g'\underline{x} = 0$, i. e. $f'x = 0$ whenever $x \geqq \underline{x}$, $x \in P$, or $g'x = 0$ whenever $x \geqq \underline{x}$, $x \in P$, by the definition of x and P.

Thus we have proved (ii) of the lemma. The obvious verification of the remaining items of the thesis of the lemma now can be omitted. Note only that a): in the item (v), the desired extension $\tilde{\mathfrak{R}}$ of $\mathfrak{R}$ is obtained by the obvious replacing of (mutually different) cosets (as elements of $\mathfrak{F}/\mathfrak{P}$) of constant functions f_x (with $f_x' t = x$) by the corresponding constant values $x \in R$; and b): in the item (vi), we have to work with the given cofinal subset $\underline{S} = S$ of R without changing either the assumptions or the results of the construction.

Remark to lemma XXXI. Our construction essentially is that of Skolem, see [Sk]. In spite of a great deal of effort, the author was unable to give an immediate generalization of the just described extension process as holding for any (not necessarily countable) power of R or of F resp.

Let us return to the needed preparation of a concrete use of the lemma XXXI in the case of s-t-rings. This is given by the last and important

Convention XX (The set of the s. c. elementary functions of an s-t-ring). Let $\mathfrak{S} = \langle RF_1F_2F_3F_4D_1D_2\rangle$ be a given s-t-ring in the sense of the definition IV (see also definitions I, II and III, as well as the theorem I including its proof).

(1) Then the following functions on R, resp. on $R \times R$ into R are called *basic primitive operations*: F_1 (the ring-addition) F_2 (the ring-multiplication), F_3 (the signum function), F_4 (the exponentiation of 2 — this last function as formally extended to the whole R by means of $F'_4x = 0$ for $x < 0$, in accordance with $F'_4x = [2^x]$ in the sense of the convention VI) and D_1, D_2 (the first and the second domain operation) as formally extended to negative elements of R by assigning them the value -1. The following operations are called *basic secondary operations*:

(2a) The ring subtraction F_5 (of the conv. II), the operations F_6 and F_6^* (with $F'_6\,\langle xy\rangle$ = the integral part of y divided by 2^x, $F_6^{*'}\langle xy\rangle$ = the remainder of y divided by 2^x, both as formally extended e. g. by $F'_6\,\langle xy\rangle = 0$, $F_6^{*'}\langle xy\rangle = 0$ for the previously excluded case of $x < 0$ and characterized by the inequality (1) and the identity (2) of the lemma IX); further, the function F_7 with $F'_7x = W(x)$ = the dyadic value of x for $x \neq 0$, as characterized by (I) and (II) of the lemma XII and as formally extended to the previously excluded case $x = 0$ e. g. by $F'_70 = -1$.

Remark. In order to enumerate further operations, let us note that the auxiliary functions on R into R, as given by the terms ${}^1u$, ${}^2u$ of the conv. XIX and serving to the definition of the "first" resp. of the "second member" of an "ordered pair" $u = \langle {}^1u\,{}^2u\rangle$, now can be redefined on the whole R as composed of the just mentioned basic operations. Put e. g.

$$I'u = \left[\frac{u}{2^{F'_7u}}\right] - 1\,, \quad G'u = |F'_7u - 2^{|F'_7F_7'u|}|\,.$$

$$H'u = F'_7F'_7u\,.\,[2^{-G'u}] - \mathrm{sg}\,(G'u)\,,$$

Then

$$ {}^1u = H'u\,.\,\mathrm{sg}\,(1 + H'I'u) - [2^{-(1+H'I'u)}]\,,$$

$$ {}^2u = H'I'u\,.\,\mathrm{sg}\,[1 + H'u) - [2^{-(1+H'u)}]$$

in accordance with their previous definition in conv. XIX. Note further that the function previously given on $R^{\leqq} \times R^{\leqq}$ into $\{0, 1\}$ by the term C_y^x (the s. c. "characteristical function" of the convention XII) now is defined on the whole $R \times R$ simultaneously with the defining basic operations.

