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EXPERIENCE IN STATISTICAL DECISION PROBLEMS

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In this paper we consider a statistical decision problem the solution of which is a random decision process constructed sequentially using past experience. The sequence of losses is shown to be $(C, 1)$ convergent to the Bayes loss with probability one.

1. Basic notations

Let $\mathsf{X} = (X, \mathfrak{S}, \mu)$ be a probability space. For avoiding misunderstandings, the random variables on X (i. e. absolutely integrable functions on X) or random vectors on X (i. e. finite sequences of random variables on X) will be denoted by Greek letters, the numbers or vectors (finite sequences of numbers) will be denoted by Latin letters.

An m-dimensional random vector τ generates the probability space

$$\mathsf{X}^\tau = (R_m, \mathfrak{B}_m, \mu\tau^{-1}) ,$$

where R_m is the m-dimensional Euclidean space, $\mathfrak{B}_m$ the σ-algebra of all Borel sets $B \subset R_m$, and

$$\mu\tau^{-1}(B) = \mu(\tau^{-1}(B))$$

for every $B \in \mathfrak{B}_m$. If h is a random variable on X^τ, the composite function $h\tau$ is a random variable on X. If φ is a random variable on X, then

$$\mathbf{E}\varphi = \int_X \varphi \, \mathrm{d}\mu \, , \quad \mathbf{E}^\tau h = \int_{R_m} h \, \mathrm{d}\mu\tau^{-1} \, .$$

The symbol $\mathbf{E}_\tau\varphi$ denotes the conditional expectation of φ with respect to τ; thus $\mathbf{E}_\tau\varphi$ is a random variable on X^τ. An integration variable will be never denoted by a letter; we denote it, if necessary, by an asterisk.

For the sake of simplicity, we introduce the following conventions: Let

$$\tau_1 = (\tau_{11}, \tau_{12}, \ldots, \tau_{1m_1})$$

and

$$\tau_2 = (\tau_{21}, \tau_{22}, \ldots, \tau_{2m_2})$$

be two random vectors. We denote by (τ_1, τ_2) the random vector $(\tau_{11}, \tau_{12}, \ldots, \ldots, \tau_{1m_1}, \tau_{21}, \ldots, \tau_{2m_2})$. Further we shall say that H is τ_1, τ_2 integrable if (i) H is a function on $R_{m_1} \times R_{m_2}$, (ii) for every fixed $y \in R_{m_1}$, $H(y, x)$ is a random variable on X^{τ_2}, (iii) $H(\tau_1, \tau_2)$ is a random variable on X.

We shall write $\tau_1 = \tau_2$ if $\tau_1(x) = \tau_2(x)$ with probability one; by the convergence of a sequence of random variables or vectors it is meant the convergence with probability one.

In the following section we state Theorem I, immediate consequence of which is Theorem II (section 3), the main result of this paper.

2

The following lemma stated here for convenience follows from Theorem III of [3] and the fact, that dealing with expectations of finite number of random variables we may suppose without loss of generality the basic space X to be $(R_t, \mathfrak{B}_t, \mu)$, where t is a suitably chosen integer.

Lemma 1. *Let τ_1, τ_2 be independent random vectors, H a function integrable τ_1, τ_2. Then, denoting*

$$g(y) = \mathbf{E}H(y, \tau_2)\,,$$

it holds

$$g = \mathbf{E}_{\tau_1} H(\tau_1, \tau_2)\,.$$

Theorem 1. *Let $\{\pi_n\}$, $\{\vartheta_n\}$ are two sequence of m_1 respectively m_2-dimensional random vectors. Let ϑ_n be independent of $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n$.*[1] *Let h be a function integrable π_i, ϑ_i and*

$$\mathbf{E}h(r, \vartheta_n) = b \qquad \textit{for every } r \in R_{m_1}\,,$$
$$\mathbf{E}h^2(r, \vartheta_n) < M \qquad \textit{for every } r \in R_{m_1}\,,$$

where b and M are real numbers. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} h(\pi_i, \vartheta_i) = b\,.$$

This assertion holds also if we replace the condition

$$K(r) = \mathbf{E}h(r, \vartheta_n) = b \quad \textit{(say)}$$

by the weakened condition

$$K\pi_n \to b\,.$$

Proof. First we shall assume that $\mathbf{E}h(y, \pi_n) = b = 0$. Let $0 < n_1 < n_2$ be two integers. Denote

$$\tau_1 = [\pi_{n_1}, \pi_{n_2}, \vartheta_{n_1}]\,, \quad \tau_2 = \vartheta_{n_2}\,.$$

[1]) In particular this condition of independence is satisfied if ϑ_n are independent and π_n is function at most of $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}$.

