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ON INTEGRATION BY PARTS

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A general theorem on the integration by parts for the Perron integral is proved by means of rearranging the terms in the partial sums.

We shall use the definition of the generalized Perron integral and the notations introduced in [1], section 1. It was proved in [1], section 1, 2, that the (generalized) Perron integral may be defined as a limit of partial sums (Theorem 1, 2, 1).

Let $U(\tau, t)$ be defined for $\tau \in \langle \tau_*, \tau^* \rangle$, $t \in \langle \tau_*, \tau^* \rangle$. Let us denote by $\int_{\tau_*}^{\tau^*} \mathrm{D}_t U(\tau, t)$ $(\int_{\tau_*}^{\tau^*} \mathrm{D}_t U)$ the integral which was denoted in [1] by $\int_{\tau_*}^{\tau^*} \mathrm{D} U(\tau, t)$. Let us define $\int_{\tau_*}^{\tau^*} \mathrm{D}_\tau U(\tau, t) = \int_{\tau_*}^{\tau^*} \mathrm{D}_t U_1(\tau, t)$ where $U_1(\tau, t) = U(t, \tau)$. It is simple to prove the following

Lemma. *If*

$$V(\tau, t) = V(t, \tau) \tag{1}$$

and if one of the integrals $\int_{\tau_*}^{\tau^*} \mathrm{D}_t V$, $\int_{\tau_*}^{\tau^*} \mathrm{D}_\tau V$ *exists, then the other one exists also and* $\int_{\tau_*}^{\tau^*} \mathrm{D}_t V = \int_{\tau_*}^{\tau^*} \mathrm{D}_\tau V$.

In this case we shall write $\int_{\tau_*}^{\tau^*} \mathrm{D} V$ instead of $\int_{\tau_*}^{\tau^*} \mathrm{D}_t V$ or $\int_{\tau_*}^{\tau^*} \mathrm{D}_\tau V$.

Let us put $V(\tau, t) = U(\tau, \tau) - U(\tau, t) - U(t, \tau) + U(t, t)$. Then $V(\tau, t)$ obviously fulfils (1).

Theorem. *Suppose that two of the integrals occurring in*

$$\int_{\tau_*}^{\tau^*} \mathrm{D}_t U + \int_{\tau_*}^{\tau^*} \mathrm{D}_\tau U = U(\tau^*, \tau^*) - U(\tau_*, \tau_*) - \int_{\tau_*}^{\tau^*} \mathrm{D} V \tag{2}$$

exist. Then the third one exists also and (2) *holds.*

P r o o f. Suppose for example that the integrals $\int_{\tau_*}^{\tau^*} \mathrm{D}_t U$ and $\int_{\tau_*}^{\tau^*} \mathrm{D}_\tau U$ exist. Then there exists a positive function $\delta_1(\tau)$ such that

$$\left|\int_{\tau_*}^{\tau^*} \mathrm{D}_t U - \sum_{i=1}^{s} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})]\right| \leqq \varepsilon$$

for an arbitrary subdivision $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\} \in \mathrm{A}(S_1)$, where

$$S_1 = \mathrm{E}[(t, \tau); \tau_* \leqq \tau \leqq \tau^*, \tau_* \leqq t \leqq \tau^*, \tau - \delta_1(\tau) \leqq t \leqq \tau + \delta_1(\tau)].$$

Similarly we find a function $\delta_2(t) > 0$ with respect to the integral $\int_{\tau_*}^{\tau^*} \mathrm{D}_\tau U$ and put $\delta(\tau) = \min(\delta_1(\tau), \delta_2(\tau))$,

$$S = E[(t, \tau), \tau_* \leqq \tau \leqq \tau^*, \tau_* \leqq t \leqq \tau^*, \tau - \delta(\tau) \leqq t \leqq \tau + \delta(\tau)] .$$

Let $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$ be an arbitrary subdivision subordinate to S. Then

$$\int_{\tau_*}^{\tau^*} \mathrm{D}_t U = \sum_{i=1}^{s} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})] + r_1 , \quad |r_1| < \varepsilon ;$$

$$\int_{\tau_*}^{\tau^*} \mathrm{D}_\tau U = \sum_{i=1}^{s} [U(\alpha_i, \tau_i) - U(\alpha_{i-1}, \tau_i)] + r_2 , \quad |r_2| < \varepsilon ;$$

$$\begin{aligned}
\int_{\tau_*}^{\tau^*} \mathrm{D}_t U + \int_{\tau_*}^{\tau^*} \mathrm{D}_\tau U = & \sum_{i=1}^{s} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})] + \\
& + \sum_{i=1}^{s} [U(\alpha_i, \tau_i) - U(\alpha_{i-1}, \tau_i)] + r_1 + r_2 = \\
& = \sum_{i=1}^{s} \{[-U(\alpha_i, \alpha_i) + U(\alpha_i, \tau_i) + U(\tau_i, \alpha_i) - U(\tau_i, \tau_i)] + \\
& + [U(\tau_i, \tau_i) - U(\tau_i, \alpha_{i-1}) - U(\alpha_{i-1}, \tau_i) + U(\alpha_{i-1}, \alpha_{i-1})]\} + \\
& + U(\tau^*, \tau^*) - U(\tau_*, \tau_*) + r_1 + r_2 = \\
& = U(\tau^*, \tau^*) - U(\tau_*, \tau_*) - \sum_{i=1}^{s} [V(\tau_i, \alpha_i) - V(\tau_i, \alpha_{i-1})] + \\
& + r_1 + r_2 .
\end{aligned}$$