(2b) The unary or binary operations $F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}, F_{14}, F_{15}, F_{16}$, introduced in the definition IV, on $R^{\leqq}$, resp. on $R^{\leqq} \times R^{\leqq}$ only, will now be formally extended to the whole R or $R \times R$ respectively in putting their results to be $= -1$ in all the previously undefined cases. (The "nonarithmetical" operation F_{11} of the "domain" is excepted, see remark b) to the requirement (s III).) The (so extended) operations are also called *basic operations* of $\mathfrak{S}$.

(3) (The elementary functions of $\mathfrak{S}$). We define:

(3a) The constant functions and the operations recalled or introduced sub (1), (2a) and (2b) are *elementary functions of* $\mathfrak{S}$; they are defined on the whole R or on $R \times R = R^2$, and are into R.

(3b) If Φ is an elementary function of n variables, Ψ an elementary function of m variables and G, H respectively are basic operations of one respectively of two variables, then the superposition function given by the (normal) term of the form $G'\Phi'\langle x_1 \ldots x_n\rangle$ or of the form

$$H' \langle \Phi'\langle x_1 \ldots x_n\rangle \, \Psi'\langle y_1 \ldots y_m\rangle\rangle$$

$$(x_i \in R\,, \quad y_j \in R\,; \quad i = 1, \ldots, n\,; \quad j = 1, \ldots, m)$$

is another elementary function of $\mathfrak{S}$ of $n^* \leqq n$, respectively of $k^* \leqq n + m$ variables, defined on the whole direct potency R^{n^*} or R^{k^*} into R; (n^* and k^* denote the number of different variables among the x_i, or respectively among the x_i and y_j; $m, n, i, j \in \omega_0$ (recursion in ω_0). Comp *8.73 of [G]).

(3c) No other functions are elementary functions of $\mathfrak{S}$.

Lemma XXXII.

Let $\mathfrak{S} = \langle RF_1F_2F_3F_4D_1D_2\rangle$ *be a given countable s-t-ring* ($\overline{\overline{R}} = \aleph_0$). *Then the set of the just defined elementary functions of one variable of* $\mathfrak{S}$ (*i. e. with* $n^* = 1$, *resp.* $k^* = 1$ *in* (3b) *of the preceding convention*) *can be taken for the set* F *of elements of an asymptotically semiordered ring of functions* $\mathfrak{F} = \langle F \oplus \odot \prec\rangle$ *of the lemma* XXXI, *if* $\mathfrak{R} = \langle RF_1F_2F_3\rangle$ *is the corresponding ordered ring of this lemma.*

The ring $\mathfrak{F}$ *now contains the subring of all constant functions as a subring order isomorphic with* $\mathfrak{R}$.

Further define the operations $\widehat{F}_4, \widehat{D}_1, \widehat{D}_2$ *on* F, *respectively on* $F \times F$, *as follows*:

$$\left.\begin{aligned} (\widehat{F}'_4 f)'\, x &= F'_4(f'x)\,, \\ (\widehat{D}'_1 f)'\, x &= D'_1(f'x)\,, \\ (\widehat{D}'_2\langle fg\rangle)'\, x &= D'_2\langle f'x\, g'x\rangle \end{aligned}\right\} \text{for every } x \in R,\ f \in F\,.$$

Then F *is closed with respect to all the operations* $\oplus$, $\odot$, $\widehat{F}_4$, $\widehat{D}_1$, $\widehat{D}_2$.

The function $I \in F$ with $I'x = x\,(= (x + x) - x)$ asymptotically surpasses every constant on R. The constant function with the value $1 \in R$ is the unit of $\mathfrak{F}$. The set F is countable if R is countable.

The obvious proof can be omitted.