By hypothesis τ_1 and τ_2 are independent. Clearly, from integrability π_i, ϑ_i of h it follows the integrability τ_1, τ_2 of H defined by

$$H(\tau_1, \tau_2) = h(\pi_{n_1}, \vartheta_{n_1})\, h(\pi_{n_2}, \vartheta_{n_2}) \,.$$

Thus lemma 1 applied, we get $g = 0$ and

$$\mathsf{E}h(\pi_{n_1}, \vartheta_{n_1})\, h(\pi_{n_2}, \vartheta_{n_2}) = \mathsf{E}H(\tau_1, \tau_2) = \mathsf{E}^{\tau_1}\mathsf{E}_{\tau_1} H(\tau_1, \tau_2) = \mathsf{E}^{\tau_1} g = 0 \,.$$

Thus, denoting $\zeta_n = h(\pi_n, \vartheta_n)$, we have proved that $\zeta_1, \zeta_2, \ldots$ are mutually orthogonal. Clearly $\mathsf{E}\zeta_n^2 = \sigma_n^2 < M$ and thus

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} \log^2 n < +\infty \,.$$

From [2] Theorem 5.2 it follows $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \zeta_i = 0$.

We have proved the first part of the theorem for $b = 0$. The generalization for $a \neq 0$ and the second part follow immediately.

3

Let k, s, m, be fixed positive integers. We shall consider a sequence of decision problems. Each problem is of this type:

The statistician observes (sequentially) a random vector $\varphi = [\varphi^1, \varphi^2, \ldots, \varphi^s]$. The random variable φ^i are distributed independently with the common distribution ν. The statistician knows that ν is an element of the finite set $\mathfrak{N} = \{\nu_1, \nu_2, \ldots, \nu_k\}$ and selects, using observations on φ, a decision d from the set $\mathfrak{D} = \{d_1, \ldots, d_m\}$. If a decision function δ is used, then every value of φ determines a decision $d \in \mathfrak{D}$ and the number of coordinates of φ observed. Let us denote by $l'(\delta, i, \varphi)$ the loss plus the cost of experimentation if δ is used, φ observed and ν_i is the "true" distribution.

It will be assumed that also i, the index of the true distribution ν_i is a random variable γ, unobservable for the statistician and taking the values $1, 2, \ldots, k$. Obviously

$$\nu_i(B) = \mu(\{x; \varphi^1(x) \in B\}|_{\gamma(x)=i}) \,.$$

Thus we may denote by ϑ the random vector $[\gamma, \varphi]$ and call it random decision problem. For each δ and each value $\vartheta(x) = [\gamma(x), \varphi(x)]$, the symbol $l(\delta, \vartheta(x)) = l'(\delta, \gamma(x), \varphi(x))$ denotes the loss when δ is used and the elementary event x occurs.

The conditional expected loss, given $\gamma(x) = i$, will be denoted by $L_i(\delta)$.

If the statistician estimates by $r = (r_1, \ldots, r_k)$, $r_i \geq 0$, $\sum_{i=1}^{k} r_i = 1$ the true apriori distribution p given by

$$p = (\mu\gamma^{-1}(\{1\}), \ldots, \mu\gamma^{-1}(\{k\}))\,,$$

he uses the r-Bayes decision function δ_r which minimises the expression

$$\sum_{i=1}^{k} r_i L_i(\delta) .$$

Clearly $L(\delta_p) = \min_r L(\delta_r) = \min_\delta L(\delta)$, where $L(\delta)$ denotes the expected loos, if the decision function δ is used, i. e., $L(\delta) = \mathbf{E}\, l\,(\delta, \vartheta)$.

We note that the existence of Bayes decision functions is guaranteed by the finitenesse of $\mathfrak{N}$, $\mathfrak{D}$, and because φ is finite-dimensional.