Hence it follows that $\int_{\tau_*}^{\tau^*} \mathrm{D}V$ exists and that (2) holds. The proof in the other cases is quite similar.

Example. Let $f(\tau)$, $\varphi(\tau)$ be of bounded variation on $\langle\tau_*, \tau^*\rangle$ and let us put $U(\tau, t) = f(\tau)\,\varphi(t)$. Both integrals $\int_{\tau_*}^{\tau^*} \mathrm{D}_t U$ and $\int_{\tau_*}^{\tau^*} \mathrm{D}_\tau U$ exist,*) and according to our theorem $\int_{\tau_*}^{\tau^*} \mathrm{D}V$ exists also. Let us evaluate $\int_{\tau_*}^{\tau^*} \mathrm{D}V$.

*) The Perron integral $\int_{\tau_*}^{\tau^*} g(\tau)\, \mathrm{d}\varphi(\tau)$ exists if $g(\tau) = \sum_{i=1}^{k} a_i h_i(\tau)$, $h_i(\tau) = 1$ for $\tau \in \langle\beta_i, \gamma_i\rangle$, $h_i(\tau) = 0$ for τ non $\in \langle\beta_i, \gamma_i\rangle$, $\beta_i \leqq \gamma_i$. If $g(\tau) > f(\tau) (g(\tau) < f(\tau))$, $\tau \in \langle\tau_*, \tau^*\rangle$, then $\int_{\tau_*}^{\tau} g(t)\, \mathrm{d}\varphi(t)$ is a major (minor) function of $f(\tau)$ with respect to $\varphi(\tau)$ and thus $\int_{\tau_*}^{\tau^*} \mathrm{D}U(\tau, t) = \int_{\tau_*}^{\tau^*} f(\tau)\, \mathrm{d}\varphi(\tau)$ exists.

Denote by N the set of such τ that

$$|f(\tau+) - f(\tau)||\varphi(\tau+) - \varphi(\tau)| + |f(\tau) - f(\tau-)||\varphi(\tau) - \varphi(\tau-)| > 0 .$$

(We put $f(\tau^*+) = f(\tau^*)$, etc.)

N is countable and we shall prove that

$$\int_{\tau_*}^{\tau^*} \mathrm{D}V = \sum_{\tau \in N, \tau < \tau^*} (f(\tau+) - f(\tau))(\varphi(\tau+) - \varphi(\tau)) + \\ + \sum_{\tau \in N, \tau > \tau_*} (f(\tau) - f(\tau-))(\varphi(\tau) - \varphi(\tau-)) . \tag{3}$$

(The sums on the right converge absolutely.) Let us put

$$h(\tau) = \sum_{\lambda \in N, \lambda < \tau} (f(\lambda+) - f(\lambda))(\varphi(\lambda+) - \varphi(\lambda)) + \\ + \sum_{\lambda \in N, \lambda \leqq \tau} (f(\lambda) - f(\lambda-))(\varphi(\lambda) - \varphi(\lambda-)) , \tag{4}$$

$\psi(\tau) = \operatorname*{var}_{\tau_* \leqq \lambda \leqq \tau} f(\lambda) + \operatorname*{var}_{\tau_* \leqq \lambda \leqq \tau} \varphi(\lambda)$. Let $\varepsilon > 0$. We shall prove that $h(\tau) + \varepsilon\psi(\tau)$ is a major function of V. Since $V(\tau_0, \tau_0) = 0$, we have

$$V(\tau_0, \tau) - V(\tau_0, \tau_0) = (f(\tau) - f(\tau_0))(\varphi(\tau) - \varphi(\tau_0)) .$$

If τ_0 non $\in N$, then there exists a $\delta(\tau_0) > 0$ such that

$$|V(\tau_0, \tau) - V(\tau_0, \tau_0)| \leqq \frac{\varepsilon}{2} |\psi(\tau) - \psi(\tau_0)| ,$$

$$|h(\tau) - h(\tau_0)| \leqq \frac{\varepsilon}{2} |\psi(\tau) - \psi(\tau_0)| \quad \text{for} \quad |\tau - \tau_0| \leqq \delta(\tau_0) . \tag{5}$$