Remark. 1. Note that we cannot assert that the operations $\widehat{F}_4$, $\widehat{D}_1$, $\widehat{D}_2$ have all the properties (in $\mathfrak{F}$) of the operations F_4, D_1, D_2 in $\mathfrak{S}$;

2. Every, elementary function Φ on R^n can and will be transformed "value-by-value"-wise (in the same manner as with $\widehat{F}_4$, $\widehat{D}_1$, $\widehat{D}_2$) into a function $\widehat{\Phi}$ on F^n, by setting

$$(\widehat{\Phi}'\langle f_1, \ldots, f_n\rangle)'\,x = \Phi'\langle f_1'x, \ldots, f_n'x\rangle\;.$$

These (normal) symbols will be used in the following

Lemma XXXIII. *Using the assumption and the result of the preceding lemma* XXXII, *let us form the discretely (simply) ordered coset ring $\mathfrak{F}/\mathfrak{P}$ in the sense of the lemma* XXXI. *Then the operations $\widehat{F}_3(=\widehat{\mathrm{sg}})$, $\widehat{F}_4(= 2\cdot\cdot\cdot)$, $\widehat{D}_1$, $\widehat{D}_2$ defined on the set $\widehat{F}$, (or on $\widehat{F} \times \widehat{F}$) of the cosets $\hat{f}, \hat{g}, \ldots$ (or of their ordered pairs $\langle\hat{f}\hat{g}\rangle, \ldots$) by the equations*

$$\widehat{F}_3'\hat{f} = \widehat{F_3'f}\,,\quad \widehat{F}_4'\hat{f} = \widehat{F_4'f}\,,\quad \widehat{D}_1'\hat{f} = \widehat{D_1'f}\,,$$
$$\widehat{D}_2'\langle\hat{f}\hat{g}\rangle = \widehat{D_2'\langle fg\rangle}\,,$$

together with the addition $\widehat{F}_1$ and multiplication $\widehat{F}_2$ in $\mathfrak{F}/\mathfrak{P}$, form an ordered seventuple $\langle\widehat{F}\widehat{F}_1\widehat{F}_2\widehat{F}_3\widehat{F}_4\widehat{D}_1\widehat{D}_2\rangle$ satisfying all the definitory requirements of the notion of an s-t-ring. By means of the obvious replacement of the s-t-subring of cosets of constant functions by the given isomorphic s-t-ring $\mathfrak{S}$ (of the constant values) we thus obtain an s-t-ring, say $\tilde{\mathfrak{S}} = \langle\tilde{R}\tilde{F}_1\tilde{F}_2\tilde{F}_3\tilde{F}_4\tilde{D}_1\tilde{D}_2\rangle$, as an uniquely determined (by $\mathfrak{S}$) extension of the given s-t-ring $\mathfrak{S}$, assumed $\mathfrak{S}$ is countable.

$\tilde{\mathfrak{S}}$ is the s. c. Skolemian extension of the given s t-ring $\mathfrak{S}$; also $\mathfrak{S} \neq \tilde{\mathfrak{S}}$. The term $\tilde{\mathfrak{S}}$ is normal since $\mathfrak{S}$ is such.

Proof. The principally easy and not new proof (see [Sk] for the general method) may be merely traced.

We transform every elementary function Φ on R^n (in the just stated sense) into the function $\widehat{\Phi}$ on F^n. Then we attempt to consider these functions of functions as operations on $\widehat{F}^n$ into $\widehat{F}$, in taking them modulo $\mathfrak{P}$, in the obvious sense. In view of the lemma XXXI, this is successful for the primitive as well as for the basic operations (and therefore for all the elementary functions), i. e. we clearly observe by induction that the results of these operations on $\widehat{F}^n$ indeed do not depend on the choice of a function $f \in F$ in a coset $\hat{f}$ of f modulo $\mathfrak{P}$. Now the desired verification of the definitory requirement of the notion of an $s - t$-ring rests, formally speaking, in the folowing procedure:

We systematically replace each variable running over R (or over $R^{\leq}$, $R \dot{-} \{0\}$ respectively) in each identity or resp. general inequality (d II), (d III) of the def. II, (1) and (2) of the lemma IX, (I) and (II) of the lemma XII, as well as in the identities for F_8, F_9, F_{10}, F_{12}, F_{13}, F_{14}, F_{15}, F_{16} and D_1, D_2 (in the def. IV), in a one-to-one manner by a corresponding term denoting the value of a variable elementary function of one common variable. Then we observe that each identity or inequality thus obtained holds almost everywhere on the set P of lemma XXXI, if we arbitrarily fix the variable functions in question, i. e. we see that the corresponding general identity or inequality holds in $\mathfrak{F}/\mathfrak{P}$ (modulo $\mathfrak{P}$ in $\mathfrak{F}$). Since each of the functions used is indeed an elementary function (in the sense of conv. XX) as can be easily verified by following the successive expressions for them, the lemma may be considered as proved. (Note that indeed $\mathfrak{S} \neq \tilde{\mathfrak{S}}$, because of the class $\widehat{I}$ (of $\mathfrak{F}/\mathfrak{P}$, in $\tilde{R} \dot{-} R$) of the indentity function I ($I'_x = x$) which surpasses every constant).

Lemma XXXIV. *Suppose that* $\{\mathfrak{S}\}_{\alpha<\beta} = \{\langle R_\alpha F_{1,\alpha} F_{2,\alpha} F_{3,\alpha} F_{4,\alpha} D_{1,\alpha} D_{2,\alpha}\rangle\}_{\alpha<\beta}$ *is an increasing well-ordered sequence of successively extended s-t-rings, i. e. if* $\alpha^* < \alpha$, *then* $\mathfrak{S}_{\alpha^*}$ *is a s-t-subring (in the usual sense of the inclusions* $R_{\alpha^*} \subset \subset R_\alpha$, $F_{i,\alpha^*} \subset F_{i,\alpha}$ $(i = 1, 2, 3, 4)$, $D_{j,\alpha^*} \subset D_{j,\alpha}$ $(j = 1, 2)$) *of the s-t-ring* $\mathfrak{S}_\alpha$. *Then* $\mathfrak{S} = \langle RF_1F_2F_3F_4D_1D_2\rangle$ *with* $R = \sum_{\alpha<\beta} R$, $F_i = \sum_{\alpha<\beta} F_{i,\alpha}$ $(i = 1, 2, 3, 4)$, $D_j = \sum_{\alpha<\beta} D_{j,\beta}$ *is also s-t-ring.*

The proof is obvious, in view of the fact that a set sum of successively extended discretely ordered rings is a discretely ordered ring and by the form of the other requirements in the definition of *s-t*-rings (as general inequalities or identities).

Now we come to the two conclusive main theorems of the paper.

Theorem II. *There is an s-t-ring* $\bar{\mathfrak{S}} = \langle RF_1F_2F_3F_4D_1D_2\rangle$ *such that* $\bar{\bar{R}}$ *(the power of* R*) is* $\aleph_1$.

Proof. (For the sake of brevity and better readability, we do not perform the transfinite construction along the strictly formal scheme of 7.5 of [G]; this formalization is easy to perform).

1. Put $\mathfrak{S}_1 =$ the *s-t*-ring of integers (of our basic formalized interpreting set theory of Gödel, see [G]).

2. Given $\mathfrak{S}_\alpha$ with $\alpha < \omega_1$ as a countable *s-t*-ring, take the Skolemian extension $\tilde{\mathfrak{S}}_\alpha$ (of the lemma XXXIII) of $\mathfrak{S}_\alpha$ for the $\mathfrak{S}_{\alpha+1}$.

3. If $\{\mathfrak{S}_\alpha\}_{\alpha<\beta}$ with a countable limit ordinal β is an increasing transfinite sequence of successively extended countable *s-t*-rings in the sense of the lemma XXXIV, then take the corresponding set sum *s-t*-ring of this lemma for $\mathfrak{S}_\beta$.

4. Define $\bar{\mathfrak{S}}$ as the *s-t*-ring resulting from the just defined uncountable

increasing sequence $\{\mathfrak{S}_\alpha\}_{\alpha<\omega_1}$ of countable *s-t*-rings, in the sense of the lemma XXXIV. This is the desired uncountable *s-t*-ring.