From the Theorem 3.6 p. 89 of [4] it follows further that

$$\lim_{r \to p} L(\delta_r) = L(\delta_p) . \qquad (*)$$

This suggest to consider the sequence of decisions problems of the same type and to use an estimate based on past experience to construct better and better decision functions. In Theorem II we show that not only the expected losses converge to the minimum loss $L(\delta_p)$. In fact, if we denote by λ_n the loss occuring at the n-th step of the process, then

$$\frac{1}{n} \sum_{i=1}^{n} \lambda_i \to L(\delta) .$$

Thus we shall consider a sequence $\vartheta_1 = \vartheta, \vartheta_2, \vartheta_3, \ldots$ of independently and identically distributed random vectors (i. e. random decision problems). A sequence of random vectors $\pi_1, \pi_2, \ldots$ (where for each $\pi_n = (\pi_n^1, \ldots, \pi_n^k)$, $\pi_n^j \geqq 0, \sum_{j=1}^{k} \pi_n^j = 1$ hold), will be called regular sequence estimating p if $\pi_n \to p$ and ϑ_n is independent of $\pi_1, \pi_2, \ldots, \pi_n, \vartheta_1, \ldots, \vartheta_{n-1}$ for every $n = 1, 2, \ldots$

The statistician, who uses at the n-th step the decision function δ_{π_n} pays the loss $\lambda_n = l(\delta_{\pi_n}, \vartheta_n) = H(\pi_n, \vartheta_n)$. But from the constructive definition of δ_r it follows easily that H is measurable π_n, ϑ_n. From the finitenesse of $\mathfrak{D}$, $\mathfrak{N}$ and from the fact that φ_n are finite dimensional it follows that H is bounded and therefore H satisfies the conditions of the Theorem I. Obviously, if we denote

$$K(r) = \mathbf{E} H(r, \vartheta_n) ,$$

we get

$$K(r) = L(\delta_r) ;$$

from (*) and from $\pi_n \to p$ we have finally

$$K\pi_n \to L(\delta_p) .$$ [2]

Hence from Theorem I it follows

Theorem II. *If π_n is a regular sequence estimating p, then the sequence of losses, if the sequence $\{\delta_{\pi_n}\}$ of decision functions is used, is $(C,1)$ convergent with probability one to the minimum loss $L(\delta_p)$. In symbols*

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[2]) We note that $K\pi_n$ is a random variable and is not $\mathbf{E} H(\pi_n, \vartheta_n)$.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \lambda_n = L(\delta_p) .$$

Example: Suppose $k = 2$, $m = 1$, $s = 2$, the weight function

$$W(i, j) = \begin{cases} 0 & \text{if} \quad i = j \\ 1 & \text{if} \quad i \neq j \end{cases}$$

and the cost of experimentation to be identically zero. Then if h_i is a generalized probability density of ν_i with respect to some measure ν (for $i = 1, 2$), then the decision function $\delta_{(r_1, r_2)}$ accepts d_1 if

$$\varphi_n(x) \in \{y \in R_1; r_1 h_1(y) > r_2 h_2(y)\}$$

and d_2 otherwise.

If ν_1 and ν_2 are not identical, then a regular sequence π_n estimating

$$p = (\mu\gamma_n^{-1}(\{1\}); \mu\gamma_n^{-1}(\{2\}))$$

can be defined as follows:

$$\pi_n(x) = \left[\frac{\frac{1}{n-1} \sum_{i=1}^{n-1} c_E \varphi_n(x)}{\nu_1(E) - \nu_2(E)}, \frac{\frac{1}{n-1} \sum_{i=1}^{n-1} c_{R_1 - E} \varphi_n(x)}{\nu_1(E) - \nu_2(E)]} \right],$$

where $\nu_1(E) \neq \nu_2(E)$ and c_E and $c_{R_1 - E}$ are characteristic functions of the sets E and $R_1 - E$ respectively.

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Резюме

ОПЫТ В СТАТИСТИЧЕСКИХ РЕШАЮЩИХ ПРОБЛЕМАХ

ВАЦЛАВ ФАБИАН, АНТОНИН ШПАЧЕК (VÁCLAV FABIAN, ANTONÍN ŠPAČEK),
Прага.

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В статье рассматривается статистическая проблема, решением которой является случайная последовательность решающих функций; построение этой последовательности опирается на каждом шагу о предыдущий опыт. Последовательность потерь сходится $(C, 1)$ к потере Бэйеса с вероятностью 1.