If $\tau_0 \in N$, then there exists a $\delta(\tau_0) > 0$ such that

$$|h(\tau) - h(\tau_0+)| \leqq \frac{\varepsilon}{2} [\psi(\tau) - \psi(\tau_0)] \quad \text{for} \quad \tau_0 < \tau \leqq \tau_0 + \delta(\tau_0) ,$$

$$|h(\tau_0-) - h(\tau)| \leqq \frac{\varepsilon}{2} [\psi(\tau_0) - \psi(\tau)] \quad \text{for} \quad \tau_0 - \delta(\tau_0) \leqq \tau < \tau_0 ,$$

$$|R(\tau_0, \tau)| \leqq \frac{\varepsilon}{2} |\psi(\tau) - \psi(\tau_0)| \quad \text{for} \quad 0 < |\tau - \tau_0| \leqq \delta(\tau_0) ,$$

where $R(\tau_0, \tau)$ is defined by

$$V(\tau_0, \tau) - V(\tau_0, \tau_0) = (f(\tau_0+) - f(\tau_0))(\varphi(\tau_0+) - \varphi(\tau_0)) + R(\tau_0, \tau) \quad (\tau > \tau_0), \tag{6}$$

$$V(\tau_0, \tau) - V(\tau_0, \tau_0) = (f(\tau_0) - f(\tau-))(\varphi(\tau_0) - \varphi(\tau_0-)) + R(\tau_0, \tau) \quad (\tau < \tau_0) . \tag{7}$$

From (4), (5), (6) and (7) we deduce that

$$(\tau - \tau_0)[h(\tau) + \varepsilon\psi(\tau) - h(\tau_0) - \varepsilon\psi(\tau_0)] \geqq (\tau - \tau_0)\, [V(\tau_0, \tau) - V(\tau_0, \tau_0)] ,$$

if $|\tau - \tau_0| \leqq \delta(\tau_0)$, so that $h(\tau) + \varepsilon\psi(\tau)$ is a major function of V. Similarly $h(\tau) - \varepsilon\psi(\tau)$ is a minor function of V and consequently $\int_{\tau_*}^{\tau^*} \mathrm{D}V = h(\tau^*) - h(\tau_*)$. As

$$\int_{\tau_*}^{\tau^*} \mathrm{D}_t U = \int_{\tau_*}^{\tau^*} f(\tau)\,\mathrm{d}\varphi(\tau)\,, \quad \int_{\tau_*}^{\tau^*} \mathrm{D}_\tau U = \int_{\tau_*}^{\tau^*} \varphi(\tau)\,\mathrm{d}f(\tau)\,,$$

it follows from the theorem that

$$\int_{\tau_*}^{\tau^*} f(\tau)\,\mathrm{d}\varphi(\tau) + \int_{\tau_*}^{\tau^*} \varphi(\tau)\,\mathrm{d}f(\tau) = f(\tau^*)\,\varphi(\tau^*) - f(\tau_*)\,\varphi(\tau_*) -$$
$$- \sum_{\tau \in N, \tau < \tau^*} (f(\tau+) - f(\tau))(\varphi(\tau+) - \varphi(\tau)) + \sum_{\tau \in N, \tau \leqq \tau^*} (f(\tau) - f(\tau-))(\varphi(\tau) - \varphi(\tau-))\,. \tag{8}$$

Let us note that according to a known theorem (see [2], (14·1), p. 102)

$$\int_{\tau_*}^{\tau^*} f(\tau)\,\mathrm{d}\varphi(\tau) + \int_{\tau_*}^{\tau^*} \varphi(\tau)\,\mathrm{d}f(\tau) = f(\tau^*)\,\varphi(\tau^*) - f(\tau_*)\,\varphi(\tau_*)$$

if for every $\tau \in \langle \tau_*, \tau^* \rangle$ at least one of the functions $f(\tau)$, $\varphi(\tau)$ is continuous or if both functions $f(\tau), \varphi(\tau)$ are regular ($f(\tau)$ is regular, if $2f(\tau) = f(\tau+) + f(\tau-)$ for $\tau \in (\tau_*, \tau^*)$, $f(\tau_*) = f(\tau_*-)$, $f(\tau^*) = f(\tau^*+)$). This result is a simple consequence of (8).

LITERATURE

[1] *J. Kurzweil:* Generalized Ordinary Differential Equations and Continuous Dependence on a Parameter, Czech. Math. Journal *7* (*82*) 1957, 3, 418—449.

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Резюме

ОБ ИНТЕГРИРОВАНИИ ПО ЧАСТЯМ

ЯРОСЛАВ КУРЦВЕЙЛЬ (Jaroslav Kurzweil), Прага

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В работе [1] был введен обобщенный интеграл Перрона и было доказано, что этот интеграл можно определить как предел частичных сумм. В настоящей работе доказывается общая теорема об интегрировании по частям путем присоединения (нового упорядочения) частичных сумм.