Now, by theorem I we get the

Theorem III. *Let* $\overline{\mathfrak{S}}$ *be the uncoutable s-t-ring of theorem* II, *and let* $\mathfrak{R}^*$ *be any of the weakly pseudoperfect immediate extensions of* $\overline{\mathfrak{S}}$, *in the sense of the definition* V. *Then the corresponding model* $\Delta(\overline{\mathfrak{S}}, \mathfrak{R}^*)$ *of theorem* I (*of the axiomatic theory of finite sets*) *is such that the set of "finite ordinal numbers" of the model is of the first uncountable power* $\aleph_1$.

If firstly: $\mathfrak{R}^*$ *is the minimal weakly pseudoperfect extension of* $\overline{\mathfrak{S}}$ *in the sense of lemma* XXX, *then the set of all the "classes" of the model is also of the power* $\aleph_1$.

If secondly: $\mathfrak{R}^*$ *is the* (*whole*) *pseudoperfect immediate extension of* $\mathfrak{D}$ *in the sense of the convention* XVI (iv), *then the set of "classes" of the model is of the power* $2^{\aleph_1}$.

As a somewhat curious corollary to theorem III we can state: *The s. c. Hessenberg's ring*[23]) *generated by all countable ordinal numbers of the basic set theory can be taken for a subring of certain "finite ordinals" of any of the just considered models* (of course, the converse is not true).

This result follows at once from R. SIKORSKI's immersion theorem VIII (of the paper [S]), as applied to the discretely ordered ring of $\overline{\mathfrak{S}}$.

Corrections to the paper [I]

1. The requirement (V) on p. 326, line 19 from above, can easily be deduced from the remaining requirements; thus (V) can be omitted.

2. On page 327, line 7 from above, "... sequences so that the following is true:" is to read: "... sequences of elements of *G* so that the following is true:"

3. After the definition of the notion of ideal (see last lines of page 327) for the case of the abstract Lindenbaum algebra, i. e. of a free generalized $\Omega\sigma$-algebra, we have omitted the algebraical characterization of the notion of the individual variable, resp. constant, as wholly dependent on the given ideal I (of the theory in question, see previous pages 324 and 328).

Indeed, we have to define: Given an ideal I of the free $\Omega\sigma$-algebra (as represented by the corresponding Lindenbaum algebra of the lower predicate calculus, see bottom of page 326 and too of page 327), then an individual sign ξ is an individual variable relatively to I if I is invariant under all the substitution-endomorphisms (of the free $\Phi\sigma$-algebra in question), say under Λ_{ξ,ξ^*}, of the form

$$\Lambda_{\xi,\xi^*}([\Phi(\ldots\xi\ldots)]) = \left[\Phi^*\left(\ldots\begin{Bmatrix}\xi\\ \xi^*\end{Bmatrix}\ldots\right)\right].$$

(The endomorphism Λ_{ξ,ξ^*} is given by an obvious induction, in view of the characteristical property (3) on page 327 of the definition of a free $\Omega\sigma$-algebra, in replacing the individual sign ξ by the individual sign ξ^*).

[23]) See [H] and [S] for the notion; the "exponentiation" is disregarded!

In short: ξ is an individual variable relatively to I (by definition) if $\Lambda_{\xi^*\xi}(\mathrm{I}) \subseteq \mathrm{I}$. —

Remark. If we wish to avoid the use of the representing Lindenbaum algebra, we have to take ‚ξ' and ‚ξ^*' for variable values of the natural indices (of members of the generating sequences, in the sense of the definition of the abstract Lindenbaum algebra see p. 327, line 5 from above). —

Now, any individual sign η that is not an individual variable relatively to I is defined as an individual constant relatively to I.

The reader is requested to supplement the page 327 by this omitted definition — without changing anything in the sequel.

4. In the new proof of theorem 5.31 of [G], on page 336, line 12 from bottom, no metamathematical notion of "a theory $\widehat{\Theta}_1$ stronger than Θ_1" needs to be considered, since the proof indeed is a very simple usual indirect proof.

5. On page 342 line 15 from above, we have omitted the (tacitly made) assumption that to every $x \in C$, the class of all the y with $y \tilde{\in} x$ exists, and moreover, is a set. This explicit assumption is to be inserted there.

On page 343, line 8 from the bottom, instead of $C \subseteq \mathbf{P}(C)$ write $\tilde{C} \subseteq \mathbf{P}(C)$.

On page 343, line 9 from above, instead of $\tilde{\mathbf{M}}$, $\mathbf{C}ls$, $\tilde{\in}$, write $\tilde{\mathbf{M}}$ $\tilde{\mathbf{C}}ls$, $\tilde{\in}$.

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Резюме

К ГЕДЕЛЕВСКОЙ АКСИОМАТИЧЕСКОЙ ТЕОРИИ МНОЖЕСТВ, II

ЛАДИСЛАВ РИГЕР (Ladislav Rieger), Прага

(Поступило в редакцию 22/II 1958 г.)

Настоящая работа является свободным продолжением работы автора [I] (под тем же названием в том же журнале, *82* (1957), 323—357).

Дан алгебраически-арифметический метод построения ненормальных моделей геделевской аксиоматической теории конечных множеств, в смысле аксиом A — E из [G], где аксиома бесконечности C 1 заменяется аксиомой non C 1 (конечности). Метод основан на известном обобщении диадических чисел Гензеля (Hensel).

Главным результатом является доказательство существования таких моделей, в которых имеется $\aleph_1$ конечных порядковых чисел.

После вступительного § 1 в § 2 приводятся необходимые основы теории т. н. диадических колец, в смысле

Определения II. Диадическим кольцом назовем дискретно упорядоченное кольцо с единицей, в котором мы имеем ещё добавочную операцию ,,возведения числа 2 в степень" для неотрицательных показателей, удовлетворяющую следующим условиям:

(d I) : $2^1 = 2$; (d II) : $2^x . 2^y = 2^{x+y}$; (d III) : $2^x > x$;

(d IV): для каждого y и каждой степени 2^x существуют элементы q, r так, что $y = 2^x . q + r$, $0 \leqq r < 2^x$; так как q определено однозначно, пишем $q = \left[\frac{y}{2^x}\right]$ = целая часть дроби $\frac{y}{2^x}$.

(d V): Для каждого y существует степень 2^x, которая еще делит y, но 2^{x+1} уже не делит y.

Так как каждый ненулевой элемент x диадического кольца можно однозначно записать в виде $x = 2^p(2q + 1)$, можно ввести и (обобщенную) диадическую норму $p = W(x)$ в смысле общей теории нормированных полей.

В § 3 основным является понятие т. н. теоретико-множественного диадического кольца (s-t-кольца), см. определение III. Это — диадическое кольцо с двумя добавочными примитивными операциями D_1, D_2, так называемыми первой и второй операцией области, и с добавочными аксиомами (s $\emptyset$) — (s *VIII*). Добавочные аксиомы выбраны так, чтобы (после расширения данного s-t-кольца в т. н. псевдоперфектное (почти совершенное) пополнение, (которое является обобщением расширения s-t-кольца целых чисел в кольцо целых диадических чисел Гензеля) можно было показать следующее:

Бинарное отношение ϵ_*, определенное формулой

$$x \epsilon_* y \equiv \left[\frac{y}{2^x}\right] - 2\left[\frac{y}{2^{x+1}}\right] = 1$$

(сперва только для неотрицательных x, y, а потом перенесенное на случай y из упомянутого псевдоперфектного пополнения) удовлетворяет всем аксиомам геделевской аксиоматической теории конечных множеств. (Содержание первой главной теоремы I.)

Наконец, в § 4 построена несчетная возрастающая последовательность счетных s-t-колец, начинающаяся s-t-кольцом целых чисел, объединение которой и является искомым несчетным s-t-кольцом; таким образом достигается (в смысле теоремы II) главный результат работы — теорема III. Метод построения расширения основан на обобщении метода Сколема (Skolem), использованного в работе [Sk].